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Smooth Stable Manifold for Delay Differential Equations with Arbitrary Growth Rate

Lokesh Singh [1,*] and Dhirendra Bahuguna [2]

1 Department of Mathematics, University of Rijeka, HR-51000 Rijeka, Croatia
2 Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, India; dhiren@iitk.ac.in
* Correspondence: lokesh.singh@uniri.hr

Abstract: In this article, we construct a C^1 stable invariant manifold for the delay differential equation $x' = Ax(t) + Lx_t + f(t, x_t)$ assuming the ρ-nonuniform exponential dichotomy for the corresponding solution operator. We also assume that the C^1 perturbation, $f(t, x_t)$, and its derivative are sufficiently small and satisfy smoothness conditions. To obtain the invariant manifold, we follow the method developed by Lyapunov and Perron. We also show the dependence of invariant manifold on the perturbation $f(t, x_t)$.

Keywords: delay differential equation; ρ-nonuniform exponential dichotomy; stable manifold; arbitrary growth rates

MSC: 34D09; 34K19; 34K20; 37D10; 37D25

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Citation: Singh, L.; Bahuguna, D. Smooth Stable Manifold for Delay Differential Equations with Arbitrary Growth Rate. *Axioms* **2021**, *10*, 105. https://doi.org/10.3390/axioms10020105

Academic Editor: Emil Saucan

Received: 19 April 2021
Accepted: 20 May 2021
Published: 25 May 2021

1. Introduction

The invariant manifold theory started with the work of Hadamard [1] in 1901 when he constructed a manifold in the solution space of a differential equation with the property that if the trajectory of a solution starts in the manifold, it will remain in the manifold for all time $t > 0$. This proved to be of great importance for analyzing complex systems as it reduces the relevant dimension significantly. Later, Perron [2] and Lyapunov [3] developed another method to construct the invariant manifolds for autonomous differential equations. In the 1970s, due to the fundamental works of Hirsch et al. [4,5], Sacker and Sell [6], and Pesin [7,8], this theory became an important instrument for various fields like applied mathematics, biology, and engineering.

In the first method for constructing the invariant manifolds, Hadamard [1] used the geometrical properties of differential equations. He constructed the manifold over the linearized stable and unstable subspaces. However, Lyapunov [3] and Perron [2] developed an analytical method to construct the invariant manifolds. They obtained the invariant manifolds, using the variation of constants formula of the differential equations. In his approach, Perron introduced (and assumed) the notion of (uniform) exponential dichotomy for the solution operators and proved the existence of Lipschitz stable invariant manifolds for the small nonlinear perturbation of autonomous differential equations. The smoothness of these invariant manifolds is proved by Pesin [7]. In 1977, Pesin [8] generalized the notion of uniform hyperbolicity to nonuniform hyperbolicity which allows the rate of expansion and contraction to depend on initial time. Later he proved the stable manifold theorem in the finite-dimensional settings for nonhyperbolic trajectories. Pugh and Shub [9] proved a similar result for nonhyperbolic trajectories using the method developed by Hadamard. Ruelle [10] extended the result by Pesin to the Hilbert space settings in 1982.

The exponential dichotomy played an essential role in the development of invariant manifold theory for autonomous differential equations. Barreira and Valls extended the notion of exponential dichotomy for nonautonomous differential equations and called it nonuniform exponential dichotomy. With the assumption of nonuniform exponential dichotomy, they constructed the Lipschitz invariant manifold [11] and smooth invariant manifold [12] for nonautonomous differential equations. They also obtained the essential conditions for the existence of the nonuniform exponential dichotomy. The book [13] contains all the early works of Barreira and Valls.

Barreira and Valls observed that the solution operators corresponding to a class of nonautonomous differential equations show dichotomic behavior and also they have growth or decay rates of $e^{c\rho(t)}$, for some function $\rho(t)$. They named this notion as ρ-nonuniform exponential dichotomy. They showed in article [14] that the class of differential equations for which all the Lyapunov's exponents are infinite, satisfies ρ-nonuniform exponential dichotomy for $\rho(t) \neq t$. Subsequently they proved the stable manifold theorem for ordinary differential equations assuming ρ-nonuniform exponential dichotomy in [15]. In the article [16], Pan proved the existence of Lipschitz stable invariant manifold for impulsive nonautonomous differential equations with the assumption of ρ-nonuniform exponential dichotomy.

In this article, we consider a differential equation with the infinite delay given by,

$$x' = Ax(t) + Lx_t + f(t, x_t), \qquad x_s = \phi, \tag{1}$$

in a Banach space X. We assume that the solution operator associated with the corresponding linear delay differential equation

$$x' = Ax(t) + Lx_t, \qquad x_s = \phi, \tag{2}$$

satisfies ρ-nonuniform exponential dichotomy and the nonlinear perturbation $f(t, x_t)$ is sufficiently small and smooth. With these assumptions, we prove the existence of a C^1 stable invariant manifold for the delay differential Equation (1) following the approach of Perron and Lyapunov. We also showed the dependence of invariant manifolds on perturbations.

Barreira and Valls used the ρ-nonuniform exponential dichotomy for the nonautonomous differential equations in [17] to construct the Lipschitz stable invariant manifold and in [18] to construct the smooth invariant manifold. Pan considered the impulsive differential equation in [16], to construct the Lipschitz stable invariant manifold assuming ρ-nonuniform exponential dichotomy. In the article [19], we considered the case of delay differential equations with nonuniform exponential dichotomy and constructed a Lipschitz invariant manifold. However, in this article, we are assuming a more general ρ-nonuniform exponential dichotomy for differential equation with infinite delay and we are constructing a C^1 stable invariant manifold. In the later part of the article, we also show that a small change in perturbation gives rise to a small variation in the manifold.

The paper is arranged in the following manner. Our setup and some preliminary results are given in Section 2. In the next section, we provide a few examples of differential equations satisfying the ρ-nonuniform exponential dichotomy. Section 3 contains the proof of the existence of the C^1 stable invariant manifold, and in Section 4, we prove the dependency of the manifold on perturbation. In the end, we present a few more examples satisfying the assumptions of our main theorem.

2. Preliminaries

Let $(X, \| \, \|_X)$ be a Banach space. For any interval $J \subset \mathbb{R} := (-\infty, \infty)$, we denote $C(J, X)$ as a space of X-valued continuous function on J. For a function $x : (-\infty, a] \to X$ and $t \leq a$, we define a function $x_t : \mathbb{R}^- := (-\infty, 0] \to X$ by $x_t(\theta) := x(t + \theta)$ for $\theta \in \mathbb{R}^-$. Furthermore, let C_γ be a space of continuous functions defined by

$$C_\gamma := \left\{ \psi \in C(\mathbb{R}^-, X) : \lim_{\theta \to -\infty} \|\psi(\theta)\|_X e^{\gamma\theta} = 0 \right\},$$

for $\gamma > 0$. We define a norm on the phase space C_γ,

$$\|\psi\|_{C_\gamma} := \sup_{\theta \in \mathbb{R}^-} \|\psi(\theta)\|_X e^{\gamma\theta}, \qquad \psi \in C_\gamma.$$

Finally, we consider a linear delay differential equation in the Banach space X,

$$x' = Ax(t) + Lx_t, \qquad x_s = \phi,$$

for $(s, \phi) \in \mathbb{R}^+ \times C_\gamma$. The linear operator $A : X \to X$ generates a strongly continuous compact semigroup $\{T(t,s)\}_{t \geq 0}$ and $L : C_\gamma \to X$ is a bounded linear operator. Let the evolution operator corresponding to the above differential equation is denoted by $V(t,s)$ and for $\phi \in C_\gamma$, $V(t,s)$ is given by

$$[V(t,s)\phi](\theta) = x_t(\theta, s, \phi, 0), \qquad \theta \in \mathbb{R}^-, t \geq s. \tag{3}$$

One can easily see that the evolution operator defines a strongly continuous semigroup and for every $t \geq s \geq r \geq 0$, it satisfies the semigroup property given by:

$$V(t,s)V(s,r) = V(t,r), \quad \text{and} \quad V(t,t) = I.$$

Here, I is an identity operator on C_γ. For a continuous function $p : \mathbb{R} \to X$, we consider the perturbed system of delay differential equation:

$$x' = Ax(t) + Lx_t + p(t), \qquad x_s = \phi. \tag{4}$$

Let the solution of the above delay differential equation is denoted by $(x_t(s, \phi, p)$. Now, we give a representation of $x_t(s, \phi, p)$ depending on the evolution operator $\{V(t,s)\}_{t \geq s}$. Let us introduce a function Γ^n given by

$$\Gamma^n(\theta) = \begin{cases} (n\theta + 1)I_X, & \frac{-1}{n} \leq \theta \leq 0, \\ 0, & \theta < \frac{-1}{n}, \end{cases}$$

where n is any positive integer and I_X is the identity operator on X. It is easy to verify that for $y \in X$,

$$\Gamma^n y \in C_\gamma \qquad \text{and} \qquad \|\Gamma^n y\|_{C_\gamma} \leq \max\{1, e^{\frac{-\gamma}{n}}\} \| y \|_X \leq \| y \|_X. \tag{5}$$

The next result by Hino and Naito [20] establishes the variation of constants formula for the delay differential Equation (4) in the phase space C_γ.

Proposition 1. *Let $(s, \phi) \in \mathbb{R}^+ \times C_\gamma$ be given. Then the segment $x_t(s, \phi, p)$ of solution $x(\cdot, s, \phi, p)$ of non-homogeneous functional differential Equation (4) satisfies the following relation in C_γ:*

$$x_t(s, \phi, p) = V(t,s)\phi + \lim_{n \to \infty} \int_s^t V(t,\tau)\, \Gamma^n\, p(\tau)\, d\tau, \qquad t \geq s. \tag{6}$$

Definition 1. *[ρ-nonuniform exponential dichotomy] Let $\rho : [0, \infty) \to [0, \infty)$ be an increasing function with $\rho(t) \to \infty$ as $t \to \infty$. We say that the linear Equation (2) admits a ρ-nonuniform exponential dichotomy if for every $t \geq s \geq 0$, there exist projection maps $P(t) : C_\gamma \to C_\gamma$, constants $a < 0 \leq b, \epsilon \geq 0$ and $K > 1$, such that :*

(i) $P(t)V(t,s) = V(t,s)P(s)$;
(ii) $V_Q(t,s) := V(t,s) : Q(s)C_\gamma \to Q(t)C_\gamma$ *is invertible, where* $Q(t) = I - P(t)$ *is the complementary projection ;*
(iii)

$$\|V(t,s)P(s)\| \leq Ke^{a(\rho(t) - \rho(s)) + \epsilon\rho(s)}, \qquad \|V_Q(t,s)^{-1}Q(t)\| \leq Ke^{-b(\rho(t) - \rho(s)) + \epsilon\rho(t)}. \tag{7}$$

Now, for each $t \geq 0$, we define

$$E(t) := P(t)(C_\gamma) \quad \text{and} \quad F(t) := Q(t)(C_\gamma).$$

We call $E(t)$ and $F(t)$ stable and unstable subspace respectively. Let us take examples of some differential equations which satisfy ρ-nonuniform exponential dichotomy.

Remark 1. *Here we are presenting a few examples of differential equations that satisfy the ρ-nonuniform exponential dichotomy. In the first example, we consider a case where differential equation satisfies ρ-nonuniform exponential dichotomy for $\rho(t) = t$.*

Example 1. *[21]*

$$x'(t) = \big(-\alpha - \frac{\cos t}{a + \sin t} + \beta\, t \sin t\big)x(t), \tag{8}$$

for $\alpha > 2\beta > 0$.

Let $V(t,s)$ be the solution operator of the above problem, then

$$V(t,s) = e^{-\alpha(t-s) - \ln(a+\sin t) + \ln(a+\sin s) + \beta(s\cos s - t\cos t) + \beta(\sin t - \sin s)}.$$

$$\begin{aligned} \|V(t,s)\| &= e^{-\alpha(t-s) - \ln(a+\sin t) + \ln(a+\sin s) + \beta(s\cos s - t\cos t) + \beta(\sin t - \sin s)} \\ &\leq e^{2\beta + 2(\ln(a+1))}\, e^{(-\alpha+\beta)(t-s) + 2\beta s} \\ &= Ce^{(-\alpha+\beta)(t-s) + 2\beta s}. \end{aligned}$$

Note that $(-\alpha + \beta) < 0$ and C is a constant. Hence, the differential equation satisfies ρ-nonuniform exponential dichotomy with $\rho(t) = t$.

In the next example, we consider a system of two differential equations which satisfies ρ-nonuniform exponential dichotomy with $\rho(t) = t^2$.

Example 2.

$$\begin{aligned} x'(t) &= \Big(\frac{-2\omega t}{1 + e^{-t^2}} + 2\epsilon\, t\big(t^2 \sin t^2 - 1\big)\Big)x(t), \\ y'(t) &= \Big(\frac{2\omega t}{1 + e^{-t^2}} - 2\epsilon\, t\big(t^2 \cos t^2 - 1\big)\Big)y(t), \end{aligned}$$

for some $\omega > 2\epsilon > 0$.

The evolution operator for linear system above is given by

$$T(t,s) = \begin{bmatrix} U(t,s) & 0 \\ 0 & V(t,s) \end{bmatrix},$$

where

$$U(t,s) = e^{-\omega(t^2 - s^2) - \omega \ln \frac{(1+e^{-t^2})}{(1+e^{-s^2})} + \epsilon(s^2 \cos s^2 - t^2 \cos t^2) + \epsilon(\sin t^2 - \sin s^2) - \epsilon(t^2 - s^2)}$$

and

$$V(t,s) = e^{\omega(t^2 - s^2) + \omega \ln \frac{(1+e^{-t^2})}{(1+e^{-s^2})} + \epsilon(s^2 \sin s^2 - t^2 \sin t^2) + \epsilon(\cos t^2 - \cos s^2) - \epsilon(t^2 - s^2)}.$$

Now, for the projection map $P(t)(x,y) = (x,0)$,

$$\|T(t,s)P(t)\| = \|U(t,s)\| \leq e^{2\epsilon + \omega \ln 2} e^{-\omega(t^2 - s^2) + 2\epsilon s^2},$$

and

$$\|T(t,s)^{-1}Q(t)\| = \|V(t,s)^{-1}\| \leq e^{2\epsilon+\omega\ln 2}e^{-\omega(t^2-s^2)+2\epsilon t^2},$$

for $t \geq s \geq 0$. *Hence, the linear system satisfies* ρ*-nonuniform exponential dichotomy with* $a = -\omega < 0$ *and* $b = \omega > 0$ *where* $\rho(t) = t^2$.

3. Stable Manifold Theorem

This section is dedicated to the construction of the C^1 stable invariant manifold for the delay differential Equation (1). Let Ω be the space of continuous functions $f(t,\phi) : \mathbb{R}^+ \times C_\gamma \to X$ such that

(i) $f(t,0) = 0$ and $\frac{\partial f}{\partial \phi}(t,0) = 0$ for all $t \geq 0$;
(ii) There exists a positive function $b : [0,\infty) \to (0,\infty)$ such that,

$$\|\frac{\partial^j f}{\partial \phi^j}(t,\phi) - \frac{\partial^j f}{\partial \phi^j}(t,\psi)\|_X \leq b(t)\|\phi-\psi\|_{C_\gamma}, \tag{9}$$

for $j = 0,1$.

Proposition 1 ensures the existence of global solution $x_t(\cdot,s,\phi,f)$ of the differential Equation (1) with the above mentioned properties of perturbation $f(t,x_t)$. We also assume that the solution operator $V(t,s)$ satisfies ρ-nonuniform exponential dichotomy. Therefore, Using the projection maps, we can project the global solution $x_t = (u_t,v_t) \in E(t) \times F(t)$ with initial condition $(u_s,v_s) \in E(s) \times F(s)$ and it satisfy

$$u_t = V(t,s)u_s + \lim_{n\to\infty}\int_s^t P(t)V(t,\tau)\Gamma^n f(\tau,u_\tau,v_\tau)d\tau, \tag{10}$$

$$v_t = V(t,s)v_s + \lim_{n\to\infty}\int_s^t Q(t)V(t,\tau)\Gamma^n f(\tau,u_\tau,v_\tau)d\tau, \tag{11}$$

for $t \geq s$, here $u(t)$ and $v(t)$ are called stable solution and unstable solution respectively.

Since we want our manifold to be in the form of a graph of some C^1 function, therefore let us consider a space of C^1 functions χ consisting of $\Phi(s,\cdot) : E(s) \to C_\gamma$ such that for each $s \geq 0$,

1. $\Phi(s,0) = 0$; $(\partial\Phi/\partial\phi)(s,0) = 0$ and $\Phi(s,E(s)) \subset F(s)$.
2. For every $\phi,\psi \in E(s)$ and for $j = 0,1$,

$$\|\frac{\partial^j \Phi}{\partial \phi^j}(s,\phi) - \frac{\partial^j \Phi}{\partial \phi^j}(s,\psi)\|_{C_\gamma} \leq \|\phi-\psi\|_{C_\gamma}. \tag{12}$$

Using the result of [22], χ is a Banach space with the norm,

$$|\Phi|' := \sup\left\{\frac{\|\Phi(s,\phi)\|_{C_\gamma}}{\|\phi\|_{C_\gamma}} : s \geq 0 \text{ and } \phi \in E(s)\setminus\{0\}\right\}. \tag{13}$$

Given $\Phi \in \chi$, we consider the graph

$$W_\Phi = \{(s,\phi,\Phi(s,\phi)) : (s,\phi) \in [0,\infty) \times E(s)\}. \tag{14}$$

Furthermore, for each $\kappa \in \mathbb{R}^+$, let Ψ_κ be the semiflow generated by the autonomous equation

$$t' = 1, \qquad x' = Ax(t) + Lx_t + f(t,x_t).$$

Given $\kappa = t - s \geq 0$ and $(s, u_s, v_s) \in \mathbb{R}^+ \times E(s) \times F(s)$, we have

$$\Psi_\kappa(s, u_s, v_s) = (s + \kappa, u_{s+\kappa}, v_{s+\kappa}) = (t, u_t, v_t), \tag{15}$$

where u_t and v_t are solutions on stable and unstable subspaces respectively given by Equations (10) and (11).

Here are our assumptions to obtain the C^1 manifold for the Equation (1).

(H1) $a + \epsilon \leq 0$.
(H2) $0 < M = \sup_{s \geq 0}\{\int_s^\infty b(\tau)e^{\epsilon\rho(\tau)}d\tau\} < \infty$ with $4KM < 1$.
(H3) $0 < \bar{M} = \sup_{s \geq 0}\{e^{(b-a+\epsilon)\rho(s)} \int_s^\infty e^{(a-b+\epsilon)\rho(\tau)} b(\tau)d\tau\} < \infty$ with $12K^2\bar{M} < 1$.

In the following Theorem, we give the existence of a C^1 stable invariant manifold for the perturbed Equation (1).

Theorem 1. *Assume that the linear Equation (2) satisfies ρ-nonuniform exponential dichotomy with $(H1)$ and the perturbation in (1) satisfies (9) with $(H2)$, $(H3)$, then there exists a C^1 function $\Phi \in \chi$ such that the set W_Φ, defined by the Equation (14), is forward invariant under the semiflow Ψ_κ, in the sense that for each $\kappa \geq 0, \Psi_\kappa(W_\Phi) \subseteq W_\Phi$. Furthermore, for every $\kappa = t - s \geq 0$; $\phi, \psi \in E(s)$ there exists $D > 0$ such that*

$$\|\Psi_\kappa(s, \phi, \Phi(s, \phi)) - \Psi_\kappa(s, \psi, \Phi(s, \psi))\|_{C_\gamma} \leq De^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}\|\phi - \psi\|_{C_\gamma}. \tag{16}$$

Outline: We want our manifold to be invariant under the semiflow, this means that the trajectory of the solutions should remain in the manifold for all time $t \geq 0$, provided it starts in the manifold. Since our choice of manifold is graph of some C^1 function Φ, therefore, in the manifold, the solution $x(t)$ must take the form, $x(t) = (u(t), \Phi(t, u(t))) \in E(t) \times F(t)$ with $(u_s, v_s) \in E(s) \times F(s)$ where $u(t), \Phi(t, u(t))$ satisfy

$$u_t = V(t,s)u_s + \lim_{n\to\infty} \int_s^t P(t)V(t,\tau)\Gamma^n f(\tau, u_\tau, \Phi(\tau, u_\tau))d\tau, \tag{17}$$

$$\Phi(t, u_t) = V(t,s)v_s + \lim_{n\to\infty} \int_s^t Q(t)V(t,\tau)\Gamma^n f(\tau, u_\tau, \Phi(\tau, u_\tau))d\tau. \tag{18}$$

Now, we are going to prove some lemmas which will be helpful in giving the existence of C^1 smooth manifold. In our first lemma, we establish the existence of solution in the stable direction given by Equation (17).

Lemma 1. *Given $(s, \phi, \Phi) \in [0, \infty) \times E(s) \times \chi$, there exists a unique C^1 function $u : (-\infty, \infty) \times E(s) \to X$ with $u_s = \phi$ and $u_t \in E(t)$ for each $t \geq s$. The function u_t also satisfies Equation (17) for every $t \geq s$ and for each $\phi \in E(s)$ the segment u_t satisfies:*

$$\|u_t\|_{C_\gamma} \leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}\|\phi\|_{C_\gamma}. \tag{19}$$

Proof. Let Ω^* be the space of all C^1 functions $u : (-\infty, \infty) \times E(s) \to X$ such that the following properties satisfies:

(1) $u_s = \phi$, and $u_t \in E(t)$ for every $t \geq s$ and for each $\phi \in E(s)$ with the norm,

$$\|u\|_* := \sup_{t \geq s, \phi \in E(s)\setminus\{0\}} \left\{ \frac{\|u_t\|_{C_\gamma}}{\|\phi\|_{C_\gamma}} e^{-\sigma(t)} \right\} \leq 2K. \tag{20}$$

(2) The function $u \in \Omega^*$ satisfies

$$\alpha(u_t) := \sup_{t\geq s, \phi\in E(s)\setminus\{0\}} \left\{ \left\| \frac{\partial u_t}{\partial \phi} \right\| e^{-\sigma(t)} \right\} \leq 2K, \tag{21}$$

$$\beta(u_t) := \sup_{t\geq s, \phi_1,\phi_2\in E(s)\setminus\{0\}} \left\{ \left\| \frac{\partial u_t}{\partial \phi}(\phi_1) - \frac{\partial u_t}{\partial \phi}(\phi_2) \right\| \frac{e^{-\sigma(t)-\epsilon\rho(s)}}{\|\phi_1 - \phi_2\|} \right\} \leq 2K, \tag{22}$$

where $\sigma(t) = a(\rho(t) - \rho(s)) + \epsilon\rho(s)$. It follows from result in [22] that Ω^* is a complete metric space with norm in (20). Given $(s, \phi) \in [0, \infty) \times E(s)$ and $\Phi \in \chi$, we define an operator $L : \Omega^* \to \Omega^*$ as

$$(Lu)_t = V(t,s)\phi + \lim_{n\to\infty} \int_s^t V(t,\tau)P(\tau)\Gamma^n f(\tau, u_\tau, \Phi(\tau, u_\tau))d\tau.$$

Note that $(Lu)_s = \phi$ and $(Lu)_t \in E(t)$ for every $t \geq s$. Now using Equations (5), (7), (9), (12) and (20), we have

$$\begin{aligned}
&\|(Lu)_t\|_{C_\gamma} \\
&\leq \|V(t,s)\phi\|_{C_\gamma} + \lim_{n\to\infty} \int_s^t \|V(t,\tau)P(\tau)\| \|\Gamma^n f(\tau, u_\tau, \Phi(\tau, u_\tau))\|_{C_\gamma} d\tau \\
&\leq Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)} \|\phi\|_{C_\gamma} + \int_s^t Ke^{a(\rho(t)-\rho(\tau))+\epsilon\rho(\tau)} b(\tau) \|(u_\tau, \Phi(\tau, u_\tau))\|_{C_\gamma} d\tau \\
&\leq Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)} \|\phi\|_{C_\gamma} + 2K \int_s^t e^{a(\rho(t)-\rho(\tau))+\epsilon\rho(\tau)} b(\tau) \|u_\tau\|_{C_\gamma} d\tau \\
&\leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)} \|\phi\|_{C_\gamma} \left(\frac{1}{2} + 2K\|u\|_* \int_s^t b(\tau) e^{\epsilon\rho(\tau)} d\tau \right).
\end{aligned}$$

Therefore,

$$\|(Lu)\|_* \leq \frac{1}{2} + 2KM\|u\|_* < \infty. \tag{23}$$

Hence, $L(\Omega^*) \subset \Omega^*$. Now, let $u, v \in \Omega^*$, then again using Equations (5), (7), (9), (12) and (20),

$$\begin{aligned}
&\|(Lu)_t - (Lv)_t\|_{C_\gamma} \\
&\leq \int_s^t Ke^{a(\rho(t)-\rho(\tau))+\epsilon\rho(\tau)} \|f(\tau, u_\tau, \Phi(\tau, u_\tau)) - f(\tau, v_\tau, \Phi(\tau, v_\tau))\|_X d\tau \\
&\leq 2K \int_s^t e^{a(\rho(t)-\rho(\tau))+\epsilon\rho(\tau)} b(\tau) \|u_\tau - v_\tau\|_{C_\gamma} d\tau \\
&\leq 4K^2 \|\phi\|_{C_\gamma} e^{a(\rho(t)-\rho(s))+\epsilon\rho(s)} \|u - v\|_* \int_s^t b(\tau) e^{\epsilon\rho(\tau)} d\tau.
\end{aligned}$$

Furthermore we can write, $\|Lu - Lv\|_* \leq 2KM\|u - v\|_*$.

Since $KM < \frac{1}{4}$, therefore L is a contraction map in Ω^* which ensures a unique fixed point function $u \in \Omega^*$ such that $Lu = u$. Using Equation (23), we have,

$$\|u\|_* \leq \frac{1}{2} + 2KM\|u\|_*.$$

Hence, for every $t \geq s$,

$$\|u_t\|_{C_\gamma} \leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)} \|\phi\|_{C_\gamma}.$$

□

Next, we give the alternate form for our manifold map.

Lemma 2. *Given $\Phi \in \chi$, let u_t be the stable solution given by Lemma 1 for $(s,\phi) \in [0,\infty) \times E(s)$. Then, for Φ, the following properties holds,*

1. *If $(s,\phi) \in [0,\infty) \times E(s)$ and for every $t \geq s$*

$$\Phi(t,u_t) = V(t,s)\Phi(s,\phi) + \lim_{n\to\infty} \int_s^t V(t,\tau)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau, \tag{24}$$

then for all $(s,\phi) \in [0,\infty) \times E(s)$,

$$\Phi(s,\phi) = -\lim_{n\to\infty} \int_s^\infty V_Q^{-1}(\tau,s)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau. \tag{25}$$

2. *If Equation (25) holds for all $(s,\phi) \in [s,\infty) \times E(s)$, then Equation (24) holds for all $t \geq s$ and $u_t = u_t(s,\phi,\Phi)$ while $(s,\phi) \in [s,\infty) \times E(s)$.*

Proof. Let us first show the validity of the integral in (25) for each fixed $n > 0$. Using ρ-nonuniform exponential dichotomy and Equations (9), (12), (19), we have

$$\begin{aligned}
&\| \int_s^\infty V_Q^{-1}(\tau,s)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau\|_{C_\gamma} \\
&\qquad\leq \int_s^\infty Ke^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)}\|f(\tau,u_\tau,\Phi(\tau,u_\tau))\|_X d\tau \\
&\qquad\leq 2K \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)} b(\tau)\|u_\tau\|_{C_\gamma} d\tau \\
&\qquad\leq 4K^2\|\phi\|_{C_\gamma} \int_s^\infty b(\tau)e^{(-b+\epsilon+a)\rho(\tau)}e^{(b-a+\epsilon)\rho(s)}d\tau \\
&\qquad\leq 4K^2\|\phi\|_{C_\gamma}\bar{M} < \infty.
\end{aligned}$$

Hence, for each fixed $n > 0$, the integral in Equation (25) is valid. Now if Equation (24) holds for every $(s,\phi) \in [0,\infty) \times E(s)$ and $t \geq s$, then

$$\Phi(s,\phi) = V_Q^{-1}(t,s)\Phi(t,u_t) - \lim_{n\to\infty} \int_s^t V_Q^{-1}(\tau,s)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau.$$

Using Equations (7), (12) and (19),

$$\begin{aligned}
\|V_Q^{-1}(t,s)\Phi(t,u_t)\|_{C_\gamma} &\leq Ke^{-b(\rho(t)-\rho(s))+\epsilon\rho(t)}\|u_t\|_{C_\gamma} \\
&\leq 2K^2e^{-b(\rho(t)-\rho(s))+\epsilon\rho(t)}e^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}\|\phi\|_{C_\gamma} \\
&\leq 2K^2e^{(b-a+\epsilon)\rho(s)}e^{(a-b+\epsilon)\rho(t)}\|\phi\|_{C_\gamma}.
\end{aligned}$$

Note that $(a+\epsilon-b) < 0$, and taking $t \to \infty$ proves our result. Additionally, note that the Equation (25) is obtained by operating the invertible map $V_Q^{-1}(t,s)$ on the solution $x(t) = (u(t), \Phi(t,u(t)))$, therefore there is no issue of convergence here.

Now assume Equation (25) holds for all $(s,\phi) \in [0,\infty) \times E(s)$. Note that $(t,u_t) \in [s,\infty) \times E(t)$, therefore it follows from Equation (25),

$$
\begin{aligned}
V(t,s)\Phi(s,\phi) &= -\lim_{n\to\infty}\int_s^\infty V(t,\tau)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau \\
&= -\lim_{n\to\infty}\int_s^t V(t,\tau)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau \\
&\quad -\lim_{n\to\infty}\int_t^\infty V^{-1}(\tau,t)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau \\
&= -\lim_{n\to\infty}\int_s^t V(t,\tau)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau + \Phi(t,u_t).
\end{aligned}
$$

Hence we get,

$$
\Phi(t,u_t) = V(t,s)\Phi(s,\phi) + \lim_{n\to\infty}\int_s^t V(t,\tau)Q(\tau)\Gamma^n f(\tau,u_\tau,\Phi(\tau,u_\tau))d\tau.
$$

□

The next result shows the dependency of the stable solution u_t on the history function $\phi \in C_\gamma$.

Lemma 3. *Let the stable solutions u_t, v_t be obtained using Lemma 1 for (s,ϕ,Φ) and $(s,\psi,\Phi) \in [0,\infty) \times E(s) \times \chi$ respectively. We have the following estimates for every $t \geq s$:*

$$
\|u_t - v_t\|_{C_\gamma} \leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}\|\phi-\psi\|_{C_\gamma}. \tag{26}
$$

Proof. Using Lemma 1 and Equations (7), (9), (19) in (17),

$$
\begin{aligned}
&\|u_t - v_t\|_{C_\gamma} \\
&\leq \|V(t,s)(\phi-\psi)\|_{C_\gamma} + \int_s^t \|V(t,\tau)P(\tau)\|\|f(\tau,u_\tau,\Phi(\tau,u_\tau)) - f(\tau,v_\tau,\Phi(\tau,v_\tau))\|_X d\tau \\
&\leq Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}\|\phi-\psi\|_{C_\gamma} + 2K\int_s^t e^{a(\rho(t)-\rho(\tau))+\epsilon\rho(\tau)}b(\tau)\|u_\tau - v_\tau\|_{C_\gamma}d\tau \\
&\leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}\|\phi-\psi\|_{C_\gamma}\left\{\frac{1}{2} + 2K\|u-v\|_*\int_s^t e^{\epsilon\rho(\tau)}b(\tau)d\tau\right\}.
\end{aligned}
$$

Therefore we have,

$$
\begin{aligned}
\|u-v\|_* &\leq \frac{1}{2} + 2KM\|u-v\|_*, \\
\|u-v\|_* &\leq \frac{1}{2(1-2KM)} < 1.
\end{aligned}
$$

Hence we get the desired result,

$$
\|u_t - v_t\|_{C_\gamma} \leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}\|\phi-\psi\|_{C_\gamma}, \qquad \text{for every } t \geq s.
$$

□

Lemma 4. *Let the stable solutions u_t, v_t be obtained using Lemma 1 for (s,ϕ,Φ) and $(s,\phi,\Psi) \in [0,\infty) \times E(s) \times \chi$ respectively. We have the following estimates for every $t \geq s$:*

$$
\|u_t - v_t\|_{C_\gamma} \leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}|\Phi-\Psi|'\|\phi\|_{C_\gamma}. \tag{27}
$$

Proof. Using Equations (12) and (13), we have

$$
\|\Phi(\tau,u_\tau) - \Psi(\tau,v_\tau)\|_{C_\gamma} \leq \|u_\tau - v_\tau\|_{C_\gamma} + |\Phi-\Psi|'\|v_\tau\|_{C_\gamma}. \tag{28}
$$

Now, using above estimate and Equations (7), (9) and (19) in (17),

$$\begin{aligned}\|u_t - v_t\|_{C_\gamma} &\leq \int_s^t Ke^{a(\rho(t)-\rho(\tau))+\epsilon\rho(\tau)} b(\tau) \|(u_\tau, \Phi(\tau, u_\tau)) - (v_\tau, \Psi(\tau, v_\tau))\|_{C_\gamma} d\tau \\ &\leq K \int_s^t e^{a(\rho(t)-\rho(\tau))+\epsilon\rho(\tau)} b(\tau) \Big\{ 2\|u_\tau - v_\tau\|_{C_\gamma} + |\Phi - \Psi|' \|v_\tau\|_{C_\gamma} \Big\} d\tau \\ &\leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)} \|\phi\|_{C_\gamma} \{2\|u-v\|_* + |\Phi - \Psi|'\} K \int_s^t b(\tau) e^{\epsilon\rho(\tau)} d\tau.\end{aligned}$$

Therefore we have,

$$\begin{aligned}\|u-v\|_* &\leq \big\{2\|u-v\|_* + |\Phi-\Psi|'\big\} KM, \\ \|u-v\|_* &\leq \frac{KM}{1-2KM} |\Phi - \Psi|'.\end{aligned}$$

Since $2KM < 1/2$, therefore, for every $t \geq s$, we get our desired result:

$$\|u_t - v_t\|_{C_\gamma} \leq 2Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)} |\Phi - \Psi|' \|\phi\|_{C_\gamma}.$$

□

The following result proves the existence of a C^1 smooth map $\Phi \in \chi$ satisfying Equation (25).

Lemma 5. *Let the assumptions $a + \epsilon \leq 0$,(H2) and (H3) hold. Then, there exists a unique function $\Phi \in \chi$ such that Equation (25) holds for each $(s, \phi) \in [0, \infty) \times E(s)$.*

Proof. Consider an operator $J : \chi \to \chi$ given by,

$$(J\Phi)(s,\phi) := -\lim_{n\to\infty} \int_s^\infty V_Q^{-1}(\tau,s) Q(\tau) \Gamma^n f(\tau, u_\tau, \Phi(\tau, u_\tau)) d\tau, \tag{29}$$

for each $(s, \phi) \in [0, \infty) \times E(s)$, where u_t is the unique function given by Lemma 1 for (s, ϕ, Φ). Using the Equation (19), $(J\Phi)(s, 0) = 0$, $\forall\ s > 0$. For $s \in [0, \infty)$ and $\phi, \psi \in E(s)$, let u_t, v_t denote the stable solutions obtained using Lemma 1 for (s, ϕ, Φ) and (s, ψ, Φ) respectively. Now, using Equations (7), (9), (12) and (26) and Lemma 3 we have,

$$\begin{aligned}&\|(J\Phi)(s,\phi) - (J\Phi)(s,\psi)\|_{C_\gamma} \\ &\leq \int_s^\infty Ke^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)} b(\tau) \|(u_\tau, \Phi(\tau, u_\tau)) - (v_\tau, \Phi(\tau, v_\tau))\|_{C_\gamma} d\tau \\ &\leq 2K \int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)} b(\tau) \|u_\tau - v_\tau\|_{C_\gamma} d\tau \\ &\leq 4K^2 \|\phi - \psi\|_{C_\gamma} \int_s^\infty e^{(a-b+\epsilon)\rho(\tau)} b(\tau) e^{(b-a+\epsilon)\rho(s)} d\tau \\ &\leq 4K^2 \bar{M} \|\phi - \psi\|_{C_\gamma}.\end{aligned}$$

Since $4K^2\bar{M} < 1$,

$$\|(J\Phi)(s,\phi) - (J\Phi)(s,\psi)\|_{C_\gamma} \leq \|\phi - \psi\|_{C_\gamma}.$$

Hence, $J(\chi) \subset \chi$. For $(s, \phi) \in [0, \infty) \times E(s)$; $\Phi, \Psi \in \chi$ and u_t, v_t are the corresponding stable solutions, then,

$$
\begin{aligned}
&\|(J\Phi)(s,\phi)-(J\Psi)(s,\phi)\|_{C_\gamma}\\
&\leq \int_s^\infty Ke^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)}b(\tau)\|(u_\tau-v_\tau,\Phi(\tau,u_\tau)-\Psi(\tau,v_\tau))\|_{C_\gamma}d\tau\\
&\leq K\int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)}b(\tau)\Big(2\|u_\tau-v_\tau\|_{C_\gamma}+|\Phi-\Psi|'\|v_\tau\|_{C_\gamma}\Big)d\tau\\
&\leq 4K^2|\Phi-\Psi|'\|\phi\|_{C_\gamma}e^{(b-a+\epsilon)\rho(s)}\int_s^\infty e^{(a-b+\epsilon)\rho(\tau)}b(\tau)d\tau\\
&\leq 4K^2\bar{M}|\Phi-\Psi|'\|\phi\|_{C_\gamma}.
\end{aligned}
$$

Since $4K^2\bar{M}<1$, hence, J is a contraction map which gives the existence of a unique fixed function $\Phi\in\chi$ such that $J\Phi=\Phi$. □

Now we give the proof of Theorem 1 using the lemmas proved in this section.

Proof. To prove Theorem 1 we need to find a function $\Phi\in\chi$ satisfying Equations (24) and (25), also, the graph of Φ, i.e., W_Φ, should be invariant under the semiflow Ψ_κ given by Equation (15). For each $(s,\phi)\in[0,\infty)\times E(s)$, Lemma 5 gives the existence of $\Phi\in\chi$ satisfying Equations (24) and (25). Furthermore, from Lemma 1 for each $(s,\phi,\Phi)\in[0,\infty)\times E(s)\times\chi$ there exists a unique function u_t satisfying Equation (17). Note that $(t,u_t)\in[0,\infty)\times E(t)$, therefore

$$\Psi_{t-s}(s,\phi,\Phi(s,\phi))=(t,u_t,\Phi(t,u_t))\ \in W_\Phi,\qquad \text{for all } t\geq s.$$

Now for each $(s,\phi),(s,\psi)\in[0,\infty)\times E(s)$ and $\kappa=t-s\geq 0$ by Lemma 3, Lemma 4 and Equation (15) we have,

$$
\begin{aligned}
\|\Psi_\kappa(s,\phi,\Phi(s,\phi))-\Psi_\kappa(s,\psi,\Phi(s,\psi))\| &\leq \|(t,u_t^\phi,\Phi(t,u_t^\phi))-(t,u_t^\psi,\Phi(t,u_t^\psi))\|\\
&\leq 2\|u_t^\phi-u_t^\psi\|_{C_\gamma}\\
&\leq 4Ke^{a(\rho(t)-\rho(s))+\epsilon\rho(s)}\|\phi-\psi\|_{C_\gamma}.
\end{aligned}
$$

This completes the proof of the theorem. □

4. Stable Manifold and Perturbations

In this section we are showing how the manifold varies with the change in perturbation. We defined the space of perturbation Ω in the beginning of Section 3. Now we define a norm on Ω, for $f\in\Omega$,

$$\|f\|_\Omega := \sup\left\{\frac{\|f(t,\phi)\|_X}{\|\phi\|_{C_\gamma}}, t\geq 0, \phi\in C_\gamma\setminus\{0\}\right\}. \tag{30}$$

We also assume one more condition:

(H4) $0<\hat{M}=\sup_{s\geq 0}\{e^{(b-a+\epsilon)\rho(s)}\int_s^\infty e^{(a-b+\epsilon)\rho(\tau)}d\tau\}\leq\infty$ with $8K^2\hat{M}<1$.

Theorem 2. *Assume that the differential Equation (1) admits ρ-nonuniform exponential dichotomy and also the assumptions in $(H1)$–$(H4)$ hold, then for each set (s,ϕ,f) and (s,ϕ,g) the maps Φ and Ψ satisfies*

$$|\Phi-\Psi|'\leq\|f-g\|_\Omega. \tag{31}$$

Proof. For every $(s,\phi,f)$and (s,ϕ,g), we obtain unique functions Φ and Ψ using Lemma 5. Now using Lemma 1 for (s,ϕ,f,Φ) and (s,ϕ,g,Ψ) we obtain unique functions $u^{\Phi}=u$ and $u^{\Psi}=v$. From Equation (25) we have,

$$
\begin{aligned}
&\|\Phi(s,\phi)-\Psi(s,\phi)\|_{C_\gamma} \\
&\leq \int_s^\infty \|V_Q^{-1}(\tau,s)Q(\tau)\big(f(\tau,u_\tau,\Phi(\tau,u_\tau))-g(\tau,v_\tau,\Psi(\tau,v_\tau))\big)\|_{C_\gamma}d\tau \\
&\leq \int_s^\infty Ke^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)}\|f(\tau,u_\tau,\Phi(\tau,u_\tau))-g(\tau,v_\tau,\Psi(\tau,v_\tau))\|_X d\tau \\
&\leq K\int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)}\Big\{b(\tau)\big(2\|u_\tau-v_\tau\|_{C_\gamma}+|\Phi-\Psi|'\|u_\tau\|_{C_\gamma}\big) \\
&\qquad\qquad +\|f-g\|_\Omega\,\|(u_\tau,\Phi(\tau,u_\tau))\|_{C_\gamma}\Big\}d\tau \\
&\leq K\int_s^\infty e^{-b(\rho(\tau)-\rho(s))+\epsilon\rho(\tau)}\Big\{6Kb(\tau)e^{a(\rho(\tau)-\rho(s))+\epsilon\rho(s)}|\Phi-\Psi|'\|\phi\|_{C_\gamma} \\
&\qquad\qquad +4K\|f-g\|_\Omega\, e^{a(\rho(\tau)-\rho(s))+\epsilon\rho(s)}\|\phi\|_{C_\gamma}\Big\}d\tau \\
&\leq 6K^2\|\phi\|_{C_\gamma}|\Phi-\Psi|'\bar{M}+4K^2\|\phi\|_{C_\gamma}\|f-g\|_\Omega\hat{M}.
\end{aligned}
$$

Therefore we have,

$$|\Phi-\Psi|'\leq\frac{4K^2\hat{M}}{1-6K^2\bar{M}}\|f-g\|_\Omega.$$

Since $\frac{4K^2\hat{M}}{1-6K^2\bar{M}}<1$, using (H4), hence we got our result. □

Thus we have shown that a small change in the perturbation gives arise to the small variation in manifold map.

5. Examples

In this example, we consider a delay differential equation admitting the dichotomic behavior with the growth rate of type $e^{\rho(t)}$.

Example 3.

$$x'(t)=\Big(-\omega\rho'(t)-\frac{\cos t}{a+\sin t}+\epsilon\,\rho'(t)\rho(t)\sin\rho(t)\Big)\,x(t)+e^{-3\epsilon(t-1)}x^2(t-1), \tag{32}$$

for $\omega>2\epsilon>0$ and $a>2$.

For the associated linear problem, the solution $x(t)$ is given by $x(t)=V(t,s)x(s)$, where,

$$
\begin{aligned}
&V(t,s)\\
&=e^{-\omega(\rho(t)-\rho(s))-\ln(a+\sin t)+\ln(a+\sin s)+\epsilon(\rho(s)\cos\rho(s)-\rho(t)\cos\rho(t))+\epsilon(\sin\rho(t)-\sin\rho(s))}.
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
&\|V(t,s)\|\\
&=e^{-\omega(\rho(t)-\rho(s))-\ln(a+\sin t)+\ln(a+\sin s)+\epsilon(\rho(s)\cos\rho(s)-\rho(t)\cos\rho(t))+\epsilon(\sin\rho(t)-\sin\rho(s))}\\
&\leq e^{2\epsilon+2(\ln(a+1))}\,e^{(-\omega+\epsilon)(\rho(t)-\rho(s))+2\epsilon\rho(s)}\\
&=Ce^{(-\omega+\epsilon)(\rho(t)-\rho(s))+2\epsilon\rho(s)}.
\end{aligned}
$$

Note that $(-\omega+\epsilon)<0$ and C is a constant, therefore the solution operator $V(t,s)$ admits dichotomic behavior with growth rate $e^{\rho(t)}$. Additionally, the nonlinear perturbation satisfies Equation (9). Hence, Theorem 1 ensures that the differential Equation (32) admits C^1 stable invariant manifold.

Now, we consider a system of delay differential equations which satisfies all the assumptions of Theorem 1 and admits C^1 stable invariant manifold.

Example 4.

$$x'(t) = \Big(\frac{-\omega\rho'(t)}{1+e^{-\rho(t)}} + \epsilon\,\rho'(t)\big(\rho(t)\sin\rho(t)-1\big)\Big)x(t) + \sin(t-1)\,e^{-5\epsilon(t-1)}y^2(t-1),$$
$$y'(t) = \Big(\frac{\omega\rho(t)}{1+e^{-\rho(t)}} - \epsilon\,\rho'(t)\big(\rho(t)\cos\rho(t)-1\big)\Big)y(t) + \cos(t-1)\,e^{-7\epsilon(t-1)}x^2(t-1),$$

for some positive constants ω, ϵ and for $\phi = (\phi_1, \phi_2)$; $\phi_1, \phi_2 \in C_\gamma$ also the function $\rho(t)$ satisfies the conditions in (7). Let the non homogeneous term be,

$$f(t,\phi) = \big(\sin t\, e^{-5\epsilon t}\phi_2^2(t), \cos t\, e^{-7\epsilon t}\phi_1^2(t)\big).$$

The evolution operator for the associated linear system is given by

$$T(t,s) = \begin{bmatrix} U(t,s) & 0 \\ 0 & V(t,s) \end{bmatrix},$$

where

$$U(t,s) = e^{-\omega(\rho(t)-\rho(s))-\omega\ln\frac{(1+e^{-\rho(t)})}{(1+e^{-\rho(s)})}+\epsilon(\rho(s)\cos\rho(s)-\rho(t)\cos\rho(t))+\epsilon(\sin\rho(t)-\sin\rho(s))-\epsilon(\rho(t)-\rho(s))}$$

and

$$V(t,s) = e^{\omega(\rho(t)-\rho(s))+\omega\ln\frac{(1+e^{-\rho(t)})}{(1+e^{-\rho(s)})}+\epsilon(\rho(s)\sin\rho(s)-\rho(t)\sin\rho(t))+\epsilon(\cos\rho(t)-\cos\rho(s))-\epsilon(\rho(t)-\rho(s))}.$$

Consider the projection map $P(t)(x,y) = (x,0)$ for all $t \geq 0$. Then,

$$\|T(t,s)P(t)\| = \|U(t,s)\| \leq e^{2\epsilon+\omega\ln 2}e^{-\omega(\rho(t)-\rho(s))+2\epsilon\rho(s)},$$

and

$$\|T(t,s)^{-1}Q(t)\| = \|V(t,s)^{-1}\| \leq e^{2\epsilon+\omega\ln 2}e^{-\omega(\rho(t)-\rho(s))+2\epsilon\rho(t)}.$$

for $t \geq s \geq 0$. Hence for $a = -\omega < 0$ and $b = \omega > 0$, the associated linear system satisfies ρ-nonuniform exponential dichotomy.

Furthermore, $f(t,\phi)$ admits the conditions in Equation (9), with $f(t,0) = 0$. Hence, Theorem 1 ensures that the differential Equation (4) admits C^1 stable invariant manifold.

6. Conclusions

In this article, we have constructed a C^1 stable invariant manifold for the differential equation with infinite delay (1). We gave the dependence of manifold on the perturbations. We have also included general examples of differential equations which satisfy ρ-nonuniform exponential dichotomy. Thus, we have shown that our result is applicable to a large class of delay differential equations.

Author Contributions: Both the authors worked together to conceptualize and investigate the project. First author has written the article with the guidance of second author. Both authors have read and agreed to the published version of the manuscript.

Funding: This work is supported in part by the Croatian Science Foundation under the project IP-2019-04-1239 (Lokesh Singh) and by the Indian Institute of Technology Kanpur (D. Bahuguna), MHRD, India.

Conflicts of Interest: The authors declare no conflict of interest.

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