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Common Fixed Point Results for Almost $\mathcal{R}_g$-Geraghty Type Contraction Mappings in b_2-Metric Spaces with an Application to Integral Equations

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Abstract: In this paper, we define almost $\mathcal{R}_g$-Geraghty type contractions and utilize the same to establish some coincidence and common fixed point results in the setting of b_2-metric spaces endowed with binary relations. As consequences of our newly proved results, we deduce some coincidence and common fixed point results for almost g-α-η Geraghty type contraction mappings in b_2-metric spaces. In addition, we derive some coincidence and common fixed point results in partially ordered b_2-metric spaces. Moreover, to show the utility of our main results, we provide an example and an application to non-linear integral equations.

Keywords: b_2-metric space; fixed point; binary relation; almost $\mathcal{R}_g$-Geraghty type contraction

MSC: 47H10; 54H25

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1. Introduction

The extension of fixed point theory to generalized structures, such as cone metric spaces, partial metric spaces, b-metric spaces and 2-metric spaces has received much attention. 2-metric space is a generalized metric space which was introduced by Gähler in [1]. Unlike the ordinary metric, the 2-metric is not a continuous function. The topology induced by 2-metric space is called 2-metric topology which is generated by the set of all open spheres with two centers. It is easy to observe that 2-metric space is not topologically equivalent to an ordinary metric. Hence, there is not any relationship between the results obtained in 2-metric spaces and the correspondence results in metric spaces. For fixed point results in the setting of 2-metric spaces, the readers may refer to [2–5] and references therein.

The concept of b-metric spaces was introduced by Czerwik [6,7] which is a generalization of the usual metric spaces and 2-metric spaces as well. Several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces have been obtained (see, e.g., [8–10]).

In 2014, Mustafa et al. [11] introduced the notion of b_2-metric spaces, as a generalization of both 2-metric and b-metric spaces.

On the other hand, the branch of related metric (metric space endowed with a binary relation) fixed point theory is a relatively new area was initiated by Turinici [12]. Recently, this direction of research is undertaken by several researchers such as: Bhaskar and Lakshmikantham [13], Samet and Turinici [14], Ben-El-Mechaiekh [15], Imdad et al. [16,17] and some others.

The aims of this paper are as follows:

- to define almost $\mathcal{R}_g$-Geraghty type contractions;

- to establish some coincidence and common fixed point results in the setting of b_2-metric spaces endowed with binary relations;
- to deduce some fixed point and common fixed point results in partially ordered b_2-metric spaces;
- to provide an example which shows the utility of our main results;
- to apply our newly proven results to non-linear integral equations.

2. Preliminaries

Definition 1 ([11]). *Let X be a non-empty set, $s \geq 1$ a given real number and $d : X^3 \to \mathbb{R}$ be a map satisfying the following conditions:*

(i) *for every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$;*
(ii) *if at least two of three points x, y, z are the same, then $d(x, y, z) = 0$;*
(iii) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$, *for all $x, y, z \in X$;*
(iv) $d(x, y, z) \leq s[d(x, y, w) + d(y, z, w) + d(z, x, w)]$, *for all $x, y, z, w \in X$.*

Then d is called a b_2-metric on X and (X, d) is called a b_2-metric space with parameter s.

Obviously, for $s = 1$, b_2-metric reduces to 2-metric.

Example 1. *Let (X, d) be a 2-metric space and $\rho(x, y, w) = (d(x, y, w))^p$, where $p \geq 1$ is a real number. We see that ρ is a b_2-metric with $s = 3^{p-1}$. In view of the convexity of $f(x) = x^p$, on $[0, \infty)$ for $p \geq 1$ and Jensen inequality, we have*

$$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p).$$

Therefore, condition (iv) of Definition 1 is satisfied and ρ is a b_2-metric on X.

Definition 2 ([11]). *Let $\{x_n\}$ be a sequence in a b_2-metric space (X, d). Then*

(i) *$\{x_n\}$ is said to be b_2-convergent and converges to $x \in X$, written $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$.*
(ii) *$\{x_n\}$ is said to be b_2-Cauchy in X if for all $a \in X$, $\lim_{m,n\to\infty} d(x_m, x_n, a) = 0$.*
(iii) *(X, d) is said to be b_2-complete if every b_2-Cauchy sequence is a b_2-convergent sequence.*

Definition 3 ([11]). *Let (X, d) and $(\bar{X}, \bar{d})$ be two b_2-metric spaces and let $f : X \to \bar{X}$ be a mapping. Then f is said to be b_2-continuous at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z, x, a) < \delta$ for all $a \in X$ imply that $\bar{d}(fz, fx, a) < \varepsilon$. The mapping f is b_2-continuous on X if it is b_2-continuous at all $z \in X$.*

Proposition 1 ([11]). *Let (X, d) and $(\bar{X}, \bar{d})$ be two b_2-metric spaces. Then a mapping $f : X \to \bar{X}$ is b_2-continuous at a point $x \in X$ if it is b_2-sequentially continuous at x, that is, whenever $\{x_n\}$ is b_2-convergent to x, $\{f(x_n)\}$ is b_2-convergent to $f(x)$.*

Lemma 1 ([11]). *Let (X, d) be a b_2-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are b_2-converge to x and y, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y, a) \leq \liminf_{n\to\infty} d(x_n, y_n, a) \leq \limsup_{n\to\infty} d(x_n, y_n, a) \leq s^2 d(x, y, a) \qquad \text{for all } a \in X.$$

In particular, if $y_n = y$, is constant, then

$$\frac{1}{s} d(x, y, a) \leq \liminf_{n\to\infty} d(x_n, y, a) \leq \limsup_{n\to\infty} d(x_n, y, a) \leq s d(x, y, a) \qquad \text{for all } a \in X.$$

Definition 4. *Let f and g be two self mappings on a non-empty set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.*

Definition 5 ([18])**.** *Two self mappings f and g are said to be weakly compatible if they commute at their coincidence points, that is, $fx = gx$ implies that $fgx = gfx$.*

Lemma 2 ([19])**.** *Let f and g be weakly compatible self mappings of a non-empty set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

A non-empty subset $\mathcal{R}$ of $X \times X$ is said to be a binary relation on X. Trivially, $X \times X$ is a binary relation on X known as the universal relation. For simplicity, we will write $x\mathcal{R}y$ whenever $(x, y) \in \mathcal{R}$ and write $x\mathcal{R}^{\neq}y$ whenever $x\mathcal{R}y$ and $x \neq y$. Observe that $\mathcal{R}^{\neq}$ is also a binary relation on X and $\mathcal{R}^{\neq} \subseteq \mathcal{R}$. The elements x and y of X are said to be $\mathcal{R}$-comparable if $x\mathcal{R}y$ or $y\mathcal{R}x$, this is denoted by $[x, y] \in \mathcal{R}$.

Definition 6. *A binary relation $\mathcal{R}$ on X is said to be:*

(i) *reflexive if $x\mathcal{R}x$ for all $x \in X$;*
(ii) *transitive if, for any $x, y, z \in X$, $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$; antisymmetric if, for any $x, y \in X$, $x\mathcal{R}y$ and $y\mathcal{R}x$ imply $x = y$;*
(iii) *preorder if it is reflexive and transitive;*
(iv) *partial order if it is reflexive, transitive and antisymmetric.*

Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and $Y \subseteq X$. Then the restriction of $\mathcal{R}$ to Y is denoted by $\mathcal{R}|_Y$ and is defined by $\mathcal{R} \cap Y^2$. The inverse of $\mathcal{R}$ is denoted by $\mathcal{R}^{-1}$ and is defined by $\mathcal{R}^{-1} = \{(x, y) \in X \times X : (y, x) \in \mathcal{R}\}$ and $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$.

Definition 7 ([20])**.** *Let X be a non-empty set and $\mathcal{R}$ a binary relation on X. A sequence $\{x_n\} \subseteq X$ is said to be an $\mathcal{R}$-preserving sequence if $x_n\mathcal{R}x_{n+1}$ for all $n \in \mathbb{N}_0$.*

Definition 8 ([20])**.** *Let X be a non-empty set and $f : X \to X$. A binary relation $\mathcal{R}$ on X is said to be f-closed if for all $x, y \in X$, $x\mathcal{R}y$ implies $fx\mathcal{R}fy$.*

Definition 9 ([20])**.** *Let X be a non-empty set and $f, g : X \to X$. A binary relation $\mathcal{R}$ on X is said to be (f, g)-closed if for all $x, y \in X$, $gx\mathcal{R}gy$ implies $fx\mathcal{R}fy$.*

Definition 10 ([20])**.** *Let (X, d) be a metric space and $\mathcal{R}$ a binary relation on X. We say that X is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges to a limit in X.*

Remark 1. *Every complete metric space is $\mathcal{R}$-complete, whatever the binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $\mathcal{R}$-completeness coincides with the usual completeness.*

Definition 11 ([21])**.** *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X, $f : X \to X$ and $x \in X$. We say that f is $\mathcal{R}$-continuous at x if, for any $\mathcal{R}$-preserving sequence $\{x_n\} \subseteq X$ such that $x_n \to x$, we have $fx_n \to fx$. Moreover, f is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.*

Remark 2. *Every continuous mapping is $\mathcal{R}$-continuous, whatever the binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $\mathcal{R}$-continuity coincides with the usual continuity.*

Definition 12 ([21])**.** *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X, $f, g : X \to X$ and $x \in X$. We say that f is $(g, \mathcal{R})$-continuous at x if, for any sequence $\{x_n\} \subseteq M$ such that $\{gx_n\}$ is $\mathcal{R}$-preserving and $gx_n \to gx$, we have $fx_n \to fx$. Moreover, f is called $(g, \mathcal{R})$-continuous if it is $(g, \mathcal{R})$-continuous at each point of X.*

Observe that on setting $g = I$, Definition 12 reduces to Definition 11.

Remark 3. *Every g-continuous mapping is $(g, \mathcal{R})$-continuous, whatever the binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $(g, \mathcal{R})$-continuity coincides with the usual g-continuity.*

Definition 13 ([21])**.** *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation on X and $f, g : X \to X$. We say that the pair (f, g) is $\mathcal{R}$-compatible if for any sequence $\{x_n\} \subseteq X$ such that $\{fx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$-preserving and $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = x \in X$, we have $\lim_{n\to\infty} d(gfx_n, fgx_n) = 0$.*

Remark 4. *Every compatible pair is $\mathcal{R}$-compatible, whatever the binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $\mathcal{R}$-compatibility coincides with the usual compatibility.*

Definition 14 ([20])**.** *Let (X, d) be a metric space. A binary relation $\mathcal{R}$ on X is said to be d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\} \subseteq X$ such that $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.*

3. Common Fixed Point Results for Almost $\mathcal{R}_g$-Geraghty Type Contraction Mappings

Lemma 3. *Let (X, d) be a b_2-metric space endowed with a binary relation $\mathcal{R}$ and $f, g : X \to X$ such that $f(X) \subseteq g(X)$, with $\mathcal{R}$ is (f, g)-closed and $\mathcal{R}|_{g(X)}$ is transitive. Assume that there exists $x_0 \in X$ such that $gx_0 \mathcal{R} fx_0$. Define a sequence $\{x_n\}$ in X by $fx_n = gx_{n+1}$ for $n \geq 0$. Then*

$$gx_m \mathcal{R} gx_n \text{ and } fx_m \mathcal{R} fx_n \quad \text{for all } m, n \in \mathbf{N}_0 \text{ with } m < n.$$

Proof. Since there exists $x_0 \in X$ such that $gx_0 \mathcal{R} fx_0$, $fx_n = gx_{n+1}$, and $\mathcal{R}$ is (f, g)-closed, we deduce that $gx_0 \mathcal{R} gx_1$, then $gx_1 = fx_0 \mathcal{R} fx_1 = gx_2$. By continuing this process, we get $gx_n \mathcal{R} gx_{n+1}$ for all $n \in \mathbf{N}$. Suppose that $m < n$, so $gx_m \mathcal{R} gx_{m+1}$ and $gx_{m+1} \mathcal{R} gx_{m+2}$, by $\mathcal{R}$ is g-transitive we have $gx_m \mathcal{R} gx_{m+2}$. Again, since $gx_m \mathcal{R} gx_{m+2}$ and $gx_{m+2} \mathcal{R} gx_{m+3}$, we get that $gx_m \mathcal{R} gx_{m+3}$. By continuing this process, we obtain $gx_m \mathcal{R} gx_n$. for all $m, n \in \mathbf{N}$ with $m < n$. In similar way and since $f(X) \subseteq g(X)$, we conclude $fx_m \mathcal{R} fx_n$ for all $m, n \in \mathbf{N}$ with $m < n$. $\square$

In 1973, Geraghty [22] introduced the class $\mathcal{F}$ of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy that $\lim\limits_{n\to\infty} \beta(t_n) = 1$ implies $\lim\limits_{n\to\infty} t_n = 0$. In addition, the author proved a fixed point result, generalizing the Banach contraction principle. Afterwards, there are many results about fixed point theorems by using such functions in this class. Đukić et al. [23] obtained fixed point results of this kind in b-metric and from [23] we denote Ω to the family of all functions $\beta_s : [0, \infty) \to [0, \frac{1}{s})$ for a real number $s \geq 1$, which satisfy the condition

$$\lim_{n\to\infty} \beta_s(t_n) = \frac{1}{s} \qquad \text{implies} \qquad \lim_{n\to\infty} t_n = 0.$$

Definition 15. *Let (X, d) be a b_2-metric space and $f, g : X \to X$. Suppose for all $x, y, a \in X$,*

$$M(x, y, a) = \max\left\{ d(gx, gy, a), d(gx, fx, a), d(gy, fy, a), \frac{d(gx, fy, a) + d(gy, fx, a)}{2s} \right\},$$

and

$$N(x, y, a) = \min\{ d(gx, fx, a), d(gy, fy, a), d(gx, fy, a), d(gy, fx, a) \}.$$

We say that f is almost $\mathcal{R}_g$-Geraghty type contraction mapping if there exist $L \geq 0$ and $\beta_s \in \Omega$ such that

$$d(fx, fy, a) \leq \beta_s(M(x, y, a)) M(x, y, a) + L N(x, y, a), \tag{1}$$

for all $x, y, a \in X$, with $gx \mathcal{R} gy$, $fx \mathcal{R}^{\#} fy$.

Definition 16. *Let (X,d) be a b_2-metric space and $f : X \to X$. Suppose for all $x, y, a \in X$,*

$$M(x,y,a) = \max\left\{d(x,y,a), d(x,fx,a), d(y,fy,a), \frac{d(x,fy,a)+d(y,fx,a)}{2s}\right\},$$

and

$$N(x,y,a) = \min\{d(x,fx,a), d(y,fy,a), d(x,fy,a), d(y,fx,a)\}.$$

We say that f is almost $\mathcal{R}$-Geraghty type contraction mapping if there exist $L \geq 0$ and $\beta_s \in \Omega$ such that

$$d(fx,fy,a) \leq \beta_s(M(x,y,a))M(x,y,a) + LN(x,y,a), \tag{2}$$

for all $x, y, a \in X$, with $x\mathcal{R}y$, $fx\mathcal{R}^{\neq}fy$.

Now, we present our main result as follows:

Theorem 1. *Let (X,d) be a b_2-metric space endowed with a binary relation $\mathcal{R}$ and $f, g : X \to X$ such that $f(X) \subseteq g(X)$, $g(X)$ is a b_2-complete subspace of X. Assume that f is almost $\mathcal{R}_g$-Geraghty type contraction mapping and the following conditions hold:*

(i) there exists x_0 in X such that $gx_0\mathcal{R}fx_0$;
(ii) $\mathcal{R}$ is (f,g)-closed and $\mathcal{R}|_{g(X)}$ is transitive;
(iii) $\mathcal{R}|_{g(X)}$ is d-self closed provided (1) holds for all $x, y, a \in X$ with $gx\mathcal{R}gy$ and $fx\mathcal{R}^{\neq}fy$.

Then f and g have a coincidence point in X.

Proof. Let $x_0 \in X$ such that $gx_0\mathcal{R}fx_0$. The proof is finished if $gx_0 = fx_0$ and x_0 is a coincidence point of f and g. Let us take $gx_0 \neq fx_0$, then since $f(X) \subseteq g(X)$ we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, we can define a sequence $\{gx_n\}$ in X by $fx_n = gx_{n+1}$, for all $n \in \mathbf{N}_0$.

We divide the proof into three steps as follows.

Step 1: We claim that $\lim\limits_{n\to\infty} d(gx_n, gx_{n+1}, a) = 0$. From Lemma 3, we have $\{gx_n\}$ is $\mathcal{R}$-preserving sequence that is $gx_n\mathcal{R}gx_{n+1}$ and $fx_n\mathcal{R}fx_{n+1}$, for all $n \in \mathbf{N}_0$. If $fx_{n_0} = fx_{n_0+1}$, for some $n_0 \in \mathbf{N}_0$, then x_{n_0+1} is a coincidence point of f and g. Suppose that $fx_n \neq fx_{n+1}$, for all $n \in \mathbf{N}_0$. Therefore, from (1), we obtain

$$\begin{aligned} d(gx_{n+1}, gx_{n+2}, gx_n) &= d(fx_n, fx_{n+1}, gx_n) \\ &\leq \beta_s(M(x_n, x_{n+1}, gx_n))M(x_n, x_{n+1}, gx_n) + LN(x_n, x_{n+1}, gx_n) \to (*) \end{aligned}$$

where

$$\begin{aligned} M(x_n, x_{n+1}, gx_n) &= \max\left\{d(gx_n, gx_{n+1}, gx_n), d(gx_n, fx_n, gx_n), d(gx_{n+1}, fx_{n+1}, gx_n),\right. \\ &\quad \left.\frac{d(gx_n, fx_{n+1}, gx_n) + d(gx_{n+1}, fx_n, gx_n)}{2s}\right\} \\ &= \max\{d(gx_n, gx_{n+1}, gx_n), d(gx_n, gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}, gx_n), \\ &\quad \frac{d(gx_n, gx_{n+2}, gx_n) + d(gx_{n+1}, gx_{n+1}, gx_n)}{2s}\} \\ &= d(gx_{n+1}, gx_{n+2}, gx_n), \end{aligned}$$

and

$$\begin{aligned} N(x_n, x_{n+1}, gx_n) &= \min\{d(gx_n, fx_n, gx_n), d(gx_{n+1}, fx_{n+1}, gx_n), d(gx_n, fx_{n+1}, gx_n), \\ &\quad d(gx_{n+1}, fx_n, gx_n)\} = 0. \end{aligned}$$

If $d(gx_{n+1}, gx_{n+2}, gx_n) \neq 0$ for some $n \in \mathbf{N}_0$, then we have (due to (*))

$$d(gx_{n+1}, gx_{n+2}, gx_n) \leq \beta_s(d(gx_{n+1}, gx_{n+2}, gx_n))d(gx_{n+1}, gx_{n+2}, gx_n),$$

yielding that

$$d(gx_{n+1}, gx_{n+2}, gx_n) - \beta_s(d(gx_{n+1}, gx_{n+2}, gx_n))d(gx_{n+1}, gx_{n+2}, gx_n) \leq 0,$$

or

$$d(gx_{n+1}, gx_{n+2}, gx_n)[1 - \beta_s(d(gx_{n+1}, gx_{n+2}, gx_n))] \leq 0 \to (**).$$

Divide both sides in (**) by $d(gx_{n+1}, gx_{n+2}, gx_n) \neq 0$, we obtain

$$1 - \beta_s(d(gx_{n+1}, gx_{n+2}, gx_n)) \leq 0,$$

or

$$\beta_s(d(gx_{n+1}, gx_{n+2}, gx_n)) \geq 1,$$

a contradiction [as $\beta_s : [0, \infty) \to [0, \frac{1}{s})$ and $s \geq 1$ so $\beta_s(c) < \frac{1}{s} \leq 1$, that is $\beta_s(c) < 1$ for all $c \in [0, \infty)$]. Therefore, we must have

$$d(gx_{n+1}, gx_{n+2}, gx_n) = 0, \quad \text{for all } n \in \mathbf{N}_0. \tag{3}$$

Thus, by the rectangle inequality and (3) we get

$$d(gx_n, gx_{n+2}, a) \leq s[d(gx_n, gx_{n+1}, a) + d(gx_{n+1}, gx_{n+2}, a)], \tag{4}$$

for all $n \in \mathbf{N}_0$, $a \in X$. Using (4), Lemma 3 and (1) we have

$$\begin{aligned} d(gx_{n+1}, gx_{n+2}, a) &= d(fx_n, fx_{n+1}, a) \\ &\leq \beta_s(M(x_n, x_{n+1}, a))M(x_n, x_{n+1}, a) + LN(x_n, x_{n+1}, a). \end{aligned} \tag{5}$$

Observe that

$$M(x_n, x_{n+1}, a) = \max\{d(gx_n, gx_{n+1}, a), d(gx_{n+1}, gx_{n+2}, a)\},$$

and

$$\begin{aligned} N(x_n, x_{n+1}, a) &= \min\{d(gx_n, fx_n, a), d(gx_{n+1}, fx_{n+1}, a), d(gx_n, fx_{n+1}, a), d(gx_{n+1}, fx_n, a)\} \\ &= \min\{d(gx_n, gx_{n+1}, a), d(gx_{n+1}, gx_{n+2}, a), d(gx_n, gx_{n+2}, a), d(gx_{n+1}, gx_{n+1}, a)\} \\ &= 0. \end{aligned}$$

Now, if $M(x_n, x_{n+1}, a) = d(gx_{n+1}, gx_{n+2}, a)$, then from (5) we have

$$d(gx_{n+1}, gx_{n+2}, a) \leq \beta_s(d(gx_{n+1}, gx_{n+2}, a))d(gx_{n+1}, gx_{n+2}, a) < d(gx_{n+1}, gx_{n+2}, a),$$

a contradiction. Hence, $M(x_n, x_{n+1}, a) = d(gx_n, gx_{n+1}, a)$, and

$$d(gx_{n+1}, gx_{n+2}, a) \leq \beta_s(d(gx_n, gx_{n+1}, a))d(gx_n, gx_{n+1}, a) < d(gx_n, gx_{n+1}, a), \tag{6}$$

for all $n \in \mathbf{N}_0$ and $a \in X$, which implies that the sequence $\{d(gx_n, gx_{n+1}, a)\}$ is strictly decreasing of positive numbers. Hence, there exists $\delta \geq 0$ such that $\lim_{n\to\infty} d(gx_n, gx_{n+1}, a) = \delta$. Suppose that $\delta > 0$. So, taking the limit as $n \to \infty$, from (6) we obtain

$$\frac{1}{s}\delta \leq \delta \leq \lim_{n\to\infty} \beta_s(d(gx_n, gx_{n+1}, a))\delta \leq \frac{1}{s}\delta.$$

Hence,

$$\lim_{n\to\infty}\beta_s(d(gx_n,gx_{n+1},a))=\frac{1}{s}.$$

From the property of β_s, we conclude that $\lim_{n\to\infty} d(gx_n,gx_{n+1},a)=0$ a contradiction, hence, $\delta=0$ and

$$\lim_{n\to\infty} d(gx_n,gx_{n+1},a)=0. \tag{7}$$

Step 2: We claim that $d(gx_i,gx_j,gx_k)=0$ for all $i,j,k\in\mathbf{N}_0$. Since $\{d(gx_n,gx_{n+1},a)\}$ is strictly decreasing and $d(gx_0,gx_1,gx_0)=0$, we conclude that $d(gx_n,gx_{n+1},gx_0)=0$, for all $n\in\mathbf{N}_0$.

Since $d(gx_{m-1},gx_m,gx_m)=0$ for all $m\in\mathbf{N}_0$ and $\{d(gx_n,gx_{n+1},a)\}$ is strictly decreasing we obtain that

$$d(gx_n,gx_{n+1},gx_m)=0, \quad \text{for all } n\geq m-1. \tag{8}$$

For $0\leq n<m-1$, we have $m-1\geq n+1$, so from (8) we have

$$d(gx_{m-1},gx_m,gx_{n+1})=d(gx_{m-1},gx_m,gx_n)=0. \tag{9}$$

Thus, by the rectangle inequality, $d(gx_n,gx_{n+1},gx_{n+1})=0$, and using (9) we obtain

$$\begin{aligned} d(gx_n,gx_{n+1},gx_m) &\leq s[d(gx_n,gx_{n+1},gx_{m-1})+d(gx_{n+1},gx_m,gx_{m-1})+d(gx_m,gx_n,gx_{m-1})]\\ &= sd(gx_n,gx_{n+1},gx_{m-1})\\ &\leq sd(gx_n,gx_{n+1},gx_{n+1})=0. \end{aligned}$$

Therefore, we get

$$d(gx_n,gx_{n+1},gx_m)=0, \quad \text{for all } 0\leq n<m-1. \tag{10}$$

Hence, from (8) and (10) we have

$$d(gx_n,gx_{n+1},gx_m)=0, \quad \text{for all } n,m\in\mathbf{N}_0.$$

Now, for all $i,j,k\in\mathbf{N}_0$, $i<j$ and $d(gx_i,gx_j,gx_{j-1})=d(gx_k,gx_j,gx_{j-1})=0$, applying the rectangle inequality we get

$$\begin{aligned} d(gx_i,gx_j,gx_k) &\leq s[d(gx_i,gx_j,gx_{j-1})+d(gx_j,gx_k,gx_{j-1})+d(gx_k,gx_i,gx_{j-1})]\\ &= sd(gx_k,gx_i,gx_{j-1})\\ &\leq s^2d(gx_k,gx_i,gx_{j-2})\leq\ldots\leq s^{j-i}d(gx_k,gx_i,gx_i)=0. \end{aligned}$$

Therefore, for all $i,j,k\in\mathbf{N}_0$, we have

$$d(gx_i,gx_j,gx_k)=0. \tag{11}$$

Step 3: We show that $\{gx_n\}$ is a b_2-Cauchy sequence. Suppose to the contrary that $\{gx_n\}$ is not a b_2-Cauchy sequence. Then there is $\varepsilon>0$ such that for an integer k there exist integers $n(k)$, $m(k)$ with $n(k)>m(k)>k$ such that

$$d(gx_{m(k)},gx_{n(k)},a)\geq\varepsilon, \tag{12}$$

for every integer k, let $n(k)$ be the least positive integer with $n(k)>m(k)$, satisfying (12) and such that

$$d(gx_{m(k)},gx_{n(k)-1},a)<\varepsilon. \tag{13}$$

Using the rectangle inequality, (11) and (12) we have

$$\varepsilon\leq d(gx_{m(k)},gx_{n(k)},a)\leq s[d(gx_{m(k)},gx_{n(k)-1},a)+d(gx_{n(k)},gx_{n(k)-1},a)].$$

Again, using the rectangle inequality and (11) in the above inequality, it follows that

$$\varepsilon \leq s^2[d(gx_{m(k)}, gx_{m(k)-1}, a) + d(gx_{m(k)-1}, gx_{n(k)-1}, a)] + sd(gx_{n(k)}, gx_{n(k)-1}, a)].$$

In addition,

$$d(gx_{m(k)-1}, gx_{n(k)-1}, a) \leq s[d(gx_{m(k)-1}, gx_{m(k)}, a) + d(gx_{n(k)-1}, gx_{m(k)}, a)].$$

Taking the upper limit as $k \to \infty$, in the above three inequalities and from (7) and (13) it follows that

$$\varepsilon \leq \limsup_{k\to\infty} d(gx_{m(k)}, gx_{n(k)}, a) < s\varepsilon, \tag{14}$$

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(gx_{m(k)-1}, gx_{n(k)-1}, a) < s\varepsilon, \tag{15}$$

$$\frac{\varepsilon}{s^3} \leq \limsup_{k\to\infty} d(gx_{m(k)}, gx_{n(k)-1}, a) < \varepsilon. \tag{16}$$

Again, using the rectangle inequality, (11) and (12) we get

$$d(gx_{m(k)-1}, gx_{n(k)}, a) \leq s[d(gx_{m(k)-1}, gx_{n(k)-1}, a) + d(gx_{n(k)}, gx_{n(k)-1}, a)],$$

$$\varepsilon \leq d(gx_{m(k)}, gx_{n(k)}, a) \leq s[d(gx_{m(k)}, gx_{m(k)-1}, a) + d(gx_{n(k)}, gx_{m(k)-1}, a)].$$

Taking the upper limit as $k \to \infty$, in the above two inequalities and from (7) and (15), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(gx_{m(k)-1}, gx_{n(k)}, a) < s^2\varepsilon. \tag{17}$$

Now, from Lemma 3 we have $fx_{m(k)-1}\mathcal{R}^{\#}fx_{n(k)-1}$ for all $m(k), n(k) \in \mathbf{N}_0$ with $m(k) < n(k)$. Hence, from (1) we conclude that

$$\begin{aligned} d(gx_{m(k)}, gx_{n(k)}, a) &= d(fx_{m(k)-1}, fx_{n(k)-1}, a) \\ &\leq \beta_s(M(x_{m(k)-1}, x_{n(k)-1}, a))M(x_{m(k)-1}, x_{n(k)-1}, a) + LN(x_{m(k)-1}, x_{n(k)-1}, a), \end{aligned} \tag{18}$$

where

$$\begin{aligned} & M(x_{m(k)-1}, x_{n(k)-1}, a) = \max\{d(gx_{m(k)-1}, gx_{n(k)-1}, a), d(gx_{m(k)-1}, fx_{m(k)-1}, a), \\ & d(gx_{n(k)-1}, fx_{n(k)-1}, a), \frac{d(gx_{m(k)-1}, fx_{n(k)-1}, a) + d(gx_{n(k)-1}, fx_{m(k)-1}, a)}{2s}\}, \\ = \; & \max\Big\{d(gx_{m(k)-1}, gx_{n(k)-1}, a), d(gx_{m(k)-1}, gx_{m(k)}, a), d(gx_{n(k)-1}, gx_{n(k)}, a), \\ & \frac{d(gx_{m(k)-1}, gx_{n(k)}, a) + d(gx_{n(k)-1}, gx_{m(k)}, a)}{2s}\Big\}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} N(x_{m(k)-1}, x_{n(k)-1}, a) \;=\; & \min\{d(gx_{m(k)-1}, gx_{m(k)}, a), d(gx_{n(k)-1}, gx_{n(k)}, a), d(gx_{m(k)-1}, gx_{n(k)}, a), \\ & d(gx_{n(k)-1}, gx_{m(k)}, a)\}. \end{aligned} \tag{20}$$

Taking the upper limit as $k \to \infty$, in (19), (20) and using (7), (15)–(17) it follows that

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, a) < s\varepsilon, \tag{21}$$

and

$$\limsup_{k\to\infty} N(x_{m(k)-1}, x_{n(k)-1}, a) = 0. \tag{22}$$

Now, taking the upper limit as $k \to \infty$ in (18) and using (14), (21) and (22), we conclude that

$$\frac{1}{s} = \frac{\varepsilon}{s\varepsilon} \le \frac{\limsup_{k\to\infty} d(gx_{m(k)}, gx_{n(k)}, a)}{\limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, a)} \le \limsup_{k\to\infty} \beta_s(M(x_{m(k)-1}, x_{n(k)-1}, a)) \le \frac{1}{s}.$$

Thus, $\limsup_{k\to\infty} \beta_s(M(x_{m(k)-1}, x_{n(k)-1}, a)) = \frac{1}{s}$. Hence, $\limsup_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, a) = 0$, which is a contradiction. Therefore, $\{gx_n\}$ is a b_2-Cauchy sequence. As $g(X)$ is b_2-complete subspace of X, then there exist $z \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = gz. \tag{23}$$

Now, we show that z is a point of coincidence of f and g. From condition (iii), we have $\mathcal{R}|_{g(X)}$ is d-self closed and (1) holds for all $x, y, a \in X$ with $gx\mathcal{R}gy$ and $fx\mathcal{R}^{\neq}fy$. As $\{gx_n\} \subseteq g(X)$, $\{gx_n\}$ is $\mathcal{R}|_{g(X)}$-preserving and $gx_n \to gz$ so there exists a subsequence $\{gx_{n(k)}\} \subseteq \{gx_n\}$ such that $gx_{n(k)}\mathcal{R}|_{g(X)}gz$ for all $k \in \mathbf{N}_0$ and since $\mathcal{R}$ is (f, g)-closed then $fx_{n(k)}\mathcal{R}|_{g(X)}fz$ for all $k \in \mathbf{N}_0$.

If $fx_{n(k)} = fz$ for all $k > k_0$, and $k_0, k \in \mathbf{N}_0$, then $\lim_{k\to\infty} fx_{n(k)} = fz$, and since $\lim_{n\to\infty} fx_n = gz$, we have $fz = gz$, that is z is a coincidence point of f and g.

On other hand, if $fx_{n(k)} \ne fz$ for all $k > k_0$, and $k_0, k \in \mathbf{N}_0$, then $fx_{n(k)}\mathcal{R}|_{g(X)}fz$ and $fx_{n(k)} \ne fz$ for all $k > k_0$, and $k_0, k \in \mathbf{N}_0$. Thus, $gx_{n(k)}\mathcal{R}|_{g(X)}gz$ and $fx_{n(k)}\mathcal{R}^{\neq}|_{g(X)}fz$, and from (1), we have

$$d(gx_{n(k)+1}, fz, a) = d(fx_{n(k)}, fz, a) \le \beta_s(M(x_{n(k)}, z, a))M(x_{n(k)}, z, a) + LN(x_{n(k)}, z, a), \tag{24}$$

where

$$\begin{aligned} M(x_{n(k)}, z, a) \quad = \quad & \max\Big\{d(gx_{n(k)}, gz, a), d(gx_{n(k)}, gx_{n(k)+1}, a), d(gz, fz, a), \\ & \frac{d(gx_{n(k)}, fz, a) + d(gz, gx_{n(k)+1}, a)}{2s}\Big\}, \end{aligned} \tag{25}$$

and

$$N(x_{n(k)}, z, a) \quad = \quad \min\{d(gx_{n(k)}, gx_{n(k)+1}, a), d(gz, fz, a), d(gx_{n(k)}, fz, a), d(gz, gx_{n(k)+1}, a)\}. \tag{26}$$

Letting $k \to \infty$ in (25), (26), we get

$$\limsup_{k\to\infty} M(x_{n(k)}, z, a) = \max\Big\{d(gz, fz, a), \frac{\limsup_{k\to\infty} d(gx_{n(k)}, fz, a)}{2s}\Big\},$$

and

$$\limsup_{k\to\infty} N(x_{n(k)}, z, a) = 0. \tag{27}$$

From Lemma 1, we have

$$\frac{d(gz, fz, a)}{s} \le \limsup_{k\to\infty} d(gx_{n(k)}, fz, a) \le sd(gz, fz, a). \tag{28}$$

Thus,

$$\max\{d(gz, fz, a), \frac{d(gz, fz, a)}{2s^2}\} \le \limsup_{k\to\infty} M(x_{n(k)}, z, a) \le \max\{d(gz, fz, a), \frac{d(gz, fz, a)}{2}\},$$

yields,

$$\limsup_{k\to\infty} M(x_{n(k)}, z, a) = d(gz, fz, a), \tag{29}$$

Again, taking the upper limit as $k \to \infty$, in (24) and using Lemma 1, (27) and (29), we get

$$\begin{aligned}
\frac{d(gz, fz, a)}{s} &\leq \limsup_{k\to\infty} d(gx_{n(k)+1}, fz, a) \\
&\leq \limsup_{k\to\infty} \beta_s(M(x_{n(k)}, z, a)) \limsup_{k\to\infty} M(x_{n(k)}, z, a) \\
&\leq \limsup_{k\to\infty} \beta_s(M(x_{n(k)}, z, a)) d(gz, fz, a) \\
&\leq \frac{1}{s} d(gz, fz, a).
\end{aligned}$$

Hence, $\limsup_{k\to\infty} \beta_s(M(x_{n(k)}, z, a)) = \frac{1}{s}$, so from the property of β_s we conclude that $\limsup_{k\to\infty} M(x_{n(k)}, z, a) = 0$ implies $d(gz, fz, a) = 0$ for all $a \in X$. That is, $gz = fz$. This shows that f and g have a coincidence point. □

The next theorem shows that under some additional hypotheses we can deduce the existence and uniqueness of a common fixed point.

Theorem 2. *In addition to the hypotheses of Theorem 1, suppose that f and g are weakly compatible and for all coincidence points u, v of f and g, there exists $w \in X$ such that $gu\mathcal{R}gw$ and $gv\mathcal{R}gw$. Then f and g have a unique common fixed point.*

Proof. The set of coincidence points of f and g is not empty due to Theorem 1. Suppose that u and v are two coincidence points of f and g, that is, $fu = gu$ and $fv = gv$. We will show that $gu = gv$. By our assumption, there exists $w \in X$ such that

$$gu\mathcal{R}gw \quad \text{and} \quad gv\mathcal{R}gw. \tag{30}$$

Now, proceeding similarly to the proof of Theorem 1, we can define a sequence $\{w_n\}$ in X as $fw_n = gw_{n+1}$ for all $n \in \mathbf{N}_0$ and $w_0 = w$, with $\lim_{n\to\infty} d(gw_n, gw_{n+1}, a) = 0$. Since $gu\mathcal{R}gw_0$ $(gv\mathcal{R}gw_0)$ and $\mathcal{R}$ is (f, g)-closed, we conclude that $fu\mathcal{R}fw_0(fv\mathcal{R}fw_0)$. Hence, $gu\mathcal{R}gw_1(gv\mathcal{R}gw_1)$. By induction, we have

$$gu\mathcal{R}gw_n \quad \text{and} \quad gv\mathcal{R}gw_n, \quad \forall n \in \mathbf{N}_0. \tag{31}$$

From (1) and using (31), we obtain

$$d(gu, gw_{n+1}, a) = d(fu, fw_n, a) \leq \beta_s(M(u, w_n, a))M(u, w_n, a) + LN(u, w_n, a), \tag{32}$$

where

$$\begin{aligned}
M(u, w_n, a) &= \max\left\{d(gu, gw_n, a), d(gu, fu, a), d(gw_n, fw_n, a), \frac{d(gu, fw_n, a) + d(gw_n, fu, a)}{2s}\right\}, \\
&= \max\left\{d(gu, gw_n, a), d(gw_n, gw_{n+1}, a), \frac{d(gu, gw_{n+1}, a) + d(gw_n, gu, a)}{2s}\right\},
\end{aligned}$$

and

$$\begin{aligned}
N(u, w_n, a) &= \min\{d(gu, fu, a), d(gw_n, fw_n, a), d(gu, fw_n, a), d(gw_n, fu, a)\} \\
&= \min\{d(gu, gu, a), d(gw_n, gw_{n+1}, a), d(gu, gw_{n+1}, a), d(gw_n, gu, a)\} = 0.
\end{aligned}$$

Hence,

$$\begin{aligned} d(gu, gw_{n+1}, a) &\leq \beta_s(M(u, w_n, a))M(u, w_n, a) \\ &< \frac{1}{s}M(u, w_n, a) \leq M(u, w_n, a). \end{aligned}$$

Since,

$$\begin{aligned} d(gu, gw_{n+1}, a) &< M(u, w_n, a) \\ &= \max\left\{d(gu, gw_n, a), d(gw_n, gw_{n+1}, a), \frac{d(gu, gw_{n+1}, a) + d(gw_n, gu, a)}{2s}\right\} \\ &= \max\left\{d(gu, gw_n, a), d(gw_n, gw_{n+1}, a)\right\}. \end{aligned}$$

Thus,

$$M(u, w_n, a) = \max\left\{d(gu, gw_n, a), d(gw_n, gw_{n+1}, a)\right\}.$$

(Case1): if $M(u, w_n, a) = d(gu, gw_n, a)$, then

$$d(gu, gw_{n+1}, a) \leq \beta_s(d(gu, gw_n, a))d(gu, gw_n, a) < \frac{1}{s}d(gu, gw_n, a) \leq d(gu, gw_n, a), \quad (33)$$

it follows that $d(gu, gw_{n+1}, a) < d(gu, gw_n, a)$. Thus, $\{d(gu, gw_n, a)\}$ is strictly decreasing. Hence, there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} d(gu, gw_n, a) = \gamma$. Letting $n \to \infty$ in (33), we obtain

$$\begin{aligned} \frac{\gamma}{s} \leq \gamma = \lim_{n\to\infty} d(gu, gw_{n+1}, a) &\leq \lim_{n\to\infty} \beta_s(d(gu, gw_n, a)) \lim_{n\to\infty} d(gu, gw_n, a) \\ &\leq \lim_{n\to\infty} \beta_s(d(gu, gw_n, a))\gamma \\ &\leq \frac{\gamma}{s}, \end{aligned}$$

this implies

$$\frac{1}{s} \leq \lim_{n\to\infty} \beta_s(d(gu, gw_n, a)) < \frac{1}{s}.$$

Thus,

$$\lim_{n\to\infty} \beta_s(d(gu, gw_n, a)) = \frac{1}{s}.$$

From the property of β_s, we conclude that $\lim_{n\to\infty} d(gu, gw_n, a) = 0$.
(Case2): If $M(u, w_n, a) = d(gw_n, gw_{n+1}, a)$, then

$$d(gu, gw_{n+1}, a) \leq \beta_s(d(gw_n, gw_{n+1}, a))d(gw_n, gw_{n+1}, a).$$

Therefore,

$$\lim_{n\to\infty} d(gu, gw_{n+1}, a) \leq \lim_{n\to\infty} \beta_s(d(gw_n, gw_{n+1}, a)) \lim_{n\to\infty} d(gw_n, gw_{n+1}, a) = 0.$$

This yields $\lim_{n\to\infty} d(gu, gw_{n+1}, a) = 0$. Therefore, from all cases we conclude that

$$\lim_{n\to\infty} d(gu, gw_n, a) = 0. \quad (34)$$

Similarly, we can show that

$$\lim_{n\to\infty} d(gv, gw_n, a) = 0. \quad (35)$$

Hence, from (34) and (35), we obtain $gu = gv$. That is, f and g have a unique point of coincidence. From Lemma 2 f and g have a unique common fixed point. $\square$

Now, we give an example to justify the hypotheses of Theorem 1.

Example 2. *Let $X = \{p, q, r, t\}$ be a set with b_2-metric $d : X^3 \to \mathcal{R}$ defined by*

$$d(p,q,r) = 0, \quad d(p,q,t) = 4, \quad d(p,r,t) = 1, \quad d(q,r,t) = 6,$$

with symmetry in all variables and if at least two of the arguments are equal then $d(x, y, a) = 0$. Then (X, d) is a complete b_2-metric space with $s = \frac{6}{5}$. Define a binary relation $\mathcal{R}$ on X by

$$\mathcal{R} = \{(p,p),(q,q),(r,r),(p,q),(q,r),(p,r),(r,p),(r,q)\}.$$

Define $f, g : X \to X$ and $\beta : (0, \infty) \to [0, 1)$ as follows:

$$f = \begin{pmatrix} p & q & r & t \\ p & p & r & t \end{pmatrix}, \quad g = \begin{pmatrix} p & q & r & t \\ p & r & q & t \end{pmatrix}, \quad \beta_s(t) = \frac{1+t}{s(1+2t)}$$

We show that all the hypotheses of Theorem 1 are satisfied. Clearly, (X, d) is a complete b_2-metric space and $f(X) \subseteq g(X)$, $g(X)$ is a b_2-complete subspace of X. $\mathcal{R} = \mathcal{R}|_{g(X)}$ is transitive. There is $r \in X$ such that $q = gr\mathcal{R}fr = r$. Since $\mathcal{R}|_{g(X)}$ is finite, then it is d-self closed. We show that $\mathcal{R}$ is (f, g)-closed, we study the nontrivial cases:

- $gp\mathcal{R}gr = p\mathcal{R}q \Rightarrow fp\mathcal{R}fr = p\mathcal{R}r \in \mathcal{R}, gr\mathcal{R}gq = q\mathcal{R}r \Rightarrow fr\mathcal{R}fq = r\mathcal{R}p,$
- $gp\mathcal{R}gq = p\mathcal{R}r \Rightarrow fp\mathcal{R}fq = p\mathcal{R}p, gq\mathcal{R}gp = r\mathcal{R}p \Rightarrow fq\mathcal{R}fp = p\mathcal{R}p,$
- $gq\mathcal{R}gr = r\mathcal{R}q \Rightarrow fq\mathcal{R}fr = p\mathcal{R}r.$

Now, we check the contractive condition 2. The nontrivial cases are when $a = t$, $(gp\mathcal{R}gr$ and $fp\mathcal{R}fr)$, $(gr\mathcal{R}gq$ and $fr\mathcal{R}fq)$ and $(gq\mathcal{R}gr$ and $fq\mathcal{R}fr)$.

In all three cases, we get $M(p,r,t) = M(r,q,t) = M(q,r,t) = 6$, $N(p,r,a) = N(r,q,t) = N(q,r,t) = 0$, and then

$$1 = d(fp, fr, t) = d(p, r, t) \leq \frac{35}{13} = \beta_s(6)6 = \beta_s(M(p,r,a))M(p,r,a) + LN(p,r,a),$$

$$1 = d(fr, fq, t) = d(r, p, t) \leq \frac{35}{13} = \beta_s(6)6 = \beta_s(M(r,q,a))M(r,q,a) + LN(r,q,a),$$

$$1 = d(fq, fr, t) = d(p, r, t) \leq \frac{35}{13} = \beta_s(6)6 = \beta_s(M(q,r,a))M(q,r,a) + LN(q,r,a),$$

Therefore, all the hypotheses of Theorem 1 are satisfied. Then f and g have two coincidence fixed points p and t. Noting that p, t are not $\mathcal{R}$-comparable so the uniqueness of coincidence point is not fulfilled.

By taking $g = I$ in Theorems 1 and 2 we deduce the following result.

Corollary 1. *Let (X, d) be a complete b_2-metric space endowed with a transitive binary relation $\mathcal{R} : X \to X$ and $f : X \to X$. Assume that f is almost $\mathcal{R}$-Geraghty type contraction mapping and the following conditions hold:*

(i) *there exists x_0 in X such that $x_0\mathcal{R}fx_0$;*
(ii) *$\mathcal{R}$ is f-closed;*
(iii) *$\mathcal{R}$ is d-self closed provided (2) holds for all $x, y, a \in X$ with $fx\mathcal{R}^{*}fy$.*

Then f has a fixed point. Moreover, if for $u, v \in Fix(f)$, there exists $w \in X$ such that $u\mathcal{R}w$ and $v\mathcal{R}w$, then f has a unique fixed point.

4. Results for Almost g-α-η Geraghty Type Contraction Mappings in b_2-Metric Spaces

Fathollahi et al. [4] introduced the concepts of triangular 2-α-η-admissible mappings as follows.

Definition 17 ([4]). *Let (X,d) be a 2-metric space, $f : X \to X$ and $\alpha, \eta : X^3 \to [0,\infty)$. We say that f is a triangular 2-α-η-admissible mapping if for all $a \in X$,*

(i) $\alpha(x,y,a) \geq \eta(x,y,a)$ *implies* $\alpha(fx,fy,a) \geq \eta(fx,fy,a)$, $x,y \in X$,

(ii) $\begin{cases} \alpha(x,y,a) \geq \eta(x,y,a), \\ \alpha(y,z,a) \geq \eta(y,z,a), \end{cases}$ *implies* $\alpha(x,z,a) \geq \eta(x,z,a)$.

If we take $\eta(x,y,a) = 1$, then we say that f is a triangular 2-α-admissible mapping. In addition, if we take $\alpha(x,y,a) = 1$, then we say that f is a triangular 2-η-subadmissible mapping.

Motivated by Fathollahi [4], we define the following concepts.

Definition 18. *Let (X,d) be a b_2-metric space, $f,g : X \to X$ and $\alpha, \eta : X^3 \to [0,\infty)$. We say that f is a triangular g-b_2-α-η-admissible mapping if for all $a \in X$,*

(i) $\alpha(gx,gy,a) \geq \eta(gx,gy,a)$ *implies* $\alpha(fx,fy,a) \geq \eta(fx,fy,a)$, $\quad x,y \in X$,

(ii) $\begin{cases} \alpha(gx,gy,a) \geq \eta(gx,gy,a), \\ \alpha(gy,gz,a) \geq \eta(gy,gz,a), \end{cases}$ *implies* $\alpha(gx,gz,a) \geq \eta(gx,gz,a)$, $\quad x,y,z \in X$.

When $\eta(gx,gy,a) = 1$, we say that f is a triangular g-b_2-α-admissible mapping. In addition, when $\alpha(gx,gy,a) = 1$, we say that f is a triangular g-b_2-η-subadmissible mapping.

Definition 19. *Let (X,d) be a b_2-metric space with $s \geq 1$ and $f,g : X \to X$, $\alpha,\eta : X^3 \to [0,\infty)$. Suppose for all $x,y,a \in X$,*

$$M(x,y,a) = \max\left\{d(gx,gy,a), d(gx,fx,a), d(gy,fy,a), \frac{d(gx,fy,a) + d(gy,fx,a)}{2s}\right\},$$

and

$$N(x,y,a) = \min\{d(gx,fx,a), d(gy,fy,a), d(gx,fy,a), d(gy,fx,a)\}.$$

We say that f is almost g-α-η Geraghty type contraction mapping if there exist $L \geq 0$ and $\beta_s \in \Omega$ such that

$$\begin{aligned} &\forall x,y \in X,\ \alpha(gx,gy,a) \geq \eta(gx,gy,a) \\ &\Rightarrow d(fx,fy,a) \leq \beta_s(M(x,y,a))M(x,y,a) + LN(x,y,a), \end{aligned} \tag{36}$$

for all $a \in X$.

Now, we state the following corollaries

Corollary 2. *Let (X,d) be a complete b_2-metric space and $f,g : X \to X$, such that $f(X) \subseteq g(X)$, $g(X)$ is a b_2-complete subspace of X. Assume that f is almost g-α-η Geraghty type contraction mapping and the following conditions hold:*

(i) *there exists x_0 in X such that $\alpha(gx_0, fx_0, a) \geq \eta(gx_0, fx_0, a)$ for all $a \in X$;*

(ii) *f is a triangular g-b_2-α-η-admissible mapping;*

(iii) *if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}, a) \geq \eta(gx_n, gx_{n+1}, a)$ for all $a \in X$, $n \in \mathbf{N}_0$ and $gx_n \to gz$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gz, a) \geq \eta(gx_{n(k)}, gz, a)$ for all $k \in \mathbf{N}_0$ and all $a \in X$.*

Then f and g have a coincidence point in X. Moreover, suppose that for all coincidence points u,v of f and g, there exists $w \in X$ such that $\alpha(gu,gw,a) \geq \eta(gu,gw,a)$ and $\alpha(gv,gw,a) \geq \eta(gv,gw,a)$ for all $a \in X$ and f,g are weakly compatible. Then f and g have a unique common fixed point.

Proof. Define $\mathcal{R}$ on X as

$$x\mathcal{R}y \Longleftrightarrow \alpha(x,y,a) \geq \eta(x,y,a).$$

We note the following:

- since there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0, a) \geq \eta(gx_0, fx_0, a)$ for all $a \in X$ then $gx_0\mathcal{R}fx_0$;
- if $gx\mathcal{R}gy$ then $\alpha(gx,gy,a) \geq \eta(gx,gy,a)$. As f is a triangular g-b_2-α-η-admissible mapping, $\alpha(fx,fy,a) \geq \eta(fx,fy,a)$ and so $fx\mathcal{R}fy$. Thus, $\mathcal{R}$ is (f,g)-closed;
- if $gx\mathcal{R}gy$ and $gy\mathcal{R}gz$, then $\alpha(gx,gy,a) \geq \eta(gx,gy,a)$ and $\alpha(gy,gz,a) \geq \eta(gy,gz,a)$. As f is a triangular g-b_2-α-η-admissible mapping, $\alpha(gx,gz,a) \geq \eta(gx,gz,a)$, that is, $gx\mathcal{R}gz$. Therefore, $\mathcal{R}|_{g(X)}$ is transitive;
- if $gx\mathcal{R}gy$, $fx\mathcal{R}^{\#}fy$, then $\alpha(gx,gy,a) \geq \eta(gx,gy,a)$, $\alpha(fx,fy,a) \geq \eta(fx,fy,a)$. Since f is almost g-α-η Geraghty type contraction, so (1) holds;
- from (iii), we have $gx_n\mathcal{R}gx_{n+1}$ for all $n \in \mathbf{N}_0$ and $gx_n \to gz$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $gx_{n(k)}\mathcal{R}gz$ for all $k \in \mathbf{N}_0$. Hence, all conditions of Theorem 1 are satisfied. Thus, f and g have a point of coincidence in X.

Finally, if for all coincidence points u,v of f and g, there exists $w \in X$ such that $\alpha(gu,gw,a) \geq \eta(gu,gw,a)$ and $\alpha(gv,gw,a) \geq \eta(gv,gw,a)$, then $gu\mathcal{R}gw$ and $gv\mathcal{R}gw$. That is, all hypotheses of Theorem 1 are satisfied. Therefore, f and g have a unique common fixed point. $\square$

By taking $g = I$ in Definitions 18 and 19, we say that f is a triangular b_2-α-η-admissible mapping and f is almost α-η Geraghty type contraction mapping.

Now, we have the following corollary.

Corollary 3. *Let (X,d) be a complete b_2-metric space and $f : X \to X$. Assume that f is almost α-η Geraghty type contraction mapping and the following conditions hold:*

(i) *there exists x_0 in X such that $\alpha(x_0, fx_0, a) \geq \eta(x_0, fx_0, a)$ for all $a \in X$;*

(ii) *f is a triangular b_2-α-η-admissible mapping;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq \eta(x_n, x_{n+1}, a)$ for all $a \in X$, $n \in \mathbf{N}_0$ and $x_n \to z$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z, a) \geq \eta(x_{n(k)}, z, a)$ for all $k \in \mathbf{N}_0$ and all $a \in X$.*

Then f has a fixed point in X. Moreover, if for $u,v \in Fix(f)$ there exists $w \in X$ such that $\alpha(u,w,a) \geq \eta(u,w,a)$ and $\alpha(v,w,a) \geq \eta(v,w,a)$ for all $a \in X$, then f has a unique fixed point.

5. Fixed Point Results in Partially Ordered b_2-Metric Spaces

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations. This trend was started by Turinici [12] in 1986. Ran and Reurings in [24] extended the Banach contraction principle in partially ordered sets with some applications to matrix equations. The obtained result in [24] was extended and refined by many authors (see, e.g., [25–27] and references therein). The aim of this section is to deduce our results in the context of partially ordered b_2-metric spaces. At first, we need to recall some concepts. Let X be a nonempty set. Then $(X, \preceq, d)$ is called a partially ordered b_2-metric space with $s \geq 1$ if (X,d) is a b_2-metric space and $(X, \preceq)$ is a partially ordered set.

Definition 20. *Let $(X, \preceq)$ be a partially ordered set and $x,y \in X$. Then x and y are called comparable if $x \preceq y$ or $y \preceq x$ holds.*

Definition 21. *Let $(X, \preceq)$ be a partially ordered set. A mapping f on X is said to be monotone non-decreasing if for all $x,y \in X$, $x \preceq y$ implies $fx \preceq fy$.*

Definition 22. *Let $(X, \preceq)$ be a partially ordered set and $f, g : X \to X$. One says f is g-non-decreasing if for $x, y \in X$,*

$$g(x) \preceq g(y) \quad \textit{implies} \quad f(x) \preceq f(y).$$

By putting $\mathcal{R} = \preceq$ in Theorems 1 and 2, we get the following results.

Corollary 4. *Let $(X, d, \preceq)$ be a complete partially ordered b_2-metric space. Assume that $f, g : X \to X$, are two mappings such that $f(X) \subseteq g(X)$, $g(X)$ is a b_2-complete subspace of X and f is a g-non-decreasing mapping. Suppose that there exists a function $\beta_s \in \Omega$ and $L \geq 0$ such that*

$$d(fx, fy, a) \leq \beta_s(M(x, y, a))M(x, y, a) + LN(x, y, a), \tag{37}$$

where

$$M(x, y, a) = \max\left\{d(gx, gy, a), d(gx, fx, a), d(gy, fy, a), \frac{d(gx, fy, a) + d(gy, fx, a)}{2s}\right\},$$

and

$$N(x, y, a) = \min\{d(gx, fx, a), d(gy, fy, a), d(gx, fy, a), d(gy, fx, a)\},$$

for all $x, y, a \in X$ with $gx \preceq gy$. In addition, suppose that the following conditions hold:

(i) there exists x_0 in X such that $gx_0 \preceq fx_0$,
(ii) if $\{gx_n\}$ is a non-decreasing sequence in X with $gx_n \to gz$ as $n \to \infty$, then $gx_n \preceq gz$ for all $n \in \mathbf{N}_0$.

Then f and g have a coincidence point in X. Moreover, suppose that for all coincidence points u, v of f and g, there exists $w \in X$ such that $gu \preceq gw$ or $gv \preceq gw$ and f, g are weakly compatible. Then f and g have a unique common fixed point.

By taking $g = I$ in Corollary 4, we obtain the following corollary.

Corollary 5. *Let $(X, d, \preceq)$ be a complete partially ordered b_2-metric space. Assume that $f : X \to X$ is a mapping satisfying the following conditions*

(i) f is non-decreasing mapping;
(ii) there exist a function $\beta_s \in \Omega$ and $L \geq 0$ such that

$$d(fx, fy, a) \leq \beta_s(M(x, y, a))M(x, y, a) + LN(x, y, a), \tag{38}$$

where

$$M(x, y, a) = \max\left\{d(x, y, a), d(x, fx, a), d(y, fy, a), \frac{d(x, fy, a) + d(y, fx, a)}{2s}\right\},$$

and

$$N(x, y, a) = \min\{d(x, fx, a), d(y, fy, a), d(x, fy, a), d(y, fx, a)\},$$

for all $x, y, a \in X$ with $x \preceq y$;

(iii) there exists x_0 in X such that $x_0 \preceq fx_0$;
(iv) if $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to z$ as $n \to \infty$, then $x_n \preceq z$ for all $n \in \mathbf{N}_0$.

Then f has a fixed point. Moreover, if $u, v \in Fix(f)$ such that there exists $w \in X$ with $u \preceq w$ and $v \preceq w$, then f has a unique fixed point. Then f has a fixed point. Moreover, if for every pair (u, v) of fixed points of f such that there exists $w \in X$ with $u \preceq w$ and $v \preceq w$, then f has a unique fixed point.

6. Application to Integral Equations

In this section, we study the existence of a solution for an integral equation using the results proved in Section 3. Let $X = (C[a,b], R)$ be the space of all real continuous functions on $[a,b]$ and $\rho : X \times X \to R^+$ defined by

$$\rho(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|, \qquad \forall x, y \in X.$$

Equip X with the 2-metric given by $\sigma : X^3 \to R^+$ which is defined by

$$\sigma(x,y,a) = \min\{\rho(x,y), \rho(y,a), \rho(a,x)\}, \qquad \forall x, y, a \in X.$$

As (X, ρ) is a complete metric space, (X, σ) is a complete 2-metric space, according to Example 1, we define a b_2-metric on X by

$$d(x,y,a) = (\sigma(x,y,a))^2, \qquad \forall x, y, a \in X.$$

It follows that (X, d) is a complete b_2-metric space with $s = 3$. Define a binary relation $\mathcal{R}$ on X by

$$\mathcal{R} = \{(x,y) \in X^2 : x(t) \le y(t) \text{ for all } t \in [a, \infty)\}. \tag{39}$$

Now, consider the integral equation:

$$x(t) = q(t) + \int_a^b h(t,s) A(s, x(s)) ds, \tag{40}$$

where $t \in [a,b] \subseteq R^+$. A solution of the Equation (40) is a function $x \in X = C[a,b]$. Assume that

(i) $h : [a,b] \times [a,b] \to [0, \infty)$, $q : [a,b] \to R$ and $A : [a,b] \times R \to R$ are continuous functions on $[a,b]$;

(ii) $\int_a^b h(t,s) dt \le r \le 1$;

(iii) there exists $x_0 \in X$ such that

$$x_0(t) \le q(t) + \int_a^b h(t,s) A(s, x_0(s)) ds.$$

(iv) A is nondecreasing in the second variable and for all $x, y, a \in X$, $s \in [a,b]$ there exists $0 < k < \frac{1}{\sqrt{3}}$ such that

$$\begin{aligned} \min \quad & \{|A(s,x(s)) - A(s,y(s))|, |A(s,x(s)) - a(s)|, |A(s,y(s)) - a(s)|\} \\ \le \quad & |A(s,x(s)) - A(s,y(s))| \\ \le \quad & k e^{-M(x,y,a)} \min\{|x(s) - y(s)|, |x(s) - a(s)|, |y(s) - a(s)|\}, \end{aligned}$$

where

$$M(x,y,a) = \max\left\{ d(x,y,a), d(x,fx,a), d(y,fy,a), \frac{d(x,fy,a) + d(y,fx,a)}{2s} \right\}.$$

Now, we are equipped to state and prove our main result in this section.

Theorem 3. *Under the assumptions (i)–(iv), the integral Equation (40) has a solution in X.*

Proof. Define $f : X \to X$ by

$$fx(t) = q(t) + \int_a^b h(t,s)A\big(s, x(s)\big)ds.$$

Observe that x is a solution for (40) if and only if x is a fixed point of f. Let $x, y, a \in X$ such that $x\mathcal{R}y$ for all $t \in [a,b]$. Since A is nondecreasing in the second variable, we have

$$\begin{aligned} fx(t) &= q(t) + \int_a^b h(t,s)A\big(s, x(s)\big)ds \\ &\leq q(t) + \int_a^b h(t,s)A\big(s, y(s)\big)ds \\ &= fy(t) \end{aligned}$$

Hence, $fx\mathcal{R}fy$ and $\mathcal{R}$ is f-closed. From Condition (iii), we conclude that $x_0 \leq fx_0$ for all $t \in [a,b]$, then $x_0\mathcal{R}fx_0$. Now, for any $x, y, a \in X$ such that $fx\mathcal{R}^{\sharp}fy$ we get

$$\begin{aligned} |fx(t) - fy(t)| &= |\int_a^b h(t,s)\big(A\big(s, x(s)\big) - A\big(s, y(s)\big)\big)ds| \\ &\leq \int_a^b |h(t,s)||A\big(s, x(s)\big) - A\big(s, y(s)\big)|ds \\ &\leq ke^{-M(x,y,a)} \int_a^b |h(t,s)| \min\{|x(s) - y(s)|, |x(s) - a(s)|, |y(s) - a(s)|\}ds \\ &\leq ke^{-M(x,y,a)} \int_a^b |h(t,s)| \min\{\max_{s\in[a,b]} |x(s) - y(s)|, \max_{s\in[a,b]} |x(s) - a(s)|, \\ &\quad \max_{s\in[a,b]} |y(s) - a(s)|\}ds \\ &\leq ke^{-M(x,y,a)} \int_a^b |h(t,s)| \min\{\rho(x(s), y(s)), \rho(x(s), a(s)), \rho(y(s), a(s))\}ds \\ &\leq ke^{-M(x,y,a)} \int_a^b |h(t,s)|\sigma(x,y,a)ds \leq rke^{-M(x,y,a)}\sigma(x,y,a). \end{aligned}$$

Therefore,

$$\sigma(fx, fy, a) \leq \max_{t\in[a,b]} |fx(t) - fy(t)| \leq rke^{-M(x,y,a)}\sigma(x,y,a).$$

It follows that

$$d(fx, fy, a) \leq r^2k^2e^{-2M(x,y,a)}d(x,y,a) \leq r^2k^2e^{-2M(x,y,a)}M(x,y,a) \leq \frac{e^{-2M(x,y,a)}}{3}M(x,y,a).$$

Thus,

$$d(fx, fy, a) \leq \frac{e^{-2M(x,y,a)}}{3}M(x,y,a) + LN(x,y,a),$$

where

$$M(x,y,a) = \max\left\{d(x,y,a), d(x,fx,a), d(y,fy,a), \frac{d(x,fy,a)+d(y,fx,a)}{2s}\right\},$$

and

$$N(x,y,a) = \min\{d(x,fx,a), d(y,fy,a), d(x,fy,a), d(y,fx,a)\},$$

with $\beta_s(t) = \frac{e^{-2t}}{3}$ and $L \geq 0$. Then f is almost a $\mathcal{R}$-Geraghty type contraction. In addition, if $\{x_n\} \in X$ is an $\mathcal{R}$-preserving sequence such that $\lim_{n\to\infty} x_n = x \in X$, then $x_n \leq x$ for all n. Hence, $x_n \mathcal{R} x$, for all n. Therefore, all the hypotheses of Corollary 1 are satisfied. Hence, f has a fixed point which is a solution for the integral Equation (40) in $X = C([a,b], R)$. $\square$

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