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A Semigroup Is Finite Iff It Is Chain-Finite and Antichain-Finite

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Abstract: A subset A of a semigroup S is called a *chain* (*antichain*) if $ab \in \{a,b\}$ ($ab \notin \{a,b\}$) for any (distinct) elements $a, b \in A$. A semigroup S is called *periodic* if for every element $x \in S$ there exists $n \in \mathbb{N}$ such that x^n is an idempotent. A semigroup S is called (*anti*)*chain-finite* if S contains no infinite (anti)chains. We prove that each antichain-finite semigroup S is periodic and for every idempotent e of S the set $\sqrt[\infty]{e} = \{x \in S : \exists n \in \mathbb{N}\ \ (x^n = e)\}$ is finite. This property of antichain-finite semigroups is used to prove that a semigroup is finite if and only if it is chain-finite and antichain-finite. Furthermore, we present an example of an antichain-finite semilattice that is not a union of finitely many chains.

Keywords: semigroup; semilattice; chain; antichain

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1. Introduction

It is well-known that a partially ordered set X is finite iff all chains and antichains in X are finite. The notions of chain and antichain are well-known in the theory of order (see, e.g., ([1] (O-1.6)) or [2]). In this paper we present a similar characterization of finite semigroups in terms of finite chains and antichains.

Let us recall that a *magma* is a set S endowed with a binary operation $S \times S \to S$, $\langle x, y\rangle \mapsto xy$. If the binary operation is associative, then the magma S is called a *semigroup*. A *semilattice* is a commutative semigroup whose elements are idempotents. Each semilattice S carries a natural partial order $\leq$ defined by $x \leq y$ iff $xy = yx = x$. Observe that two elements x, y of a semilattice are comparable with respect to the partial order $\leq$ if and only if $xy \in \{x, y\}$. This observation motivates the following algebraic definition of chains and antichains in any magma.

A subset A of a magma S is defined to be

- a *chain* if $xy \in \{x, y\}$ for any elements $x, y \in A$;
- an *antichain* if $xy \notin \{x, y\}$ for any distinct elements $x, y \in A$.

The definition implies that each chain consists of idempotents.

A magma S is defined to be (*anti*)*chain-finite* if it contains no infinite (anti)chains.

Let us note that chain-finite semilattices play an important role in the theory of complete topological semigroups. In [3], Stepp showed that for each homomorphism $f : X \to Y$ from a chain-finite semilattice X to a Hausdorff topological semigroup Y, the image $f[X]$ is closed in Y. Banakh and Bardyla [4] extended the result of Stepp to the following characterization:

Theorem 1. *For a semilattice X the following conditions are equivalent:*

- *X is chain-finite;*
- *X is closed in each Hausdorff topological semigroup containing X as a discrete subsemigroup;*
- *For each homomorphism $f : X \to Y$ into a Hausdorff topological semigroup Y, the image $f[X]$ is closed;*

For other completeness properties of chain-finite semilattices see [4–6]. Antichain-finite posets and semilattices were investigated by Yokoyama [7].

The principal result of this note is the following theorem characterizing finite semigroups.

Theorem 2. *A semigroup S is finite if and only if it is chain-finite and antichain-finite.*

A crucial step in the proof of this theorem is the following proposition describing the (periodic) structure of antichain-finite semigroups.

A semigroup S is called *periodic* if for every $x \in S$ there exists $n \in \mathbb{N}$ such that x^n is an idempotent of S. In this case

$$S = \bigcup_{e \in E(S)} \sqrt[\infty]{e},$$

where $E(S) = \{x \in S : xx = x\}$ is the set of idempotents of S and

$$\sqrt[\infty]{e} = \{x \in S : \exists n \in \mathbb{N} \ \ (x^n = e)\}$$

for $e \in E(S)$.

Proposition 1. *Each antichain-finite semigroup S is periodic and for every $e \in E(S)$ the set $\sqrt[\infty]{e}$ is finite.*

Theorem 2 and Proposition 1 will be proved in the next section.

Remark 1. *Theorem 2 does not generalize to magmas. To see this, consider the set of positive integers $\mathbb{N}$ endowed with the following binary operation: $nm = n$ if $n < m$ and $nm = 1$ if $n \geq m$. This magma is infinite but each nonempty chain in the magma is of the form $\{1, n\}$ for some $n \in \mathbb{N}$, and each nonempty antichain in this magma is a singleton.*

Next we present a simple example of an antichain-finite semilattice which is not a union of finitely many chains.

Example 1. *Consider the set*

$$S = \{\langle 2n-1, 0\rangle : n \in \mathbb{N}\} \cup \{\langle 2n, m\rangle : n, m \in \mathbb{N},\ m \leq 2n\}$$

endowed with the semilattice binary operation

$$\langle x, i\rangle \cdot \langle y, j\rangle = \begin{cases} \langle x, i\rangle & \text{if } x = y \text{ and } i = j; \\ \langle x-1, 0\rangle & \text{if } x = y \text{ and } i \neq j; \\ \langle x, i\rangle & \text{if } x < y; \\ \langle y, j\rangle & \text{if } y < x. \end{cases}$$

It is straightforward to check that the semilattice S has the following properties:

1. *S is antichain-finite;*
2. *S has arbitrarily long finite antichains;*
3. *S is not a union of finitely many chains;*
4. *The subsemilattice $L = \{\langle 2n-1, 0\rangle : n \in \mathbb{N}\}$ of S is a chain;*

5. *S admits a homomorphism $r : S \to L$ such that $r^{-1}(\langle x, 0\rangle) = \{\langle y, i\rangle \in S : y \in \{x, x+1\}\}$ is finite for every element $\langle x, 0\rangle \in L$.*

Example 1 motivates the following question.

Question 1. *Let S be an antichain-finite semilattice. Is there a finite-to-one homomorphism $r : S \to Y$ to a semilattice Y which is a finite union of chains?*

A function $f : X \to Y$ is called *finite-to-one* if for every $y \in Y$ the preimage $f^{-1}(y)$ is finite.

2. Proofs of the Main Results

In this section, we prove some lemmas implying Theorem 2 and Proposition 1. More precisely, Proposition 1 follows from Lemmas 1 and 4; Theorem 2 follows from Lemma 5.

The following lemma exploit ideas of Theorem 1.9 from [8].

Lemma 1. *Every antichain-finite semigroup S is periodic.*

Proof. Given any element $x \in S$ we should find a natural number $n \in \mathbb{N}$ such that x^n is an idempotent. First we show that $x^n = x^m$ for some $n \neq m$. Assuming that $x^n \neq x^m$ for any distinct numbers n, m, we conclude that the set $A = \{x^n : n \in \mathbb{N}\}$ is infinite and for any $n, m \in \mathbb{N}$ we have $x^n x^m = x^{n+m} \notin \{x^n, x^m\}$, which means that A is an infinite antichain in S. However, such an antichain cannot exist as S is antichain-finite. This contradiction shows that $x^n = x^m$ for some numbers $n < m$ and then for the number $k = m - n$ we have $x^{n+k} = x^m = x^n$. By induction we can prove that $x^{n+pk} = x^n$ for every $p \in \mathbb{N}$. Choose any numbers $r, p \in \mathbb{N}$ such that $r + n = pk$ and observe that

$$x^{r+n}x^{r+n} = x^{r+n}x^{pk} = x^r x^{n+pk} = x^r x^n = x^{r+n},$$

which means that x^{r+n} is an idempotent and hence S is periodic. □

An element $1 \in S$ is called an *identity* of S if $x1 = x = 1x$ for all $x \in S$. For a semigroup S let $S^1 = S \cup \{1\}$ where 1 is an element such that $x1 = x = 1x$ for every $x \in S^1$. If S contains an identity, then we will assume that 1 is the identity of S and hence $S^1 = S$.

For a set $A \subseteq S$ and element $x \in S$ we put

$$xA = \{xa : a \in A\} \quad \text{and} \quad Ax = \{ax : a \in A\}.$$

For any element x of a semigroup S, the set

$$H_x = \{y \in S : yS^1 = xS^1 \ \wedge\ S^1y = S^1x\}$$

is called the *$\mathcal{H}$-class* of x. By Lemma I.7.9 [9], for every idempotent e its $\mathcal{H}$-class H_e coincides with the maximal subgroup of S that contains the idempotent e.

Lemma 2. *If a semigroup S is antichain-finite, then for every idempotent e of S its $\mathcal{H}$-class H_e is finite.*

Proof. Observe that the set $H_e \setminus \{e\}$ is an antichain (this follows from the fact that the left and right shifts in the group H_e are injective). Since S is antichain-finite, the antichain $H_e \setminus \{e\}$ is finite and so is the set H_e. □

Lemma 3. *If a semigroup S is antichain-finite, then for every idempotent e in S we have*

$$(H_e \cdot \sqrt[\infty]{e}) \cup (\sqrt[\infty]{e} \cdot H_e) \subseteq H.$$

Proof. Given any elements $x \in \sqrt[\infty]{e}$ and $y \in H_e$, we have to show that $xy \in H_e$ and $yx \in H_e$. Since $x \in \sqrt[\infty]{e}$, there exists a number $n \in \mathbb{N}$ such that $x^n = e$. Then $x^{n+1}S^1 = exS^1 \subseteq eS^1$ and $eS^1 = x^{2n}S^1 \subseteq x^{n+1}S^1$, and hence $eS^1 = x^{n+1}S$. By analogy we can prove that $S^1e = S^1x^{n+1}$. Therefore, $x^{n+1} \in H_e$.

Then $xy = x(ey) = (xe)y = (xx^n)y = x^{n+1}y \in H_e$ and $yx = (ye)x = y(ex) = yx^{n+1} \in H_e$. $\square$

For each $k \in \mathbb{N}$ by $[\mathbb{N}]^k$ we denote the set of all k-element subsets of $\mathbb{N}$. The proofs of the next two lemmas essentially use the classical Ramsey Theorem, so let us recall its formulation, see ([10] (p. 16)) for more details.

Theorem 3 (Ramsey)**.** *For any $n, k \in \mathbb{N}$ and map $\chi : [\mathbb{N}]^k \to n = \{0, \dots, n-1\}$ there exists an infinite subset $I \subseteq \mathbb{N}$ such that $\chi\big[[I]^k\big] = \{c\}$ for some number $c \in n$.*

Lemma 4. *If a semigroup S is antichain-finite, then for every idempotent $e \in E(S)$ the set $\sqrt[\infty]{e}$ is finite.*

Proof. By Lemma 2, the $\mathcal{H}$-class H_e is finite. Assuming that $\sqrt[\infty]{e}$ is infinite, we can choose a sequence $(x_n)_{n\in\omega}$ of pairwise distinct points of the infinite set $\sqrt[\infty]{e} \setminus H_e$.

Let $P = \{\langle n, m\rangle \in \omega \times \omega : n < m\}$ and $\chi : P \to 5 = \{0, 1, 2, 3, 4\}$ be the function defined by

$$\chi(n, m) = \begin{cases} 0 & \text{if } x_n x_m = x_n; \\ 1 & \text{if } x_m x_n = x_n; \\ 2 & \text{if } x_n x_m = x_m; \\ 3 & \text{if } x_m x_n = x_m; \\ 4 & \text{otherwise.} \end{cases}$$

By the Ramsey Theorem 3, there exists an infinite subset $\Omega \subseteq \omega$ such that $\chi[P \cap (\Omega \times \Omega)] = \{c\}$ for some $c \in \{0, 1, 2, 3, 4\}$.

If $c = 0$, then $x_n x_m = x_n$ for any numbers $n < m$ in Ω. Fix any two numbers $n < m$ in Ω. By induction we can prove that $x_n x_m^p = x_n$ for every $p \in \mathbb{N}$. Since $x_m \in \sqrt[\infty]{e}$, there exists $p \in \mathbb{N}$ such that $x_m^p = e$. Then $x_n = x_n x_m^p = x_n e \in H_e$ by Lemma 3. However, this contradicts the choice of x_n.

By analogy we can derive a contradiction in cases $c \in \{1, 2, 3\}$.

If $c = 4$, then the set $A = \{x_n\}_{n\in\Omega}$ is an infinite antichain in S, which is not possible as the semigroup S is antichain-finite.

Therefore, in all five cases we obtain a contradiction, which implies that the set $\sqrt[\infty]{e}$ is finite. $\square$

Our final lemma implies the non-trivial "if" part of Theorem 2.

Lemma 5. *A semigroup S is finite if it is chain-finite and antichain-finite.*

Proof. Assume that S is both chain-finite and antichain-finite. By Lemma 1, the semigroup S is periodic and hence $S = \bigcup_{e\in E(S)} \sqrt[\infty]{e}$. By Lemma 4, for every idempotent $e \in E(S)$ the set $\sqrt[\infty]{e}$ is finite. Now it suffices to prove that the set $E(S)$ is finite.

To derive a contradiction, assume that $E(S)$ is infinite and choose a sequence of pairwise distinct idempotents $(e_n)_{n\in\omega}$ in S. Let $P = \{\langle n, m\rangle \in \omega \times \omega : n < m\}$ and $\chi : P \to \{0, 1, 2, 3, 4, 5\}$ be the function defined by the formula

$$\chi(n,m) = \begin{cases} 0 & \text{if } e_n e_m \in \{e_n, e_m\} \text{ and } e_m e_n \in \{e_n, e_m\}; \\ 1 & \text{if } e_n e_m = e_n \text{ and } e_m e_n \notin \{e_n, e_m\}; \\ 2 & \text{if } e_n e_m = e_m \text{ and } e_m e_n \notin \{e_n, e_m\}; \\ 3 & \text{if } e_n e_m \notin \{e_n, e_m\} \text{ and } e_m e_n = e_n; \\ 4 & \text{if } e_n e_m \notin \{e_n, e_m\} \text{ and } e_m e_n = e_m; \\ 5 & \text{if } e_n e_m \notin \{e_n, e_m\} \text{ and } e_m e_n \notin \{e_n, e_m\}. \end{cases}$$

The Ramsey Theorem 3 yields an infinite subset $\Omega \subseteq \omega$ such that $\chi[P \cap (\Omega \times \Omega)] = \{c\}$ for some $c \in \{0,1,2,3,4,5\}$.

Depending on the value of c, we shall consider six cases.

If $c = 0$ (resp. $c = 5$), then $\{e_n\}_{n\in\omega}$ is an infinite (anti)chain in S, which is forbidden by our assumption.

Next, assume that $c = 1$. Then $e_n e_m = e_n$ and $e_m e_n \notin \{e_n, e_m\}$ for any numbers $n < m$ in Ω. For any number $k \in \Omega$, consider the set $Z_k = \{e_n e_k : k < n \in \Omega\}$. Observe that for any $e_n e_k, e_m e_k \in Z_k$ we have

$$(e_n e_k)(e_m e_k) = e_n(e_k e_m)e_k = e_n e_k e_k = e_n e_k,$$

which means that Z_k is a chain. Since S is chain-finite, the chain Z_k is finite.

By induction we can construct a sequence of points $(z_k)_{k\in\omega} \in \prod_{k\in\omega} Z_k$ and a decreasing sequence of infinite sets $(\Omega_k)_{k\in\omega}$ such that $\Omega_0 \subseteq \Omega$ and for every $k \in \omega$ and $n \in \Omega_k$ we have $e_n e_k = z_k$ and $n > k$. Choose an increasing sequence of numbers $(k_i)_{i\in\omega}$ such that $k_0 \in \Omega_0$ and $k_i \in \Omega_{k_{i-1}}$ for every $i \in \mathbb{N}$. We claim that the set $Z = \{z_{k_i} : i \in \omega\}$ is a chain. Take any numbers $i, j \in \omega$ and choose any number $n \in \Omega_{k_i} \cap \Omega_{k_j}$.

If $i \le j$, then

$$z_{k_i} z_{k_j} = (e_n e_{k_i})(e_n e_{k_j}) = e_n(e_{k_i} e_n)e_{k_j} = e_n e_{k_i} e_{k_j} = e_n e_{k_i} = z_{k_i}.$$

If $i > j$, then $k_i \in \Omega_{k_{i-1}} \subseteq \Omega_{k_j}$ and hence

$$z_{k_i} z_{k_j} = (e_n e_{k_i})(e_n e_{k_j}) = e_n(e_{k_i} e_n)e_{k_j} = e_n(e_{k_i} e_{k_j}) = e_n z_{k_j} = e_n(e_n e_{k_j}) = e_n e_{k_j} = z_{k_j}.$$

In both cases we obtain that $z_{k_i} z_{k_j} \in \{z_{k_i}, z_{k_j}\}$, which means that the set $Z = \{z_{k_i} : i \in \omega\}$ is a chain. Since S is chain-finite, the set Z is finite. Consequently, there exists $z \in Z$ such that the set $\Lambda = \{i \in \omega : z_{k_i} = z\}$ is infinite. Choose any numbers $i < j$ in the set Λ and then choose any number $n \in \Omega_{k_j} \subseteq \Omega_{k_i}$. Observe that $k_j \in \Omega_{k_{j-1}} \subseteq \Omega_{k_i}$ and hence $e_{k_j} e_{k_i} = z_{k_i} = z$. Then

$$e_{k_j} = e_{k_j} e_{k_j} = (e_{k_j} e_n)e_{k_j} = e_{k_j}(e_n e_{k_j}) = e_{k_j} z_{k_j} = e_{k_j} z = e_{k_j} z_{k_i} = e_{k_j}(e_{k_j} e_{k_i}) = e_{k_j} e_{k_i} \notin \{e_{k_j}, e_{k_j}\}$$

as $c = 1$.

By analogy we can prove that the assumption $c \in \{2,3,4\}$ also leads to a contradiction. □

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