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Nonlocal Problem for a Third-Order Equation with Multiple Characteristics with General Boundary Conditions

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Abstract: The article considers third-order equations with multiple characteristics with general boundary value conditions and non-local initial data. A regular solution to the problem with known methods is constructed here. The uniqueness of the solution to the problem is proved by the method of energy integrals. This uses the theory of non-negative quadratic forms. The existence of a solution to the problem is proved by reducing the problem to Fredholm integral equations of the second kind. In this case, the method of Green's function and potential is used.

Keywords: third order equations with multiple characteristic; boundary condition; nonlocal problem; uniqueness theorem; Fredholm integral equation of the second kind; existence theorem

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1. Introduction and Formulation of the Problem

The purpose of this work is to investigate the partial differential equations of the form

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} = 0, \tag{1}$$

in the domain of $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$ with boundary conditions

$$u(x,0) = \mu u(x,T), \quad \mu = const,\ 0 \leq x \leq 1, \tag{2}$$

$$\alpha_1(t)u(0,t) + \alpha_2(t)u_{xx}(0,t) = \varphi_1(t), \quad u_x(0,t) = \varphi_2(t), \quad 0 \leq t \leq T \tag{3}$$

$$\beta_1(t)u(1,t) + \beta_2(t)u_x(1,t) + \beta_3(t)u_{xx}(1,t) = \varphi_3(t), \quad 0 \leq t \leq T. \tag{4}$$

Equation (1) is an analogue of the linear part of the Korteweg-de Vries equation with (a = -1). Therefore, the problem under study may be of interest to those researchers who are engaged in constructing solutions to boundary value problems for non-linear and quasi-linear equations of odd order. For example, when constructing boundary value problems, the Korteweg–de Vries equation and the Zakharov–Kuznetsov equations [1] describe the motion of solitons in an inhomogeneous medium and the motion of solitary waves and ion-acoustic wave processes in a plasma.

In work E. Del Vecchio (see [2]) developed a technique for constructing fundamental solutions for equations of odd order with multiple characteristics. Equation (1) is one of the representatives of these equations.

Italian mathematician L.Cattabriga (see [3,4]) developing the results of E. Del Vecchio constructed the fundamental solution of the Equation (1)

$$U(x-\xi;t-\tau)=(t-\tau)^{-\frac{1}{3}}f\left(\frac{x-\xi}{(t-\tau)^{\frac{1}{3}}}\right),\ x\neq\xi,\ t>\tau, \tag{5}$$

$$V(x-\xi;t-\tau)=(t-\tau)^{-\frac{1}{3}}\varphi\left(\frac{x-\xi}{(t-\tau)^{\frac{1}{3}}}\right),\ x>\xi,\ t>\tau. \tag{6}$$

Here, the functions $f(z)$, $\varphi(z)$ are called the Airy functions. For these functions, the following relation is valid

$$f(z)=\int_0^\infty \cos\left(\lambda^3-\lambda z\right)d\lambda,\ -\infty<z<\infty,$$

$$\varphi(z)=\int_0^\infty \left(\exp\left(-\lambda^3-\lambda z\right)+\sin\left(\lambda^3-\lambda z\right)\right)d\lambda,\ z>0,$$

$$f''(z)+\frac{1}{3}zf(z)=0,\ \ \varphi''(z)+\frac{1}{3}z\varphi(z)=0, \tag{7}$$

$$\int_{-\infty}^{\infty} f(z)=\pi,\ \int_{-\infty}^{0} f(z)=\frac{\pi}{3},\ \int_0^\infty f(z)=\frac{2\pi}{3},\ \int_0^\infty \varphi(z)=0, \tag{8}$$

$$z=(x-\xi)(t-\tau)^{-\frac{1}{3}},$$

$$f^{(n)}(z)\sim c_n^+ z^{\frac{2n-1}{4}}\sin\left(\frac{2}{3}z^{\frac{3}{2}}\right),\ \ z\to\infty, \tag{9}$$

$$\varphi^{(n)}(z)\sim c_n^+ z^{\frac{2n-1}{4}}\sin\left(\frac{2}{3}z^{\frac{3}{2}}\right),\ \ z\to\infty, \tag{10}$$

$$f^{(n)}(z)\sim c_n^-\left|z\right|^{\frac{2n-1}{4}}\exp\left(-\frac{2}{3}|z|^{\frac{3}{2}}\right),\ \ z\to-\infty. \tag{11}$$

Further, having studied the properties of fundamental solutions of Equation (1), he developed the theory of potentials.

In his works, using the asymptotic properties of the Airy function, he proved that for the function $U(x-\xi;t-\tau)$, $V(x-\xi;t-\tau)$, i.e. for fundamental solutions of Equation (1) the following relation is valid.

$$\lim_{(x,t)\to(a-0,t)}\int_\tau^t U_{\xi\xi}(x-a;t-\tau)\alpha(\xi,\tau)d\tau=\frac{\pi}{3}\alpha(t), \tag{12}$$

$$\lim_{(x,t)\to(a+0,t)}\int_\tau^t U_{\xi\xi}(x-a;t-\tau)\alpha(\xi,\tau)d\tau=-\frac{2\pi}{3}\alpha(t), \tag{13}$$

$$\lim_{(x,t)\to(a+0,t)}\int_\tau^t V_{\xi\xi}(x-a;t-\tau)\alpha(\xi,\tau)d\tau=0, \tag{14}$$

The work of L.Cattabriga (see [4]) made it possible to construct a regular solution of boundary value problems with various boundary conditions (see [5–14]). The proof of the correctness of boundary value problems for stationary and non-stationary equations of the third order in bounded domains made it possible to establish energy estimates of the Saint–Venant principle type, which allows one to study the solution of an equation in the vicinity of irregular points of the boundary and at infinity (see [15–18]).

For example, in [17] a priori estimates like the Saint–Venant principle was established, and then using these estimates we were able to prove the uniqueness theorem in the widest class of growing functions due to the fact that in [19] the correctness of the problem was proved in a certain bounded area.

The paper is organized as follows. The system of partial differential Equation (1) with boundary conditions (2)–(4) and initial facts are presented in the Introduction. The main results of Uniqueness and Existence theorems with proofs are found in the next two chapters. The final chapter conclusions presents a summary of motivation and future possibilities of the research, obtained results and a comparison with the authors' previous paper [17].

2. Uniqueness Theorem

The following theorem presents the conditions for the existence of no more than one solution of Equation (1) with the boundary conditions (2)–(4).

Theorem 1. *Let $m < m_0 < 0$, $\exp\{mT\} - \mu^2 \geq 0$, and the following conditions be satisfied:*

$$\alpha_2 \neq 0, \beta_3 \neq 0, \; 2\beta_1\beta_3 - \beta_2^2 \geq 0, \; \frac{\alpha_1}{\alpha_2} \leq 0.$$

Then problem (1), with the conditions (2)–(4) has no more than one solution.

Proof. Let problem (1), (2)–(4) have two solutions: $u_1(x,t), u_2(x,t)$. Then putting $v(x,t) = u_1(x,t) - u_2(x,t)$ we get the following problem for the function $v(x,t)$

$$Lv \equiv \frac{\partial^3 v}{\partial x^3} - \frac{\partial v}{\partial t} = 0, \quad (x,t) \in \Omega, \tag{15}$$

$$v(x,0) = \mu v(x,T), \quad \mu = const, \tag{16}$$

$$\alpha_1(t)v(0,t) + \alpha_2(t)v_{xx}(0,t) = 0, \quad v_x(0,t) = 0, \tag{17}$$

$$\beta_1(t)v(1,t) + \beta_2(t)v_x(1,t) + \beta_3(t)v_{xx}(1,t) = 0. \tag{18}$$

Consider the identity

$$\int_0^T \int_0^1 L(v)v \exp\{mt\}dxdt = 0. \tag{19}$$

Integrating it by parts, we get

$$-\frac{m}{2}\int_0^T\int_0^1 v^2(x,t)\exp\{mt\}dxdt - \frac{1}{2}\int_0^1 \Big(\exp\{mT\} - \mu^2\Big)v^2(x,T)dx + \int_0^T \frac{\alpha_1(t)}{\alpha_2(t)}v^2(0,t)dt -$$

$$-\int_0^T \left(\frac{1}{2}v_x^2(1,t) + \frac{\beta_2}{\beta_3}v(1,t)v_x(1,t) + \frac{\beta_1}{\beta_3}v^2(1,t)\right)exp\{mt\}dt = 0. \tag{20}$$

The conditions of the theorem show that the quadratic form

$$Q(t) \equiv \frac{1}{2}v_x^2(1,t) + \frac{\beta_2}{\beta_3}v(1,t)v_x(1,t) + \frac{\beta_1}{\beta_3}v^2(1,t)$$

will be positive definite. Then the quadratic form $Q(t)$ can be written in the following form

$$\frac{1}{2}v_x^2(1,t) + \frac{\beta_2}{\beta_3}v(1,t)v_x(1,t) + \frac{\beta_1}{\beta_3}v^2(1,t) = \lambda_1 v_x^2 + \lambda_2 v^2, \;\; \lambda_1 > 0, \;\; \lambda_2 > 0.$$

Here, λ_1, λ_2—are the roots of the characteristic polynomial of a matrix of quadratic form $Q(t)$.

Then, by virtue of the conditions of the theorem, we have $v(x,t) = 0$ in $Q(t)$, and by virtue of the continuity of the function $v(x,t)$ in $\overline{\Omega}$, we obtain that $v(x,t) = 0$ in $\overline{\Omega}$. $\square$

3. Existence Theorem

The existence of solution of the system (1) with the boundary conditions (2)–(4) is studied in the following theorem.

Theorem 2. *Let $\varphi_1(t), \varphi_3(t) \in C([0,T])$, $\varphi_2(t) \in C^1([0,T])$, and the condition of Theorem 1 be satisfied. Then there exists a solution to problem (1), (2)–(4).*

Proof. The solution to the problem (1), (2)–(4) will be constructed by reducing it to the Fredholm integral equation of the second kind.

To this end, consider an auxiliary problem: Find a function that is a regular solution to the equation

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} = 0 \tag{21}$$

in the region $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$, and satisfies the following boundary conditions

$$u(x,0) = u_0(x), \tag{22}$$

$$u_{xx}(0,t) = \rho_1(t),\ \ u_x(0,t) = \varphi_2(t),\ \ u_{xx}(1,t) = \rho_2(t). \tag{23}$$

Now we construct a solution to the problem (22), (23). In this case, we use Green's function method. For this purpose, consider the following identity

$$\varphi L[\psi] - \psi L^*[\varphi] = \frac{\partial}{\partial \xi}\left(\varphi\psi_{\xi\xi} - \varphi_\xi\psi_\xi + \varphi_{\xi\xi}\psi\right) - \frac{\partial}{\partial \tau}(\varphi\psi) = 0. \tag{24}$$

Here, operator $L^*(v)$ is conjugate to the operator $L(v)$.

Now we integrate identity (24) over the domain $\Omega_\varepsilon = \{(x,t) : 0 < x < 1,$ $0 < t \leq T - \varepsilon\}$. When integrating, we first put $\varphi = U(x-\xi; t-\tau)$, $\psi = u(\xi,\tau)$, then $\varphi = W(x-\xi; t-\tau)$, $\psi = u(\xi,\tau)$. Further, passing to the limit $\varepsilon \to 0$, and denoting $G(x-\xi; t-\tau) = U(x-\xi; t-\tau) - W(x-\xi; t-\tau)$ we have

$$\begin{aligned}
\pi u(x,t) = &-\int_0^t G(x-0; t-\tau)\rho_1(\tau)d\tau \\
&+\int_0^t G_\xi(x-0; t-\tau)\varphi_2(\tau)d\tau + \int_0^t G(x-1; t-\tau)\rho_2(\tau)d\tau \\
&+\int_0^1 G(x-\xi; t-0)u_0(\xi)d\xi.
\end{aligned} \tag{25}$$

Here, the function $W(x-\xi; t-\tau)$ is the regular part of the Green's function, and is a solution to the following problem

$$L^*(W) \equiv -\frac{\partial^3 W}{\partial x^3} + \frac{\partial W}{\partial t} = 0,$$

$$U_\xi\Big|_{\xi=1} = W_\xi\Big|_{\xi=1},\ U_{\xi\xi}\Big|_{\xi=1} = W_{\xi\xi}\Big|_{\xi=1},\ U_{\xi\xi}\Big|_{\xi=0} = W_{\xi\xi}\Big|_{\xi=0},$$

$$W\Big|_{\tau=t} = 0.$$

So, by construction (25) is a solution auxiliary problem (21), (22), (23). Now we turn to the construction of problem (1), (2)–(4). For this purpose, we will satisfy the boundary condition (2)–(4), and introduce the notation:

$$u(x,T)=\gamma_0,\ \alpha_1(t)u(0,t)=\omega_1(t),\ \beta_1(t)u(1,t)+\beta_2(t)u_x(1,t)=\omega_2(t).$$

Then (25) can be written in the following form:

$$\pi u(x,T)=\pi\gamma_0(x)=\frac{1}{\mu}\int\limits_0^1 K_0(x,\xi)\gamma_0(\xi)d\xi+\Phi_0(x;\omega_1(t),\omega_2(t)), \tag{26}$$

$$\pi\alpha_1(t)u(0,t)=\pi\omega_1(t)=\int\limits_0^t K_1(t,\tau)\omega_1(\tau)d\tau+\Phi_1(t;\omega_2(t),\gamma_0(x)), \tag{27}$$

$$\beta_1(t)u(1,t)+\beta_2(t)u_x(1,t)=\omega_2(t)=\int\limits_0^t K_2(t,\tau)\omega_2(\tau)d\tau+\Phi_2(t;\omega_1(t),\gamma_0(x)). \tag{28}$$

Since the estimates of type $|K_i(t,\tau)|\le\frac{C}{|t-\tau|^{\frac{3}{4}}}$ are valid for the functions $K_1(t,\tau)$, $K_2(t,\tau)$ and the estimate function $K_0(x,\xi)$ is of type $|K_0(x,\xi)|\le\frac{C}{|x-\xi|^{\frac{1}{4}}}$, the theory of integral equations makes it possible to find the functions $u(x,T)=\gamma_0$, $\alpha_1(t)u(0,t)=\omega_1(t)$, $\beta_1(t)u(1,t)+\beta_2(t)u_x(1,t)=\omega_2(t)$, in the required class. Then expression (25) gives a solution to problem (1), (2)–(4). □

4. Conclusions

In the previous paper of authors [17], the widest class of uniqueness of solutions was identified depending on the characteristic of the domain, using the Saint–Venant principle. Now, as a continuation of this, it is necessary to construct a solution to this problem in an unbounded domain, which the authors are currently working on. To construct a solution in an unbounded domain, it is necessary that first the solution (either classical or generalized does not matter) must be constructed in some bounded domain, to then cover the unbounded domain with this solution with some law, which, probably, the authors will find in our further research. In the study of the third-order pseudo-elliptic equation, the authors rely on the work of [19], in which the correctness of the problem for the third-order equation of the pseudo-elliptic type in bounded domains was proved.

If one want to conduct a similar study, which was conducted by us in [17], for a non-stationary third-order equation with a non-local condition, then there is nothing to refer to by studying the correctness of the problem in bounded domains. Because the correctness of the problem under study in a bounded domain has not been investigated. Therefore, the proposed work from the point of view of constructing a solution to a non-stationary third-order equation in an unbounded domain in classes of functions growing at infinity is of great interest. And from the point of view of the theory of boundary value problems for the equations of a nonstationary equation, the problem under study is new, although well-known methods are used to solve this problem. As the authors noted above, it was necessary to solve this problem for our further research.

Issues relating to the equations of the third order is very broad. A nonlocal boundary-value problem for a third-order parabolic-hyperbolic equation with the degeneracy of type and order in the domain of hyperbolicity, containing second-order derivatives in the boundary conditions was considered in [20]. A different class of equations (with the Tricomi method used in the proof of the uniqueness theorem for a solution) was studied in [20] than in this presented paper. The application to physics was presented in [21].

The problem of reduction of third-order equations to equivalent problem for a system of second-order hyperbolic equations with parameters and integral relations was presented in [22]. Boundary value problems of higher-order are studied for example in papers [23,24].

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