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Solutions for Several Quadratic Trinomial Difference Equations and Partial Differential Difference Equations in $\mathbb{C}^2$

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Abstract: This article is to investigate the existence of entire solutions of several quadratic trinomial difference equations $f(z+c)^2+2\alpha f(z)f(z+c)+f(z)^2=e^{g(z)}$, and the partial differential difference equations $f(z+c)^2+2\alpha f(z+c)\frac{\partial f(z)}{\partial z_1}+\left(\frac{\partial f(z)}{\partial z_1}\right)^2=e^{g(z)}$, $f(z+c)^2+2\alpha f(z+c)\left(\frac{\partial f(z)}{\partial z_1}+\frac{\partial f(z)}{\partial z_2}\right)+\left(\frac{\partial f(z)}{\partial z_1}+\frac{\partial f(z)}{\partial z_2}\right)^2=e^{g(z)}$. We establish some theorems about the forms of the finite order transcendental entire solutions of these functional equations. We also list a series of examples to explain the existence of the finite order transcendental entire solutions of such equations. Meantime, some examples show that there exists a very significant difference with the previous literature on the growth order of the finite order transcendental entire solutions. Our results show that some functional equations can admit the transcendental entire solutions with any positive integer order. These results make a few improvements of the previous theorems given by Xu and Cao, Liu and Yang.

Keywords: Nevanlinna theory; entire solution; partial differential difference equation

MSC: 30D 35; 35M 30; 39A 45

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1. Introduction

Let $\alpha^2(\neq 0,1)$, $z+c=(z_1+c_1,z_2+c_2)$ for $z=(z_1,z_2)$ and $c=(c_1,c_2)$. This paper is devoted to investigating the transcendental entire solutions with finite order of the quadratic trinomial difference equation

$$f(z+c)^2+2\alpha f(z)f(z+c)+f(z)^2=e^{g(z)}, \tag{1}$$

and the quadratic trinomial partial differential difference equations

$$f(z+c)^2+2\alpha f(z+c)\frac{\partial f(z)}{\partial z_1}+\left(\frac{\partial f(z)}{\partial z_1}\right)^2=e^{g(z)}, \tag{2}$$

$$f(z+c)^2+2\alpha f(z+c)\left(\frac{\partial f(z)}{\partial z_1}+\frac{\partial f(z)}{\partial z_2}\right)+\left(\frac{\partial f(z)}{\partial z_1}+\frac{\partial f(z)}{\partial z_2}\right)^2=e^{g(z)}, \tag{3}$$

where c_1,c_2 are constants and $g(z)$ is a polynomial in $\mathbb{C}^2$. When $\alpha=\pm1$, Equations (1)–(3) can be turned to the following equations

$$f(z)\pm f(z+c)=\pm e^{\frac{1}{2}g(z)}, \tag{4}$$

$$f(z+c)\pm\frac{\partial f}{\partial z_1}=\pm e^{\frac{1}{2}g(z)}, \tag{5}$$

$$f(z+c) \pm (\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2}) = \pm e^{\frac{1}{2}g(z)}, \tag{6}$$

respectively. In fact, it is easy to get the entire solutions of (4)–(6); for example, $f(z) = e^{z_1+z_2}$ is a finite-order transcendental entire solution of Equation (4), if $(1 \pm e^{c_1+c_2}) = \pm e^b$ and $g(z) = e^{2(z_1+z_2)+2}$, and $f(z) = e^{a_1z_1+a_2z_2}$ is a finite order entire solution of Equations (5) and (6), if $e^{a_1c_1+a_2c_2} \pm a_1 = \pm e^b$ and $e^{a_1c_1+a_2c_2} \pm (a_1 + a_2) = \pm e^b$, respectively, where $g(z) = e^{2(a_1z_1+a_2z_2)+2b}$, $a_1(\neq 0), a_2, b$ are constants. When $\alpha = 0$ and $g(z) = 0$, Equations (1) and (2) become the Fermat type difference equations

$$f(z+c)^2 + f(z)^2 = 1 \tag{7}$$

and

$$f(z+c)^2 + \left(\frac{\partial f}{\partial z_1}\right)^2 = 1, \tag{8}$$

which were discussed by Xu and Cao [1]. They pointed out that the transcendental entire solution with finite order of Equation (7) must satisfy $f(z) = \sin(A_1z_1 + A_2z_2 + Constant)$, where $A_1^2 = 1$ and $A_1e^{i(A_1c_1+A_2c_2)} = 1$, if $c_2 \neq 0$, and any nonconstant entire solution with finite order of Equation (7) has the form of $f(z) = \cos(A_1z_1 + A_2z_2 + Constant)$, where $A_1c_1 + A_2c_2 = -\frac{\pi}{2} - 2k\pi$, $k \in \mathbb{Z}$. As is known, the study of the Fermat type functional equation

$$f^2 + g^2 = 1 \tag{9}$$

can be tracked back to about sixty years ago or even earlier (see [2–4]). In 1966, Gross [2] discussed the existence of solutions of Equation (9) and showed that the entire solutions are $f = \cos a(z), g = \sin a(z)$, where $a(z)$ is an entire function. In recent years, with the development of Nevanlinna theory and difference Nevanlinns theory of meromorphic function with one and several variables ([5–11]), there were many references focusing on the solutions of Fermat type functional equations (readers can refer to [12–19]).

In 1999 and 2004, E. G. Saleeby [20,21] studied the forms of the entire and meromorphic solutions of some partial differential equations, extended some of the above conclusions to the case of several complex variables, and obtained the following results.

Theorem 1 (see ([20], Theorem 1))**.** . *If f is an entire solution of*

$$\left(\frac{\partial f(z_1,z_2)}{\partial z_1}\right)^2 + \left(\frac{\partial f(z_1,z_2)}{\partial z_2}\right)^2 = 1 \tag{10}$$

in $\mathbb{C}^2$, then $f(z_1, z_2) = \eta_1z_1 + \eta_2z_2 + \eta$ where $\eta_1, \eta_2, \eta \in \mathbb{C}$ and $\eta_1^2 + \eta_2^2 = 1$.

Theorem 2 (see ([21], Theorem 1))**.** *The nonconstant entire and meromorphic solutions of equation*

$$f(z_1,z_2)^2 + \left(\frac{\partial f(z_1,z_2)}{\partial z_1}\right)^2 + \left(\frac{\partial f(z_1,z_2)}{\partial z_2}\right)^2 = 1$$

are of the form $f(z_1, z_2) = \sin(\eta_1z_1 + \eta_2z_2 + \eta)$, where $\eta_1, \eta_2, \eta \in \mathbb{C}$ and $\eta_1^2 + \eta_2^2 = 1$.

In 2005, Li [22] discussed the partial differential equation of Fermat-type

$$\left(\frac{\partial u}{\partial z_1}\right)^2 + \left(\frac{\partial u}{\partial z_2}\right)^2 = e^g, \tag{11}$$

where g is a polynomial or an entire function in $\mathbb{C}^2$ and obtained some results on the forms of entire solution of Equation (11) as follows:

Theorem 3 ([22], Theorem 2.1)**.** *Let g be a polynomial in $\mathbb{C}^2$. Then, u is an entire solution of the partial differential Equation (11), if and only if:*

(i) $u = f(c_1z_1 + c_2z_2)$; or
(ii) $u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$,

where f is an entire function in $\mathbb{C}$ satisfying that $f'(c_1z_1 + c_2z_2) = \pm e^{\frac{1}{2}g(z)}$, c_1 and c_2 are two constants satisfying that $c_1^2 + c_2^2 = 1$, and ϕ_1 and ϕ_2 are entire functions in $\mathbb{C}$ satisfying that $\phi_1'(z_1 + iz_2)\phi_2'(z_1 - iz_2) = \frac{1}{4}e^{g(z)}$.

Later, E. G. Saleeby [23] further investigated the entire and meromorphic solutions for the quadratic trinomial functional equations

$$f^2 + 2\alpha fg + g^2 = 1, \alpha^2 \neq 1, \alpha \in \mathbb{C}, \tag{12}$$

and obtained

Theorem 4 (see ([23], Theorem 2.1))**.** *The entire and meromorphic solutions of (12) have the form given in, respectively,*

$$f = \frac{1}{\sqrt{2}}\left(\frac{\cos h}{\sqrt{1+\alpha}} + \frac{\sin h}{\sqrt{1-\alpha}}\right), g = \frac{1}{\sqrt{2}}\left(\frac{\cos h}{\sqrt{1+\alpha}} - \frac{\sin h}{\sqrt{1-\alpha}}\right)$$

and

$$f = \frac{\alpha_1 - \alpha_2\beta^2}{(\alpha_1 - \alpha_2)\beta}, g = \frac{1 - \beta^2}{(\alpha_1 - \alpha_2)\beta},$$

where h is entire and β is meromorphic on $\mathbb{C}^n$.

Theorems 1–4 suggest the following question as an open problem.

Question 1. *What would happen to the existence and form of solution of Equation (12) when g is replaced of some special forms of f, and the right side of those equations 1 is replaced by a function e^{φ} in Theorem D, where φ is a polynomial?*

2. Results and Examples

Motivated by the above question, this article is concerned with the entire solutions for the difference Equation (1) and the partial differential difference Equations (2) and (3). The main tools are used in this paper are the Nevanlinna theory and difference Nevanlinna theory. Our results are obtained to generalize the previous theorems given by Xu and Cao, Liu, and Yang [1,24]. Here and below, let $\alpha^2 \neq 0, 1$, and

$$A_1 = \frac{1}{2\sqrt{1+\alpha}} - \frac{i}{2\sqrt{1-\alpha}}, A_2 = \frac{1}{2\sqrt{1+\alpha}} + \frac{i}{2\sqrt{1-\alpha}}. \tag{13}$$

Thus, we have $A_1A_2 = \frac{4}{1-\alpha^2}$.

The first main theorem is about the existence and the forms of the solutions for the quadratic trinomial difference Equation (1).

Theorem 5. *Let $g(z)$ be a polynomial in $\mathbb{C}^2$ and $\alpha^2 \neq 0, 1, c \in \mathbb{C}^2$. If the difference Equation (1) admits a transcendental entire solution $f(z)$ of finite order, then $g(z_1, z_2)$ must be of the form $g(z_1, z_2) = L(z_1, z_2) + H(c_2z_1 - c_1z_2)$, where $L(z_1, z_2)$ is a linear form of $L(z_1, z_2) = a_1z_1 + a_2z_2$, $H(s)$ is a polynomial in s, and $a_1, a_2 \in \mathbb{C}$. Further, $f(z_1, z_2)$ must satisfy one of the following cases:*

(i)

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}(A_1\xi + A_2\xi^{-1})e^{\frac{1}{2}[L(z_1,z_2)+H(c_2z_1-c_1z_2)]},$$

where $\xi(\neq 0) \in \mathbb{C}$ *and* $a_1, a_2, c_1, c_2, \xi, A_1, A_2$ *satisfying*

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)} = \frac{A_2\xi + A_1\xi^{-1}}{A_1\xi + A_2\xi^{-1}};$$

or

(ii)

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}\Big(A_1 e^{L_1(z_1,z_2)+H_1(c_2z_1-c_1z_2)} + A_2 e^{L_2(z_1,z_2)+H_2(c_2z_1-c_1z_2)}\Big),$$

where $L_1(z_1, z_2) = a_{11}z_1 + a_{12}z_2$, $L_2(z_1, z_2) = a_{21}z_1 + a_{22}z_2$, $H_j(s)$ *are polynomials in* s, $a_{ij} \in \mathbb{C}, (i = 1, 2; j = 1, 2)$ *satisfy* $L_1(z_1, z_2) \neq L_2(z_1, z_2)$,

$$g(z_1, z_2) = L_1(z_1, z_2) + L_2(z_1, z_2) + H_1(c_2z_1 - c_1z_2) + H_2(c_2z_1 - c_1z_2) = L(z_1, z_2) + H(c_2z_1 - c_1z_2),$$

and

$$e^{a_{11}c_1+a_{12}c_2} = \frac{A_2}{A_1}, e^{a_{21}c_1+a_{22}c_2} = \frac{A_1}{A_2}, e^{a_1c_1+a_2c_2} = 1.$$

The following examples show that the forms of solutions are precise to some extent.

Example 1. *Let* $\alpha = \frac{1}{2}$ *and* $\xi = 1$. *Then, it follows that* $A_1 = \frac{1}{\sqrt{3}}(\frac{1}{2} - \frac{\sqrt{3}}{2}i)$, $A_2 = \frac{1}{\sqrt{3}}(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$. *Let*

$$f(z) = \frac{1}{\sqrt{2}}e^{z_1+z_2+1}.$$

Thus, $\rho(f) = 1$ *and* $f(z)$ *is a transcendental entire solution of* (1) *with* $g(z) = 2(z_1 + z_2) + 2$ *and* $(c_1, c_2) = (\pi i, 3\pi i)$.

Remark 1. *Here and below,* $\rho(f)$ *is the order of the function* f*, which is defined by*

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ *is the Nevanlinna characteristic function (see [9]).*

Example 2. *Let* $\alpha = \frac{1}{2}$, $a_{11} = a_{12} = \frac{1}{3}$, $a_{21} = a_{22} = \frac{2}{3}$ *and* $b_1 = b_2 = 0$. *Then, it follows that*

$$f(z) = \frac{1}{\sqrt{3}}\Big(e^{\frac{1}{3}(z_1+z_2)-\pi^2(3z_1-z_2)^2-\frac{1}{3}\pi i} + e^{\frac{2}{3}(z_1+z_2)+\pi^4(3z_1-z_2)^4+\frac{1}{3}\pi i}\Big).$$

Thus, $\rho(f) = 4$ *and* $f(z)$ *is a transcendental entire solution of* (1) *with* $(c_1, c_2) = (\frac{\pi i}{2}, \frac{3\pi i}{2})$ *and*

$$g(z) = (z_1 + z_2) - \pi^2(3z_1 - z_2)^2 + \pi^4(3z_1 - z_2)^4.$$

When $f(z + c)$ is replaced by $\frac{\partial f}{\partial z_1}$ or $\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2}$ in (1), we obtain the second theorem as follows.

Theorem 6. *Let* $\alpha^2 \neq 0, 1, \alpha \in \mathbb{C}$ *and* $g(z)$ *be a polynomial in* $\mathbb{C}^2$. *If the partial differential equation*

$$f(z)^2 + 2\alpha f(z)\frac{\partial f(z)}{\partial z_1} + \left(\frac{\partial f(z)}{\partial z_1}\right)^2 = e^{g(z)} \tag{14}$$

admits a transcendental entire solution $f(z)$ *of finite order, then* $g(z)$ *must be of the form* $g(z) = a_1z_1 + \varphi(z_2)$, *where* $\varphi(z_2)$ *is a polynomial in* z_2 *and* $a_1 \in \mathbb{C}$.

Theorem 7. *Let $\alpha^2 \neq 0, 1, \alpha \in \mathbb{C}$ and $g(z)$ be a polynomial in $\mathbb{C}^2$. If the partial differential equation*

$$f(z)^2 + 2\alpha f(z)\left(\frac{\partial f(z)}{\partial z_1} + \frac{\partial f(z)}{\partial z_2}\right) + \left(\frac{\partial f(z)}{\partial z_1} + \frac{\partial f(z)}{\partial z_2}\right)^2 = e^{g(z)} \tag{15}$$

admits a transcendental entire solution $f(z)$ of finite order, then $g(z)$ must be of the form $g(z) = a_1z_1 + \varphi(z_2 - z_1)$, where $\varphi(z_2 - z_1)$ is a polynomial in $z_2 - z_1$ and $a_1 \in \mathbb{C}$.

The following examples show that the forms of solutions are precisely to some extent.

Example 3. *Let $g(z) = -4z_1 + z_2 + z_2^2$. Then, it is easy to get that the function*

$$f(z) = \frac{1}{2\sqrt{6}}\left[(1-\sqrt{3})e^{(-2+\sqrt{3})z_1+z_2} + (1+\sqrt{3})e^{(-2-\sqrt{3})z_1+z_2^2}\right]$$

is a transcendental entire solution of Equation (14) with $\alpha = 2$.

Example 4. *Let $g(z) = -6z_1 + (z_2 - z_1)^2 + (z_2 - z_1)^4$. Then, it is easy to get that the function*

$$f(z) = \frac{1}{4\sqrt{2}}\left[(1-\sqrt{2})e^{(-3-2\sqrt{2})z_1+(z_2-z_1)^2} + (1+\sqrt{2})e^{(2\sqrt{2}-3)z_1+(z_2-z_1)^4}\right]$$

is a transcendental entire solution of Equation (15) with $\alpha = 3$.

From Theorem 7, it is easy to get the following corollary.

Corollary 1. *Let $\alpha^2 \neq 0, 1, \alpha \in \mathbb{C}$ and $g(z)$ be a polynomial in $\mathbb{C}^2$ which the degree of z_1 is more than 2. Then, the following partial differential-difference equation*

$$f(z)^2 + 2\alpha f(z)\frac{\partial f(z)}{\partial z_1} + \left(\frac{\partial f(z)}{\partial z_1}\right)^2 = e^{g(z)} \tag{16}$$

admits no transcendental entire solution with finite order.

For the partial differential difference counterpart of Theorem 7, we have

Theorem 8. *Let $\alpha^2 \neq 0, 1, c_2 \neq 0$ and $g(z)$ be a nonconstant polynomial in $\mathbb{C}^2$, and not the form of $\phi(z_2)$. If the differential-difference Equation (2) admits a transcendental entire solution $f(z)$ of finite order, then $g(z)$ must be of the form $g(z) = a_1z_1 + a_2z_2 + b$, where $a_1(\neq 0), a_2, b \in \mathbb{C}$. Further, $f(z)$ must satisfy one of the following cases:*

(i)

$$f(z) = \frac{\sqrt{2}}{a_1}(A_1\xi^{-1} + A_2\xi)e^{\frac{1}{2}(a_1z_1+a_2z_2+b)},$$

where $\xi(\neq 0) \in \mathbb{C}$ and $a_1, a_2, b, c_1, c_2, A_1, A_2, \xi$ satisfying

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)} = \frac{a_1(A_1\xi + A_2\xi^{-1})}{2(A_2\xi + A_1\xi^{-1})};$$

or

(ii)

$$f(z) = \frac{1}{\sqrt{2}}\left(\frac{A_2}{a_{11}}e^{a_{11}z_1+a_{12}z_2+b_1} + \frac{A_1}{a_{21}}e^{a_{21}z_1+a_{22}z_2+b_2}\right),$$

where $a_j(\neq 0), b_j \in \mathbb{C}, (j = 1, 2)$ satisfy $a_{11}z_1 + a_{12}z_2 \neq a_{21}z_1 + a_{22}z_2$,

$$g(z) = (a_{11} + a_{21})z_1 + (a_{12} + a_{22})z_2 + b_1 + b_2,$$

and

$$e^{a_{11}c_1+a_{12}c_2}=\frac{A_2}{A_1}a_{11},e^{a_{21}c_1+a_{22}c_2}=\frac{A_1}{A_2}a_{21},e^{a_1c_1+a_2c_2}=a_{11}a_{21}.$$

The following examples explain the existence of transcendental entire solutions with finite order of (2).

Example 5. *Let $\alpha=-\frac{1}{2}$ and $\xi=-1$. Then, it follows that $A_1=\frac{\sqrt{2}}{\sqrt{3}}e^{-\frac{\pi}{6}i}$ and $A_2=\frac{\sqrt{2}}{\sqrt{3}}e^{\frac{\pi}{6}i}$. Let*

$$f(z)=\frac{\sqrt{2}}{2}e^{\frac{1}{2}(2z_1+3z_2+b)}.$$

Then, $f(z)$ is a transcendental entire solution of Equation (2) with $g(z)=2z_1+3z_2+b$, $(c_1,c_2)=(-\frac{\pi i}{2},\frac{3\pi i}{2})$ and $b\in\mathbb{C}$.

Example 6. *Let $\alpha=-\frac{1}{2}$, $a_{11}=a_{12}=a_{21}=1$, $a_{22}=-1$ and $b_1=b_2=0$. Let*

$$f(z)=\frac{1}{\sqrt{3}}\left(e^{z_1+z_2+\frac{\pi}{6}i}+e^{z_1-z_2-\frac{\pi}{6}i}\right),$$

then $\rho(f)=1$ and $f(z)$ is a transcendental entire solution of Equation (2) with $g(z)=2z_1$ and $(c_1,c_2)=(0,\frac{\pi}{3}i)$.

The following example shows that the condition $c_2\neq 0$ in Theorem 5 cannot be removed.

Example 7. *Let $c_2=0$, $\alpha=3$, a_{11} be a root of equation $e^x=x(2\sqrt{2}-3)$, and a_{21} be a root of equation $e^x=-x(3+2\sqrt{2})$. Thus, it follows $a_{11}\neq a_{21}$. Let*

$$f(z)=\frac{1}{4\sqrt{2}}\left(\frac{1+\sqrt{2}}{a_{11}}e^{a_{11}z_1+a_{12}z_2+z_2^2}+\frac{1-\sqrt{2}}{a_{21}}e^{a_{21}z_1+a_{22}z_2+z_2^3}\right),$$

where $a_{12},a_{22}\in\mathbb{C}$, then $\rho(f)=3$ and $f(z)$ is a transcendental entire solution of Equation (3) with $g(z)=(a_{11}+a_{21})z_1+(a_{12}+a_{22})z_2+z_2^2+z_2^3$ and $(c_1,c_2)=(1,0)$.

Remark 2. *From Example 7, we can see that $\rho(f)=3>1$ and $g(z)$ is not a linear form of z_1,z_2 when $c_2=0$ for Equation (2). These are significant differences with the condition $c_2\neq 0$ in Theorem 8. Thus, this shows that our results are precise to some extent.*

Theorem 9. *Let $\alpha^2\neq 0,1,c_1\neq c_2$ and $g(z)$ be a nonconstant polynomial in $\mathbb{C}^2$, and not the form of $\phi(z_2-z_1)$. If $f(z)$ is a finite-order transcendental entire solution of the differential-difference Equation (3), then $f(z),g(z)$ must satisfy one of the following cases:*

(i) $g(z)=a_1z_1+a_2z_2+b$ and

$$f(z)=\frac{\sqrt{2}}{a_1+a_2}(A_2\xi+A_1\xi^{-1})e^{\frac{1}{2}(a_1z_1+a_2z_2+b)},$$

where $\xi(\neq 0)\in\mathbb{C}$ and a_1,a_2,b,c,A_1,A_2,ξ satisfying

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)}=\frac{1}{2}\frac{A_1\xi+A_2\xi^{-1}}{A_2\xi+A_1\xi^{-1}}(a_1+a_2);$$

or

(ii) $g(z)=L_1(z)+L_2(z)+b_1+b_2$ and

$$f(z)=\frac{1}{\sqrt{2}}\left(\frac{A_2}{a_{11}+a_{12}}e^{L_1(z)+b_1}+\frac{A_1}{a_{21}+a_{22}}e^{L_2(z)+b_2}\right),$$

where $L_1(z) = a_{11}z_1 + a_{12}z_2$, $L_2(z) = a_{21}z_1 + a_{22}z_2$, *and* $a_{ij}, b_j \in \mathbb{C}, (i,j = 1,2)$ *satisfy* $a_{11} + a_{12} \neq 0$, $a_{21} + a_{22} \neq 0$, $L_1(z) \neq L_2(z)$, *and*

$$e^{a_{11}c_1+a_{12}c_2} = (a_{11} + a_{12})\frac{A_1}{A_2}, e^{a_{21}c_1+a_{22}c_2} = (a_{21} + a_{22})\frac{A_2}{A_1}.$$

The following examples explain the existence of transcendental entire solutions with finite order of (3).

Example 8. *Let* $\alpha = 3$ *and* $\xi = 1$. *Then, it follows that* $A_1 = \frac{1}{4} - \frac{1}{2\sqrt{2}}$ *and* $A_2 = \frac{1}{4} + \frac{1}{2\sqrt{2}}$. *Let*

$$f(z) = -\frac{\sqrt{2}}{4}e^{z_1-2z_2+b}.$$

Then, $\rho(f) = 1$ *and* $f(z)$ *is a transcendental entire solution of Equation* (3) *with* $g(z) = 2z_1 - 4z_2 + b$, $(c_1, c_2) = (3\pi i, \pi i)$ *and* $b \in \mathbb{C}$.

Example 9. *Let* $\alpha = -\frac{1}{2}$, $a_{11} = 2, a_{12} = -1, a_{21} = 2$, $a_{22} = -3$, *and* $b_1 = b_2 = 0$. *Let*

$$f(z) = \frac{1}{\sqrt{3}}\left(e^{2z_1-z_2+\frac{\pi}{6}i} - e^{2z_1-3z_2-\frac{\pi}{6}i}\right).$$

Then, $\rho(f) = 1$ *and* $f(z)$ *is a transcendental entire solution of Equation* (3) *with* $g(z) = 4z_1 - 4z_2$ *and* $(c_1, c_2) = (\frac{\pi}{12}i, -\frac{\pi}{6}i)$.

The following example shows that the condition $c_1 \neq c_2$ in Theorem 9 cannot be removed.

Example 10. *Let* $c_1 = c_2 = 1$, $\alpha = 2$, a_{11}, a_{12} *satisfy equation* $e^{a_{11}+a_{12}} = (a_{11} + a_{12})(\sqrt{3} - 2)$, *and* a_{21}, a_{22} *satisfy equation* $e^{a_{21}+a_{22}} = -(a_{21} + a_{22})(\sqrt{3} + 2)$. *Thus, it follows that* $a_{11} = a_{21}$ *and* $a_{12} = a_{22}$ *cannot hold at the same time. Let*

$$f(z) = \frac{1}{2\sqrt{6}}\left(\frac{1+\sqrt{3}}{a_{11}+a_{12}}e^{a_{11}z_1+a_{12}z_2+(z_1-z_2)^s} + \frac{1-\sqrt{3}}{a_{21}+a_{22}}e^{a_{21}z_1+a_{22}z_2+(z_1-z_2)^t}\right),$$

where $s, t \in \mathbb{Z}_+$. *Then,* $\rho(f) = \max\{s, t\}$ *and* $f(z)$ *is a transcendental entire solution of Equation* (3) *with* $g(z) = (a_{11} + a_{21})z_1 + (a_{12} + a_{22})z_2 + (z_1 - z_2)^s + (z_1 - z_2)^t$ *and* $(c_1, c_2) = (1, 1)$.

Remark 3. *By comparing the conclusions of Theorem 8 and Example 10, we can see that there are significant differences in the form of solution* $f(z)$ *and* $g(z)$ *when* $c_1 \neq c_2$ *in* (3).

3. Conclusions and Discussion

From Theorems 5 and 8, one can see that our theorems are very good supplements of the previous results given by Xu and Cao [1] because Equations (1) and (2) are more general than Equations (7) and (8). Moreover, Examples 2, 7, and 10 show that Equations (1)–(3) can admit the transcendental entire solutions with any positive integer order. However, Xu and Cao [1] showed that the order of the transcendental entire solutions of Equations (7) and (8) must be one. In fact, this is a very significant difference. Besides, by comparing with the abstract form of solution in Theorem D, we can get a more specific expression of the solution of Equation (12) when g is replaced by the special forms of f such as $g = f(z + c)$, $g = \frac{\partial f}{\partial z_1}$, etc. Finally, one can find that we only focus on the finite-order transcendental entire solutions of Equations (1)–(3) in this article; thus, the following questions can be raised naturally:

Question 2. *Do the infinite-order transcendental entire solutions of Equations* (1)–(3) *exist?*

Question 3. *How should the meromorphic solutions of Equations (1)–(3) be characterized?*

4. Some Lemmas

The following lemmas play the key roles in proving our results.

Lemma 1 ([25], Lemma 3.2)**.** *Let f be a non-constant meromorphic function in $\mathbb{C}^n$. Then, for any $I \in (\mathbf{Z}^+)^n$, $T(r, \partial^I f) = O(T(r, f))$ for all r except possibly a set of finite Lebesgue measure, and where $I = (i_1, \ldots, i_n) \in (\mathbf{Z}^+)^n$ denotes a multiple index with $|I| = i_1 + \cdots + i_n, \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, and $\partial^I f = \frac{\partial^I f}{\partial^{i_n}\zeta_n \cdots \partial^{i_1}\zeta_1}$*

Lemma 2 ([9,10])**.** *For an entire function F on $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then, there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Remark 4. *Here, denote $\rho(n_F)$ to be the order of the counting function of zeros of F.*

Lemma 3 ([4])**.** *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases:*

(a) the internal function h is a polynomial and the external function g is of finite order; or

(b) the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.

Lemma 4 ([26], Lemma 3.1)**.** *Let $f_j(\not\equiv 0), j = 1, 2, 3$, be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant, $f_1 + f_2 + f_3 = 1$, and*

$$\sum_{j=1}^{3} \left\{ N_2(r, \frac{1}{f_j}) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then, either $f_2 = 1$ or $f_3 = 1$.

Remark 5. *Here, $N_2(r, \frac{1}{f})$ is the counting function of the zeros of f in $|z| \leq r$, where the simple zero is counted once, and the multiple zero is counted twice.*

5. The Proof of Theorem 5

Proof. Assume that $f(z)$ is a transcendental entire solution with finite order of Equation (1). Set

$$f(z) = \frac{1}{\sqrt{2}}(u + v), f(z + c) = \frac{1}{\sqrt{2}}(u - v),$$

where u, v are entire functions in $\mathbb{C}^2$. Thus, Equation (1) can be written as

$$(1 + \alpha)u^2 + (1 - \alpha)v^2 = e^g. \quad (17)$$

Hence, we can deduce from (17) that

$$\left(\frac{\sqrt{1 + \alpha}u}{e^{\frac{g(z)}{2}}} \right)^2 + \left(\frac{\sqrt{1 - \alpha}v}{e^{\frac{g(z)}{2}}} \right)^2 = 1,$$

which leads to

$$\left(\frac{\sqrt{1 + \alpha}u}{e^{\frac{g(z)}{2}}} + i\frac{\sqrt{1 - \alpha}v}{e^{\frac{g(z)}{2}}} \right) \left(\frac{\sqrt{1 + \alpha}u}{e^{\frac{g(z)}{2}}} - i\frac{\sqrt{1 - \alpha}v}{e^{\frac{g(z)}{2}}} \right) = 1. \quad (18)$$

In view of the assumptions, there exists a polynomial $p(z)$ such that

$$\begin{cases} \frac{\sqrt{1+\alpha u}}{e^{\frac{g(z)}{2}}} + i\frac{\sqrt{1-\alpha v}}{e^{\frac{g(z)}{2}}} = e^{p(z)}, \\ \frac{\sqrt{1+\alpha u}}{e^{\frac{g(z)}{2}}} - i\frac{\sqrt{1-\alpha v}}{e^{\frac{g(z)}{2}}} = e^{-p(z)}. \end{cases} \tag{19}$$

Denote

$$\gamma_1(z) = \frac{g(z)}{2} + p(z), \gamma_2(z) = \frac{g(z)}{2} - p(z). \tag{20}$$

In view of (19), it follows that

$$\sqrt{1+\alpha u} = \frac{e^{\gamma_1(z)} + e^{\gamma_2(z)}}{2}, \sqrt{1-\alpha v} = \frac{e^{\gamma_1(z)} - e^{\gamma_2(z)}}{2i}.$$

This leads to

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2}}\left[\frac{e^{\gamma_1(z)} + e^{\gamma_2(z)}}{2\sqrt{1+\alpha}} + \frac{e^{\gamma_1(z)} - e^{\gamma_2(z)}}{2\sqrt{1-\alpha}i}\right] \\ &= \frac{1}{\sqrt{2}}(A_1 e^{\gamma_1(z)} + A_2 e^{\gamma_2(z)}), \end{aligned} \tag{21}$$

$$\begin{aligned} f(z+c) &= \frac{1}{\sqrt{2}}\left[\frac{e^{\gamma_1(z)} + e^{\gamma_2(z)}}{2\sqrt{1+\alpha}} - \frac{e^{\gamma_1(z)} - e^{\gamma_2(z)}}{2\sqrt{1-\alpha}i}\right] \\ &= \frac{1}{\sqrt{2}}(A_2 e^{\gamma_1(z)} + A_1 e^{\gamma_2(z)}), \end{aligned} \tag{22}$$

where A_1, A_2 are defined in (13). Thus, in view of (21) and (22), it follows that

$$\frac{A_2}{A_1}e^{\gamma_2(z+c)-\gamma_2(z)} - \frac{A_2}{A_1}e^{\gamma_1(z)-\gamma_2(z)} + e^{\gamma_1(z+c)-\gamma_2(z)} \equiv 1. \tag{23}$$

We discuss two cases below.

Case 1. Suppose that $\gamma_1(z+c) - \gamma_2(z)$ is a constant. Let $\gamma_1(z+c) - \gamma_2(z) = \kappa, \kappa \in \mathbb{C}$. In view of (20), it follows that $\gamma_1(z) - \gamma_2(z) = 2p(z)$. Substituting these into (23), we have

$$e^{2p(z)} + (1-\eta)\frac{A_1}{A_2} = e^{-2p(z+c)}\eta, \tag{24}$$

where $\eta = e^{\kappa}$. If $\eta \neq 1$, by using the Nevanlinna second fundamental theorem, we have

$$\begin{aligned} T(r, e^{2p}) &\leq N(r, \frac{1}{e^{2p}}) + N(r, \frac{1}{e^{2p} - \delta}) + S(r, e^{2p}) \\ &\leq N(r, \frac{1}{e^{-2p(z+c)}\eta}) + S(r, e^{2p}) = S(r, e^{2p}), \end{aligned}$$

where $\delta = -(1-\eta)\frac{A_1}{A_2} \neq 0$. This means that $p(z)$ is a constant. Set $\xi = e^p$. Substituting this into (21) and (22), we have

$$f(z) = \frac{1}{\sqrt{2}}(A_1\xi + A_2\xi^{-1})e^{\frac{1}{2}g(z)}, \tag{25}$$

$$f(z+c) = \frac{1}{\sqrt{2}}(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}g(z)}. \tag{26}$$

From (25) and (26), it follows that

$$(A_1\xi + A_2\xi^{-1})e^{\frac{g(z+c)-g(z)}{2}} = A_2\xi + A_1\xi^{-1}. \tag{27}$$

In view of $\alpha^2 \neq 1$, it follows that $A_2\xi + A_1\xi - 1 = 0$ and $(A_1\xi + A_2\xi^{-1}) = 0$ cannot hold at the same time. Hence, we have $A_2\xi + A_1\xi^{-1} \neq 0$ and $(A_1\xi + A_2\xi^{-1}) \neq 0$. Since $g(z)$ is a polynomial, (27) implies that $g(z+c) - g(z)$ is a constant in $\mathbb{C}$. Otherwise, we obtain a contradiction from the fact that the left of the above equation is not transcendental but the right is transcendental. Thus, it follows that $g(z) = L(z) + H(s)$, where $L(z) = a_1z_1 + a_2z_2 + H(s)$, a_1, a_2 are constants, and $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$ satisfying

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)} = \frac{A_2\xi + A_1\xi^{-1}}{(A_1\xi + A_2\xi^{-1})}. \tag{28}$$

If $\eta = 1$, it follows that $\kappa = 2k\pi i$, $k \in \mathbb{Z}$, and $\gamma_1(z+c) - \gamma_2(z) = 2k\pi i$. In view of (23), we have $\gamma_2(z+c) = \gamma_1(z)$. By combining with (20), we have $g(z+c) = 2k\pi i + g(z)$. Thus, it follows that $g(z) = L(z) + H(s)$, where $L(z) = a_1z_1 + a_2z_2$, $H(s)$ is a polynomial in s, and $a_1c_1 + a_2c_2 = 2k\pi i$. Thus, this corresponds to $\xi^4 = 1$ in (28). Thus, from the above discussion, we have

$$f(z) = \frac{1}{\sqrt{2}}(A_1\xi + A_2\xi^{-1})e^{\frac{1}{2}(L(z)+H(s))},$$

where $L(z) = a_1z_1 + a_2z_2$ and $H(s)$ is a polynomial in s satisfying (28).

Thus, this completes the proof of Theorem 5 (i).

Case 2. Suppose that $\gamma_1(z+c) - \gamma_2(z)$ is not a constant. Since $\gamma_1(z), \gamma_2(z)$ are polynomials, by applying Lemma 4 for (23), it follows that

$$-\frac{A_2}{A_1}e^{\gamma_1(z)-\gamma_2(z)} \equiv 1, or \frac{A_2}{A_1}e^{\gamma_2(z+c)-\gamma_2(z)} \equiv 1.$$

If $-\frac{A_2}{A_1}e^{\gamma_1(z)-\gamma_2(z)} \equiv 1$, it follows from (23) that $-\frac{A_1}{A_2}e^{\gamma_1(z+c)-\gamma_2(z+c)} \equiv 1$. Thus, in view of (20), we have

$$-\frac{A_2}{A_1}e^{2p(z)} \equiv 1, -\frac{A_1}{A_2}e^{2p(z+c)} \equiv 1, \tag{29}$$

which imply that $p(z)$ is a constant and $\frac{A_2}{A_1} = \frac{A_1}{A_2}$. This leads to $A_1^2 = A_2^2$, which is a contradiction with $\alpha^2 \neq 0, 1$.

If $\frac{A_2}{A_1}e^{\gamma_2(z+c)-\gamma_2(z)} \equiv 1$, then it follows that $\gamma_2(z)$ is of the form $\gamma_2(z) = L_2(z) + H_2(s)$, $L_2(z) = a_{21}z_1 + a_{22}z_2$, $H_2(s)$ is a polynomial in s, and a_{21}, a_{22}, c_1, c_2 are constants satisfying

$$e^{a_{21}c_1+a_{22}c_2} = \frac{A_1}{A_2}.$$

Moreover, it follows from (23) that $\frac{A_2}{A_1}e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$. This means that $\gamma_1(z)$ is of the form $\gamma_1(z) = L_1(z) + H_1(s)$, $L_1(z) = a_{11}z_1 + a_{12}z_2$, $H_1(s)$ is a polynomial in s, and a_{11}, a_{12}, c_1, c_2 are constants satisfying

$$e^{a_{11}c_1+a_{12}c_2} = \frac{A_2}{A_1}.$$

Since $e^{\gamma_1(z+c)-\gamma_2(z)}$ is not a constant, it follows that $L_1(z) \neq L_2(z)$. In view of (20), we have

$$e^{\gamma_1(z+c)+\gamma_2(z+c)-(\gamma_1(z)+\gamma_2(z))} \equiv e^{g(z+c)-g(z)} \equiv 1, \tag{30}$$

which means that $g(z)$ is of the form $g(z) = L(z) + H(s)$, $L(z) = a_1z_1 + a_2z_2$, and

$$L(z) = L_1(z) + L_2(z), H(s) = H_1(s) + H_2(s), e^{a_1c_1+a_2c_2} = 1.$$

Substituting these into (21), we have

$$f(z) = \frac{1}{\sqrt{2}}(A_1e^{L_1(z)+H_1(s)} + A_2e^{L_2(z)+H_2(s)}).$$

Therefore, this completes the proof of Theorem 5. □

6. Proofs of Theorems 6 and 7

6.1. The Proof of Theorem 6

Suppose that $f(z)$ is a transcendental entire solution with finite order of Equation (14). By using the same argument as in the proof of Theorem 5, we have (21) and

$$\frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}}\left[\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2\sqrt{1+\alpha}} - \frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2\sqrt{1-\alpha}i}\right] = \frac{1}{\sqrt{2}}(A_2e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}). \tag{31}$$

Thus, it follows from (21) and (31) that

$$\frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}}\left[A_1\frac{\partial\gamma_1(z)}{\partial z_1}e^{\gamma_1(z)} + A_2\frac{\partial\gamma_2(z)}{\partial z_1}e^{\gamma_2(z)}\right] = \frac{1}{\sqrt{2}}(A_2e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}),$$

which leads to

$$e^{\gamma_1(z)}\left(A_1\frac{\partial\gamma_1(z)}{\partial z_1}-A_2\right) = e^{\gamma_2(z)}\left(A_1-A_2\frac{\partial\gamma_2(z)}{\partial z_1}\right). \tag{32}$$

By combining with (20) and (32), we have

$$e^{2p(z)}(A_1\frac{\partial\gamma_1(z)}{\partial z_1}-A_2) = A_1-A_2\frac{\partial\gamma_2(z)}{\partial z_1}. \tag{33}$$

If $p(z)$ is not a constant, then it follows from (33) that

$$A_1\frac{\partial\gamma_1(z)}{\partial z_1}-A_2=0, A_1-A_2\frac{\partial\gamma_2(z)}{\partial z_1}=0.$$

Otherwise, we have

$$e^{2p(z)} = \frac{A_1\frac{\partial\gamma_1}{\partial z_1}-A_2}{A_1-A_2\frac{\partial\gamma_2}{\partial z_1}}. \tag{34}$$

Since $p(z), g(z)$ are polynomials, the left of Equation (34) is transcendental, but the right of Equation (34) is a polynomial. This is a contradiction. Hence, it follows that

$$\gamma_1(z) = \frac{A_2}{A_1}z_1+\varphi_1(z_2), \gamma_2(z) = \frac{A_1}{A_2}z_1+\varphi_2(z_2),$$

where $\varphi_1(z_2), \varphi_2(z_2)$ are polynomials in z_2. Thus, we have

$$g(z) = \gamma_1(z)+\gamma_2(z) = \left(\frac{A_2}{A_1}+\frac{A_1}{A_2}\right)z_1+\varphi(z) = -2\alpha z_1+\varphi(z_2),$$

where $\varphi(z_2) = \varphi_1(z_2)+\varphi_2(z_2)$.

If $p(z)$ is a constant, then $\frac{\partial\gamma_1}{\partial z_1} = \frac{\partial\gamma_2}{\partial z_1} = \frac{1}{2}\frac{\partial g}{\partial z_1}$. Let $\xi = e^{2p}$, in view of (33), it follows that

$$(\frac{1}{2}A_1\frac{\partial g}{\partial z_1}-A_2)\xi = (A_1-\frac{1}{2}A_2\frac{\partial g}{\partial z_1}),$$

which leads to

$$\frac{\partial g}{\partial z_1} = \frac{2(A_1+A_2\xi)}{A_1\xi+A_2}.$$

Thus, we have $g(z) = \frac{2(A_1+A_2\xi)}{A_1\xi+A_2}z_1 + \varphi(z_2)$.
Hence, $g(z)$ must be of the form $g(z) = a_1 z_1 + \varphi(z_2)$.
Therefore, this completes the proof of Theorem 6.

6.2. The Proof of Theorem 7

Suppose that $f(z)$ is a transcendental entire solution with finite order of Equation (15). Similar to the argument as in the proof of Theorem 6, we have (21) and

$$\frac{\partial f(z)}{\partial z_1} + \frac{\partial f(z)}{\partial z_2} = \frac{1}{\sqrt{2}}(A_2 e^{\gamma_1(z)} + A_1 e^{\gamma_2(z)}), \quad (35)$$

where γ_1, γ_2 are stated as in (20). Thus, it follows from (21) and (35) that

$$e^{\gamma_1(z)}\left[A_1\left(\frac{\partial\gamma_1(z)}{\partial z_1} + \frac{\partial\gamma_1(z)}{\partial z_2}\right) - A_2\right] = e^{\gamma_2(z)}\left[A_1 - A_2\left(\frac{\partial\gamma_2(z)}{\partial z_1} + \frac{\partial\gamma_2(z)}{\partial z_2}\right)\right]. \quad (36)$$

By combining with (20) and (36), we have

$$e^{2p(z)}\left[A_1\left(\frac{\partial\gamma_1(z)}{\partial z_1} + \frac{\partial\gamma_1(z)}{\partial z_2}\right) - A_2\right] = A_1 - A_2\left(\frac{\partial\gamma_2(z)}{\partial z_1} + \frac{\partial\gamma_2(z)}{\partial z_2}\right). \quad (37)$$

If $p(z)$ is not a constant, similar to the argument as in the proof of Theorem 6, we have

$$A_1\left(\frac{\partial\gamma_1(z)}{\partial z_1} + \frac{\partial\gamma_1(z)}{\partial z_2}\right) - A_2 = 0, \quad (38)$$

$$A_1 - A_2\left(\frac{\partial\gamma_2(z)}{\partial z_1} + \frac{\partial\gamma_2(z)}{\partial z_2}\right) = 0. \quad (39)$$

The characteristic equations of (38) are

$$\frac{dz_1}{dt} = 1, \frac{dz_2}{dt} = 1, \frac{d\gamma_1}{dt} = \frac{A_2}{A_1}.$$

Using the initial conditions $z_1 = 0, z_2 = s$, and $\gamma_1 = \gamma_1(0,s) := \varphi_1(s)$ with a parameter s, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = t, z_2 = t + s$,

$$\gamma_1(z) = \gamma_1(t,s) = \int_0^t \frac{A_2}{A_1}dt + \varphi_1(s) = \frac{A_2}{A_1}t + \varphi_1(s) = \frac{A_2}{A_1}z_1 + \varphi_1(z_2 - z_1), \quad (40)$$

where $\varphi_1(s)$ is a transcendental entire function with finite order in s. Similarly, we have

$$\gamma_2(z) = \frac{A_1}{A_2}z_1 + \varphi_2(z_2 - z_1), \quad (41)$$

where $\varphi_2(s)$ is a transcendental entire function with finite order in $s = z_2 - z_1$. In view of (20), (40), and (41), it follows that

$$g(z) = \gamma_1(z) + \gamma_2(z) = -2\alpha z_1 + \varphi_1(z_2 - z_1) + \varphi_2(z_2 - z_1) = -2\alpha z_1 + \varphi(z),$$

where $\varphi(z) = \varphi_1(z_2 - z_1) + \varphi_2(z_2 - z_1)$.
If $p(z)$ is a constant, then

$$\frac{\partial\gamma_1}{\partial z_1} + \frac{\partial\gamma_1}{\partial z_2} = \frac{\partial\gamma_2}{\partial z_1} + \frac{\partial\gamma_2}{\partial z_2} = \frac{1}{2}\left(\frac{\partial g}{\partial z_1} + \frac{\partial g}{\partial z_1}\right).$$

Let $\xi = e^{2p}$; in view of (37), it follows that

$$\left[\frac{1}{2}A_1\left(\frac{\partial g}{\partial z_1}+\frac{\partial g}{\partial z_2}\right)-A_2\right]\xi = A_1 - \frac{1}{2}A_2\left(\frac{\partial g}{\partial z_1}+\frac{\partial g}{\partial z_2}\right),$$

which leads to

$$\frac{\partial g}{\partial z_1}+\frac{\partial g}{\partial z_2} = \frac{2(A_1+A_2\xi)}{A_1\xi+A_2}.$$

Thus, by solving the solutions of the characteristic equations of the above equation, we have $g(z) = \frac{2(A_1+A_2\xi)}{A_1\xi+A_2}z_1 + \varphi(z_2-z_1)$, where $\varphi(z_2-z_1)$ is a polynomial in z_2-z_1.

Hence, $g(z)$ must be of the form $g(z) = a_1z_1 + \varphi(z_2-z_1)$.

Therefore, this completes the proof of Theorem 7.

7. Proofs of Theorems 8 and 9

7.1. The Proof of Theorem 8

Suppose that $f(z)$ is a transcendental entire solution with finite order of Equation (2). By using the same argument as in the proof of Theorem 5, we have (31) and

$$\begin{aligned} f(z+c) &= \frac{1}{\sqrt{2}}\left[\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2\sqrt{1+\alpha}}+\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2\sqrt{1-\alpha i}}\right] \\ &= \frac{1}{\sqrt{2}}(A_1e^{\gamma_1(z)}+A_2e^{\gamma_2(z)}), \end{aligned} \tag{42}$$

where $\gamma_1(z), \gamma_2(z)$ are stated as in (20) and $p(z)$ is a polynomial in $\mathbb{C}^2$. In view of (31) and (42), it follows that

$$\begin{aligned} \frac{\partial f(z+c)}{\partial z_1} &= \frac{1}{\sqrt{2}}\left(A_1\frac{\partial \gamma_1(z)}{\partial z_1}e^{\gamma_1(z)}+A_2\frac{\partial \gamma_2(z)}{\partial z_1}e^{\gamma_2(z)}\right) \\ &= \frac{1}{\sqrt{2}}(A_2e^{\gamma_1(z+c)}+A_1e^{\gamma_2(z+c)}). \end{aligned}$$

Thus, we have

$$\frac{A_1}{A_2}\frac{\partial \gamma_1(z)}{\partial z_1}e^{\gamma_1(z)-\gamma_1(z+c)}+\frac{\partial \gamma_2(z)}{\partial z_1}e^{\gamma_2(z)-\gamma_1(z+c)}-\frac{A_1}{A_2}e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1. \tag{43}$$

Now, we discuss two cases below.

Case 1. Suppose that $p(z)$ is a constant; then, it follows that $\gamma_1(z+c)-\gamma_2(z+c) = 2p(z+c)$ is a constant. Denote $\xi = e^p$. In view of (20) and (42), it follows that

$$f(z+c) = \frac{1}{\sqrt{2}}(A_1\xi+A_2\xi^{-1})e^{\frac{1}{2}g(z)}, \frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}}(A_2\xi+A_1\xi^{-1})e^{\frac{1}{2}g(z)}. \tag{44}$$

Thus, we can deduce from (44) that

$$(A_2\xi+A_1\xi^{-1})e^{\frac{1}{2}(g(z+c)-g(z))} = \frac{1}{2}(A_1\xi+A_2\xi^{-1})\frac{\partial g(z)}{\partial z_1}. \tag{45}$$

Since $g(z)$ is a polynomial in $\mathbb{C}^2$, and not of the form $\phi(z_2)$, it follows that $\frac{\partial g(z)}{\partial z_1} \not\equiv 0$. If $g(z+c)-g(z)$ is not a constant, we can deduce from (45) that $A_2\xi + A_1\xi^{-1} = 0$ and $A_1\xi+A_2\xi^{-1} = 0$. Otherwise, we have

$$e^{\frac{1}{2}(g(z+c)-g(z))} = \frac{1}{2}\frac{\partial g(z)}{\partial z_1}\frac{A_1\xi+A_2\xi^{-1}}{A_2\xi+A_1\xi^{-1}}. \tag{46}$$

The left of Equation (46) is transcendental, but the right of Equation (46) is a polynomial. Thus, a contradiction can be obtained from (46). The fact that $A_2\xi + A_1\xi^{-1} = 0$ and $A_1\xi + A_2\xi^{-1} = 0$ can yield that $A_1^2 = A_2^2$, which is a contradiction with $\alpha^2 = 1$.

Thus, it follows that $g(z+c) - g(z)$ is a constant. Then, we have $g(z) = L(z) + H(s)$, where $L(z)$ is a linear form of $L(z) = a_1z_1 + a_2z_2$, a_1, a_2 are constants, and $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$. From (46), it follows that

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)} = \frac{1}{2}\frac{A_1\xi + A_2\xi^{-1}}{A_2\xi + A_1\xi^{-1}}(a_1 + c_2H'). \tag{47}$$

In view of $c_2 \neq 0$, it yields that H' is a constant, which means that $\deg_s H \leq 1$. Hence, we can conclude that $L(z) + H(s)$ is a linear form in z_1, z_2. Let us still denote $g(z) = a_1z_1 + a_2z_2 + b$, which implies that $H' \equiv 0$. Thus, by combining with (44) and (47), we have

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2}}(A_1\xi + A_2\xi^{-1})e^{\frac{1}{2}(a_1z_1+a_2z_2+b)-\frac{1}{2}(a_1c_1+a_2c_2)} \\ &= \frac{\sqrt{2}}{a}(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}(a_1z_1+a_2z_2+b)}. \end{aligned} \tag{48}$$

Thus, in view of (47) and (48), this completes the proof of Theorem 8 (i).

Case 2. Suppose that $p(z)$ is not a constant. Then, we have that $\frac{\partial\gamma_1}{\partial z_1}$ and $\frac{\partial\gamma_2}{\partial z_1}$ cannot be equal to 0 at the same time. Otherwise, it yields that $\gamma_1(z+c) - \gamma_2(z+c)$ is a constant, which is a contradiction. If $\frac{\partial\gamma_1}{\partial z_1} \equiv 0$ and $\frac{\partial\gamma_2}{\partial z_1} \not\equiv 0$, it thus follows from (43) that

$$\frac{\partial\gamma_2}{\partial z_1}e^{\gamma_2(z)-\gamma_1(z+c)} - \frac{A_1}{A_2}e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1. \tag{49}$$

Obviously, $\gamma_2(z) - \gamma_1(z+c)$ is not a constant. Otherwise, $\gamma_2(z+c) - \gamma_1(z+c)$ is a constant because γ_1, γ_2 are polynomials. By applying the Nevanlinna second fundamental theorem for $e^{\gamma_2(z+c)-\gamma_1(z+c)}$, we have from (49) that

$$\begin{aligned} &T(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}) \\ \leq &N(r, \frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)}}) + N(r, \frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)} + \frac{A_2}{A_1}}) + S(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}) \\ \leq &N(r, \frac{1}{\frac{\partial\gamma_2}{\partial z_1}e^{\gamma_2(z)-\gamma_1(z+c)}}) + S(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}) = S(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}), \end{aligned}$$

which is a contradiction.

If $\frac{\partial\gamma_1}{\partial z_1} \not\equiv 0$ and $\frac{\partial\gamma_2}{\partial z_1} \equiv 0$, using the same argument as in the above, we can get a contradiction. Hence, we have $\frac{\partial\gamma_1}{\partial z_1} \not\equiv 0$ and $\frac{\partial\gamma_2}{\partial z_1} \not\equiv 0$. By Lemma 4, it follows that

$$\frac{A_1}{A_2}\frac{\partial\gamma_1}{\partial z_1}e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1, \quad or \quad \frac{\partial\gamma_2}{\partial z_1}e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1.$$

Subcase 2.1. If $\frac{A_1}{A_2}\frac{\partial\gamma_1}{\partial z_1}e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$, it yields that $\gamma_1(z) - \gamma_1(z+c)$ is a constant. This implies that $\gamma_1(z)$ is a linear form of $\gamma_1(z) = L_1(z) + H_1(s)$, where $L_1(z) = a_{11}z_1 + a_{12}z_2$, a_{11}, a_{12} are constants, and $H_1(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$. Thus, it follows that

$$\frac{A_1}{A_2}(a_{11} + c_2H_1')e^{-L_1(c)} \equiv 1,$$

which means that c_2H_1' is a constant. In view of $c_2 \neq 0$, it yields that $\deg_s H_1 \leq 1$. Hence, we can conclude that $L_1(z) + H_1(s)$ is a linear form in z_1, z_2. Let us still denote $\gamma_1(z) = L_1(z) + b_1 = a_{11}z_1 + a_{12}z_2 + b_1$, which implies that $H_1' \equiv 0$. Hence, we have

$$e^{L_1(c)} = e^{a_{11}c_1+a_{12}c_2} = a_{11}\frac{A_1}{A_2}. \tag{50}$$

In addition, in view of (43), it follows

$$\frac{A_2}{A_1}\frac{\partial\gamma_2}{\partial z_1}e^{\gamma_2(z)-\gamma_2(z+c)} \equiv 1.$$

Similarly, we have $\gamma_2(z) = L_2(z) + b_2 = a_{21}z_1 + a_{22}z_2 + b_2$, where a_{21}, a_{22}, b_2 are constants satisfying

$$e^{L_2(c)} = e^{a_{21}c_1+a_{22}c_2} = a_{21}\frac{A_2}{A_1}. \tag{51}$$

Since $\gamma_1(z+c) - \gamma_2(z+c)$ is not a constant, it follows that $L_1(z) \neq L_2(z)$. In view of (20) and (42), it follows that

$$g(z) = \gamma_1(z) + \gamma_2(z) = (a_{11} + a_{21})z_1 + (a_{12} + a_{22})z_2 + b_1 + b_2 = a_1z_1 + a_2z_2 + b,$$

and

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2}}(A_1e^{L_1(z)+b_1-L_1(c)} + A_2e^{L_2(z)+b_2-L_2(c)}) \\ &= \frac{1}{\sqrt{2}}(A_1\frac{A_2}{a_{11}A_1}e^{a_{11}z_1+a_{12}z_2+b_1} + A_2\frac{A_1}{a_{21}A_2}e^{a_{21}z_1+a_{22}z_2+b_2}) \\ &= \frac{1}{\sqrt{2}}(\frac{A_2}{a_{11}}e^{a_{11}z_1+a_{12}z_2+b_1} + \frac{A_1}{a_{21}}e^{a_{21}z_1+a_{22}z_2+b_2}). \end{aligned} \tag{52}$$

Subcase 2.2. If $\frac{\partial\gamma_2}{\partial z_1}e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$, this means that $\gamma_2(z) - \gamma_1(z+c)$ is a constant, without loss of generalization, denote

$$\gamma_2(z) - \gamma_1(z+c) = \mu_1, \tag{53}$$

where μ_1 is a constant. In view of (43), it thus follows that $\frac{\gamma_1}{\partial z_1}e^{\gamma_1(z)-\gamma_2(z+c)} \equiv 1$, this means

$$\gamma_1(z) - \gamma_2(z+c) \equiv \mu_2, \tag{54}$$

where μ_2 is a constant. In view of (53) and (54), it yields that

$$\gamma_1(z) - \gamma_2(z) + \gamma_1(z+c) - \gamma_2(z+c) = \mu_2 - \mu_1,$$

and by combining with (20), we have

$$p(z) + p(z+c) = \frac{1}{2}(\mu_2 - \mu_1),$$

this is a contradiction with the assumption that $\gamma_1(z+c) - \gamma_2(z+c) = 2p(z+c)$ is not a constant. Thus, we get the conclusions of Theorem 8 (ii) from (50)–(52).

Therefore, this completes the proof of Theorem 8.

7.2. The Proof of Theorem 9

Suppose that $f(z)$ is a transcendental entire solution with finite order of Equation (3). Similar to the argument as in the proof of Theorem 8, we have (35), (42), and

$$\frac{A_1}{A_2}(\frac{\partial\gamma_1}{\partial z_1} + \frac{\partial\gamma_1}{\partial z_2})e^{\gamma_1(z)-\gamma_1(z+c)} + (\frac{\partial\gamma_2}{\partial z_1} + \frac{\partial\gamma_2}{\partial z_2})e^{\gamma_2(z)-\gamma_1(z+c)} - \frac{A_1}{A_2}e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1, \tag{55}$$

where $\gamma_1(z), \gamma_2(z)$ are stated as in (20), and $p(z)$ is a polynomial in $\mathbb{C}^2$.

Now, we discuss two cases below.

Case 1. Suppose that $p(z)$ is a constant. In view of $\gamma_1(z+c) - \gamma_2(z+c) = 2p(z+c)$, it follows that $\gamma_1(z+c) - \gamma_2(z+c)$ is a constant. Denote $\xi = e^p$. In view of (20) and (55), it follows that

$$f(z+c) = \frac{1}{\sqrt{2}}(A_1\xi + A_2\xi^{-1})e^{\frac{1}{2}g(z)}, \quad \frac{\partial f(z)}{\partial z_1} + \frac{\partial f(z)}{\partial z_2} = \frac{1}{\sqrt{2}}(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}g(z)}. \quad (56)$$

Thus, we can deduce from (56) that

$$(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}(g(z+c)-g(z))} = \frac{1}{2}(A_1\xi + A_2\xi^{-1})\left(\frac{\partial g(z)}{\partial z_1} + \frac{\partial g(z)}{\partial z_2}\right). \quad (57)$$

If $\frac{\partial g(z)}{\partial z_1} + \frac{\partial g(z)}{\partial z_2} \equiv 0$, we have $g(z) = \phi(z_2 - z_1)$ by solving this partial differential equation, which is a contradiction with the assumption of Theorem 9. Thus, $\frac{\partial g(z)}{\partial z_1} + \frac{\partial g(z)}{\partial z_2} \not\equiv 0$. If $g(z+c) - g(z)$ is not a constant, we can deduce from (57) that $A_2\xi + A_1\xi^{-1} = 0$ and $A_1\xi + A_2\xi^{-1} = 0$. Otherwise, we have

$$e^{\frac{1}{2}(g(z+c)-g(z))} = \frac{1}{2}\left(\frac{\partial g(z)}{\partial z_1} + \frac{\partial g(z)}{\partial z_2}\right)\frac{A_1\xi + A_2\xi^{-1}}{A_2\xi + A_1\xi^{-1}}. \quad (58)$$

The left of Equation (58) is transcendental, but the right of Equation (58) is a polynomial. Thus, a contradiction can be obtained from (58). The fact that $A_2\xi + A_1\xi^{-1} = 0$ and $A_1\xi + A_2\xi^{-1} = 0$ can yield that $A_1^2 = A_2^2$, which is a contradiction with $\alpha^2 = 1$.

Thus, it follows that $g(z+c) - g(z)$ is a constant. Then, we have $g(z) = L(z) + H(s)$, where $L(z)$ is a linear form of $L(z) = a_1z_1 + a_2z_2$, a_1, a_2 are constants, and $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$. From (58), it follows that

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)} = \frac{1}{2}\frac{A_1\xi + A_2\xi^{-1}}{A_2\xi + A_1\xi^{-1}}[a_1 + a_2 + (c_2 - c_1)H']. \quad (59)$$

In view of $c_2 \neq c_1$, it yields that H' is a constant, which means that $\deg_s H \leq 1$. Hence, we can conclude that $L(z) + H(s)$ is a linear form in z_1, z_2. Let us still denote $g(z) = a_1z_1 + a_2z_2 + b$, which implies that $H' \equiv 0$. Thus, by combining with (56) and (59), we have

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2}}(A_1\xi + A_2\xi^{-1})e^{\frac{1}{2}(a_1z_1+a_2z_2+b)-\frac{1}{2}(a_1c_1+a_2c_2)} \\ &= \frac{\sqrt{2}}{a_1 + a_2}(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}(a_1z_1+a_2z_2+b)}, \end{aligned} \quad (60)$$

and

$$e^{\frac{1}{2}(a_1c_1+a_2c_2)} = \frac{1}{2}\frac{A_1\xi + A_2\xi^{-1}}{A_2\xi + A_1\xi^{-1}}(a_1 + a_2). \quad (61)$$

Thus, in view of (59) and (61), this completes the proof of Theorem 9 (i).

Case 2. Suppose that $p(z)$ is not a constant. Then, we have that $\frac{\partial \gamma_1}{\partial z_1} + \frac{\partial \gamma_1}{\partial z_2}$ and $\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2}$ cannot be equal to 0 at the same time. Otherwise, it yields that $\gamma_1(z+c) - \gamma_2(z+c)$ is a constant, which is a contradiction. If $\frac{\partial \gamma_1}{\partial z_1} + \frac{\partial \gamma_1}{\partial z_2} \equiv 0$ and $\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2} \not\equiv 0$, it thus follows from (55) that

$$\left(\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2}\right)e^{\gamma_2(z)-\gamma_1(z+c)} - \frac{A_1}{A_2}e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1. \quad (62)$$

Obviously, $\gamma_2(z) - \gamma_1(z+c)$ is not a constant. Otherwise, $\gamma_2(z+c) - \gamma_1(z+c)$ is a constant because γ_1, γ_2 are polynomials. By applying the Nevanlinna second fundamental theorem for $e^{\gamma_2(z+c)-\gamma_1(z+c)}$, we have from (62) that

$$
\begin{aligned}
&T(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}) \\
\leq &N(r, \frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)}}) + N(r, \frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)} + \frac{A_2}{A_1}}) + S(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}) \\
\leq &N(r, \frac{1}{(\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2})e^{\gamma_2(z)-\gamma_1(z+c)}}) + S(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}) = S(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}),
\end{aligned}
$$

which is a contradiction.

If $\frac{\partial \gamma_1}{\partial z_1} + \frac{\partial \gamma_1}{\partial z_2} \not\equiv 0$ and $\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2} \equiv 0$, using the same argument as in the above, we can get a contradiction. Hence, we have $\frac{\partial \gamma_1}{\partial z_1} + \frac{\partial \gamma_1}{\partial z_2} \not\equiv 0$ and $\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2} \not\equiv 0$. By Lemma 4, it follows that

$$
\frac{A_1}{A_2}\left(\frac{\partial \gamma_1}{\partial z_1} + \frac{\partial \gamma_1}{\partial z_2}\right)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1, or \left(\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2}\right)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1.
$$

Subcase 2.1. If $\frac{A_1}{A_2}\left(\frac{\partial \gamma_1}{\partial z_1} + \frac{\partial \gamma_1}{\partial z_2}\right)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$, it yields that $\gamma_1(z) - \gamma_1(z+c)$ is a constant. This implies that $\gamma_1(z)$ is a linear form of $\gamma_1(z) = L_1(z) + H_1(s)$, where $L_1(z) = a_{11}z_1 + a_{12}z_2$, a_{11}, a_{12} are constants, and $H_1(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$. Thus, it follows that

$$
\frac{A_1}{A_2}[a_{11} + a_{12} + (c_2 - c_1)H_1']e^{-L_1(c)} \equiv 1,
$$

which means that $(c_2 - c_1)H_1'$ is a constant. In view of $c_2 \neq c_1$, it yields that $\deg_s H_1 \leq 1$. Hence, we can conclude that $L_1(z) + H_1(s)$ is a linear form in z_1, z_2. Let us still denote $\gamma_1(z) = L_1(z) + b_1 = a_{11}z_1 + a_{12}z_2 + b_1$, which implies that $H_1' \equiv 0$. Hence, we have

$$
e^{L_1(c)} = e^{a_{11}c_1 + a_{12}c_2} = (a_{11} + a_{12})\frac{A_1}{A_2}. \tag{63}
$$

In addition, in view of (55), it follows

$$
\frac{A_2}{A_1}\left(\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2}\right)e^{\gamma_2(z)-\gamma_2(z+c)} \equiv 1.
$$

Similarly, we have $\gamma_2(z) = L_2(z) + b_2 = a_{21}z_1 + a_{22}z_2 + b_2$, where a_{21}, a_{22}, b_2 are constants satisfying

$$
e^{L_2(c)} = e^{a_{21}c_1 + a_{22}c_2} = (a_{21} + a_{22})\frac{A_2}{A_1}. \tag{64}
$$

Since $\gamma_1(z+c) - \gamma_2(z+c)$ is not a constant, it follows that $L_1(z) \neq L_2(z)$. In view of (20) and (55), it follows that

$$
g(z) = \gamma_1(z) + \gamma_2(z) = (a_{11} + a_{21})z_1 + (a_{12} + a_{22})z_2 + b_1 + b_2 = a_1z_1 + a_2z_2 + b,
$$

and

$$
\begin{aligned}
f(z) &= \frac{1}{\sqrt{2}}(A_1e^{L_1(z)+b_1-L_1(c)} + A_2e^{L_2(z)+b_2-L_2(c)}) \\
&= \frac{1}{\sqrt{2}}(\frac{A_2}{a_{11} + a_{12}}e^{a_{11}z_1 + a_{12}z_2 + b_1} + \frac{A_1}{a_{21} + a_{22}}e^{a_{21}z_1 + a_{22}z_2 + b_2}).
\end{aligned} \tag{65}
$$

Subcase 2.2. If $(\frac{\partial \gamma_2}{\partial z_1} + \frac{\partial \gamma_2}{\partial z_2})e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$, similar to the argument as the proof of Subcase 2.2 in Theorem 8, we have $p(z) + p(z+c) = \frac{1}{2}(\mu_2 - \mu_1)$, which contradicts with the assumption that $\gamma_1(z+c) - \gamma_2(z+c) = 2p(z+c)$ is not a constant. Thus, we get the conclusions of Theorem 9 (ii) from (63)–(65).

Therefore, this completes the proof of Theorem 9.

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References

1. Xu, L.; Cao, T.B. Solutions of complex Fermat-type partial difference and differential-difference equations. *Mediterr. J. Math.* **2018**, *15*, 1–14; Correction in: **2020**, *17*, 1–4. [CrossRef]
2. Gross, F. On the equation $f^n + g^n = 1$. *Bull. Am. Math. Soc.* **1966**, *72*, 86–88. [CrossRef]
3. Montel, P. *Lecons sur les Familles Normales de Fonctions Analytiques et Leurs Applications*; Gauthier-Villars: Paris, France, 1927; pp. 135–136.
4. Pólya, G. On an integral function of an integral function. *J. Lond. Math. Soc.* **1926**, *1*, 12–15. [CrossRef]
5. Cao, T.; Korhonen, R. A new version of the second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables. *J. Math. Anal. Appl.* **2016**, *444*, 1114–1132. [CrossRef]
6. Chiang, Y.M.; Feng, S.J. On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. *Ramanujan J.* **2008**, *16*, 105–129. [CrossRef]
7. Halburd, R.G.; Korhonen, R. Finite-order meromorphic solutions and the discrete Painlevé equations. *Proc. Lond. Math. Soc.* **2007**, *94*, 443–474. [CrossRef]
8. Halburd, R.G.; Korhonen, R.J. Nevanlinna theory for the difference operator. *Ann. Acad. Sci. Fenn.* **2006**, *31*, 463–478.
9. Ronkin, L.I. *Introduction to the Theory of Entire Functions of Several Variables*; Nauka: Moscow, Russian, 1971.
10. Stoll, W. *Holomorphic Functions of Finite Order in Several Complex Variables*; American Mathematical Society: Providence, RI, USA, 1974.
11. Vitter, A. The lemma of the logarithmic derivative in several complex variables. *Duke Math. J.* **1977**, *44*, 89–104. [CrossRef]
12. Cao, T.B.; Xu, L. Logarithmic difference lemma in several complex variables and partial difference equations. *Ann. Mat.* **2020**, *199*, 767–794. [CrossRef]
13. Khavinson, D. A note on entire solutions of the eiconal equation. *Am. Math. Mon.* **1995**, *102*, 159–161. [CrossRef]
14. Li, B.Q. On entire solutions of Fermat type partial differential equations. *Int. J. Math.* **2004**, *15*, 473–485 [CrossRef]
15. Li, B.Q. Entire solutions of certain partial differential equations and factorization of partial derivatives. *Tran. Amer. Math. Soc.* **2004**, *357*, 3169–3177. [CrossRef]
16. Li, B.Q. Entire solutions of eiconal type equations. *Arch. Math.* **2007**, *89*, 350–357. [CrossRef]
17. Lü, F.; Lü, W.; Li, C.; Xu, J. Growth and uniqueness related to complex differential and difference equations. *Results Math.* **2019**, *74*, 30 [CrossRef]
18. Xu, H.Y.; Liu, S.Y.; Li, Q.P. Entire solutions for several systems of nonlinear difference and partial differentialdifference equations of fermat-type. *J. Math. Anal. Appl.* **2020**, *483*, 123641. [CrossRef]
19. Zhang, J. On some special difference equations of Malmquist type. *Bull. Korean Math. Soc.* **2018**, *55*, 51–61.
20. Saleeby, E.G. Entire and meromorphic solutions of Fermat type partial differential equations. *Analysis* **1999**, *19*, 69–376. [CrossRef]
21. Saleeby, E.G. On entire and meromorphic solutions of $\lambda u^k + \sum_{i=1}^{n} u_{z_i}^m = 1$. *Complex Var. Theory Appl. Int. J.* **2004**, *49*, 101–107.
22. Li, B.Q. Entire solutions of $(u_{z_1})^m + (u_{z_2})^n = e^g$. *Nagoya Math. J.* **2005**, *178*, 151–162. [CrossRef]
23. Saleeby, E.G. On complex analytic solutions of certain trinomial functional and partial differential equations. *Aequat. Math.* **2013**, *85*, 553–562. [CrossRef]
24. Liu, K.; Yang, L.Z. A note on meromorphic solutions of Fermat types equations. *An. Ştiinţ. Univ. Al. I Cuza Iaşi Mat. (N. S.)* **2016**, *1*, 317–325.

25. Berenstein, C.A.; Chang, D.C.; Li, B.Q. On the shared values of entire functions and their partial differential polynomials in $\mathbb{C}^n$. *Forum Math.* **1996**, *8*, 379–396. [CrossRef]
26. Hu, P.-C.; Li, P.; Yang, C.-C. *Unicity of Meromorphic Mappings, Advances in Complex Analysis and its Applications*; Kluwer Academic: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 2003; Volume 1.