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Article

Mann-Type Inertial Subgradient Extragradient Rules for Variational Inequalities and Common Fixed Points of Nonexpansive and Quasi-Nonexpansive Mappings

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Abstract: Suppose that in a real Hilbert space H, the variational inequality problem with Lipschitzian and pseudomonotone mapping A and the common fixed-point problem of a finite family of nonexpansive mappings and a quasi-nonexpansive mapping with a demiclosedness property are represented by the notations VIP and CFPP, respectively. In this article, we suggest two Mann-type inertial subgradient extragradient iterations for finding a common solution of the VIP and CFPP. Our iterative schemes require only calculating one projection onto the feasible set for every iteration, and the strong convergence theorems are established without the assumption of sequentially weak continuity for A. Finally, in order to support the applicability and implementability of our algorithms, we make use of our main results to solve the VIP and CFPP in two illustrating examples.

Keywords: Mann-type inertial subgradient extragradient rule; variational inequality problem; pseudomonotone mapping; Nonexpansive and quasi-nonexpansive mappings; common fixed point

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1. Introduction

In a real Hilbert space $(H, \|\cdot\|)$, equipped with the inner product $\langle\cdot,\cdot\rangle$, we assume that C is a nonempty closed convex subset and P_C is the metric projection of H onto C. If $S : C \to H$ is a mapping on C, then we denote by $\mathrm{Fix}(S)$ the fixed-point set of S. Moreover, we denote by $\mathbf{R}$ the set of all real numbers. Given a mapping $A : H \to H$. Consider the classical variational inequality problem (VIP) of finding $x^* \in C$ such that $\langle Ax^*, x - x^*\rangle \geq 0$ for all $x \in C$. We denote by $\mathrm{VI}(C, A)$ the solution set of the VIP.

To the best of our knowledge, one of the most efficient methods to deal with the VIP is the extragradient method invented by Korpelevich [1] in 1976, that is, for any given $u_0 \in C$, $\{u_m\}$ is the sequence constructed by

$$\begin{cases} v_m = P_C(u_m - \ell A u_m), \\ u_{m+1} = P_C(u_m - \ell A v_m) \quad \forall m \geq 0, \end{cases} \tag{1}$$

with constant $\ell \in (0, \frac{1}{L})$. If $\mathrm{VI}(C, A) \neq \emptyset$, one knows that this method has only weak convergence, and only requires that A is monotone and L-Lipschitzian. The literature on the VIP is vast, and Korpelevich's extragradient method has received great attention from many authors, who improved it via various approaches so that some new iterative methods happen to solve the VIP and related optimization problems; see, e.g., [2–12] and the references therein, to name but a few.

It is worth pointing out that the extragradient method needs to calculate two projections onto the feasible set C per iteration. Without question, once one is hard to calculate the projection onto C, the minimum distance problem has to be solved twice per iteration. This perhaps affects the applicability and implementability of the method. To improve Algorithm 1, one has to reduce the number of projections per iteration. In 2011, Censor et al. [13] first suggested the subgradient extragradient method, in which the second projection onto C is replaced by a projection onto a half-space:

$$\begin{cases} v_m = P_C(u_m - \ell A u_m), \\ C_m = \{w \in H : \langle u_m - \ell A u_m - v_m, w - v_m \rangle \leq 0\}, \\ u_{m+1} = P_{C_m}(u_m - \ell A v_m) \quad \forall m \geq 0, \end{cases} \tag{2}$$

where A is a L-Lipschitzian monotone mapping and $\ell \in (0, \frac{1}{L})$.

Since then, various modified extragradient-like iterative methods have been investigated by many researchers; see, e.g., [14–19]. In 2014, combining the subgradient extragradient method and Halpern's iteration method, Kraikaew and Saejung [20] proposed the Halpern subgradient extragradient method for solving the VIP, that is, for any given $u_0 \in H$, $\{u_m\}$ is the sequence constructed by

$$\begin{cases} v_m = P_C(u_m - \ell A u_m), \\ C_m = \{v \in H : \langle u_m - \ell A u_m - v_m, v - v_m \rangle \leq 0\}, \\ w_m = P_{C_m}(u_m - \ell A v_m), \\ u_{m+1} = \alpha_m u_0 + (1 - \alpha_m) w_m \quad \forall m \geq 0, \end{cases} \tag{3}$$

where $\ell \in (0, \frac{1}{L})$, $\{\alpha_m\} \subset (0,1)$, $\lim_{m\to\infty} \alpha_m = 0$ and $\sum_{m=1}^{\infty} \alpha_m = +\infty$. They proved the strong convergence of $\{u_m\}$ to $P_{\mathrm{VI}(C,A)} u_0$.

In 2018, Thong and Hieu [21] first suggested the inertial subgradient extragradient method, that is, for any given $u_0, u_1 \in H$, the sequence $\{u_m\}$ is generated by

$$\begin{cases} w_m = u_m + \alpha_m(u_m - u_{m-1}), \\ v_m = P_C(w_m - \ell A w_m), \\ C_m = \{v \in H : \langle w_m - \ell A w_m - v_m, v - v_m \rangle \leq 0\}, \\ u_{m+1} = P_{C_m}(w_m - \ell A v_m) \quad \forall m \geq 1, \end{cases} \tag{4}$$

with constant $\ell \in (0, \frac{1}{L})$. Under suitable conditions, they proved the weak convergence of $\{u_m\}$ to an element of $\mathrm{VI}(C, A)$. Later, Thong and Hieu [22] designed two inertial subgradient extragradient algorithms with linesearch process for solving a VIP with monotone and Lipschitz continuous mapping A and a FPP of quasi-nonexpansive mapping T with a demiclosedness property in H. Under appropriate conditions, they established the weak convergence results for the suggested algorithms.

Suppose that the notations VIP and CFPP represent a variational inequality problem with Lipschitzian and pseudomonotone mapping $A : H \to H$ and a common fixed-point problem of finitely many nonexpansive mappings $\{T_i\}_{i=1}^N$ and a quasi-nonexpansive mapping T with a demiclosedness property, respectively. Inspired by the research works above, we design two Mann-type inertial subgradient extragradient iterations for finding a common solution of the VIP and CFPP. Our algorithms require only computing one projection onto the feasible set C per iteration, and the strong convergence theorems are established without the assumption of sequentially weak continuity for A on C. Finally, in order to support the applicability and implementability of our algorithms, we make use of our main results to solve the VIP and CFPP in two illustrating examples.

This paper is organized as follows: In Section 2, we recall some definitions and preliminaries for the sequel use. Section 3 deals with the convergence analysis of the proposed algorithms. Finally, in Section 4, in order to support the applicability and implementability of our algorithms, we make use of our main results to find a common solution of the VIP and CFPP in two illustrating examples.

2. Preliminaries

Throughout this paper, we assume that C is a nonempty closed convex subset of a real Hilbert space H. If $\{u_m\}$ is a sequence in H, then we denote by $u_m \to u$ (respectively, $u_m \rightharpoonup u$) the strong (respectively, weak) convergence of $\{u_m\}$ to u. A mapping $F : C \to H$ is said to be nonexpansive if $\|Fu - Fv\| \leq \|u - v\|\ \forall u, v \in C$. Recall also that $F : C \to H$ is called

(i) L-Lipschitz continuous (or L-Lipschitzian) if $\exists L > 0$ such that $\|Fu - Fv\| \leq L\|u - v\|\ \forall u, v \in C$;
(ii) monotone if $\langle Fu - Fv, u - v\rangle \geq 0\ \forall u, v \in C$;
(iii) pseudomonotone if $\langle Fu, v - u\rangle \geq 0 \Rightarrow \langle Fv, v - u\rangle \geq 0\ \forall u, v \in C$;
(iv) α-strongly monotone if $\exists \alpha > 0$ such that $\langle Fu - Fv, u - v\rangle \geq \alpha\|u - v\|^2\ \forall u, v \in C$;
(v) quasi-nonexpansive if $\mathrm{Fix}(F) \neq \emptyset$, and $\|Fu - p\| \leq \|u - p\|\ \forall u \in C,\ p \in \mathrm{Fix}(F)$;
(vi) sequentially weakly continuous on C if for $\{u_m\} \subset C$, the relation holds: $u_m \rightharpoonup u \Rightarrow Fu_m \rightharpoonup Fu$.

It is clear that every monotone operator is pseudomonotone, but the converse is not true. Next, we provide an example of a quasi-nonexpansive mapping which is not nonexpansive.

Example 1. *Let $H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. Let $T : H \to H$ be defined as $Tu := \frac{u}{2}\sin u\ \forall u \in H$. It is clear that $\mathrm{Fix}(T) = \{0\}$ and T is quasi-nonexpansive. However, we claim that T is not nonexpansive. Indeed, putting $u = 2\pi$ and $v = \frac{3\pi}{2}$, we have $\|Tu - Tv\| = \|\frac{2\pi}{2}\sin 2\pi - \frac{3\pi}{4}\sin\frac{3\pi}{2}\| = \frac{3\pi}{4} > \|2\pi - \frac{3\pi}{2}\| = \frac{\pi}{2}$.*

Definition 1 ([23])**.** *Assume that $T : H \to H$ is a nonlinear operator with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{u_n\}$ in H, the implication holds: $u_n \rightharpoonup u$ and $(I - T)u_n \to 0 \Rightarrow u \in \mathrm{Fix}(T)$.*

Very recently, Thong and Hieu gave an example to illustrate that there exists a quasi-nonexpansive mapping T, but $I - T$ is not demiclosed at zero; see ([22], Example 2). For each $u \in H$, we know that there exists a unique nearest point in C, denoted by $P_C u$, such that $\|u - P_C u\| \leq \|u - v\|\ \ \forall v \in C$. P_C is called a metric projection of H onto C.

Lemma 1 ([23])**.** *The following hold:*

(i) $\langle u - v, P_C u - P_C v\rangle \geq \|P_C u - P_C v\|^2\ \ \forall u, v \in H$;
(ii) $\langle u - P_C u, v - P_C u\rangle \leq 0\ \ \forall u \in H, v \in C$;
(iii) $\|u - v\|^2 \geq \|u - P_C u\|^2 + \|v - P_C u\|^2\ \ \forall u \in H, v \in C$;
(iv) $\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle\ \ \forall u, v \in H$;
(v) $\|\lambda u + (1 - \lambda)v\|^2 = \lambda\|u\|^2 + (1 - \lambda)\|v\|^2 - \lambda(1 - \lambda)\|u - v\|^2\ \ \forall u, v \in H,\ \lambda \in [0, 1]$.

Lemma 2 ([24])**.** *For all $u \in H$ and $\alpha \geq \beta > 0$, the inequalities hold: $\frac{\|u - P_C(u - \alpha Au)\|}{\alpha} \leq \frac{\|u - P_C(u - \beta Au)\|}{\beta}$ and $\|u - P_C(u - \beta Au)\| \leq \|u - P_C(u - \alpha Au)\|$.*

Lemma 3 ([13])**.** *Suppose that $A : C \to H$ is pseudomonotone and continuous. Then $u^* \in C$ is a solution to the VIP $\langle Au^*, u - u^*\rangle \geq 0\ \forall u \in C$, if and only if $\langle Au, u - u^*\rangle \geq 0\ \forall u \in C$.*

Lemma 4 ([25])**.** *Suppose that $\{a_m\}$ is a sequence of nonnegative numbers satisfying the conditions: $a_{m+1} \leq (1 - \lambda_m)a_m + \lambda_m\gamma_m\ \forall m \geq 1$, where $\{\lambda_m\}$ and $\{\gamma_m\}$ lie in $\mathbf{R} = (-\infty, \infty)$ such that (i) $\{\lambda_m\} \subset [0, 1]$ and $\sum_{m=1}^{\infty}\lambda_m = \infty$, and (ii) $\limsup_{m\to\infty}\gamma_m \leq 0$ or $\sum_{m=1}^{\infty}|\lambda_m\gamma_m| < \infty$. Then $\lim_{m\to\infty} a_m = 0$.*

Lemma 5 ([23]). *Suppose that $T : C \to C$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, that is, if $\{u_m\}$ is a sequence in C such that $u_m \rightharpoonup u \in C$ and $(I - T)u_m \to 0$, then $(I - T)u = 0$, where I is the identity mapping of H.*

Lemma 6 ([25]). *Suppose that $\lambda \in (0, 1]$, $T : C \to H$ is a nonexpansive mapping, and the mapping $T^\lambda : C \to H$ is defined as $T^\lambda u := Tu - \lambda\mu F(Tu)$ $\forall u \in C$, where $F : H \to H$ is κ-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, that is, $\|T^\lambda u - T^\lambda v\| \leq (1 - \lambda\ell)\|u - v\|$ $\forall u, v \in C$, where $\ell := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.*

Lemma 7 ([26]). *Suppose that $\{\Gamma_m\}$ is a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{m_k}\}$ of $\{\Gamma_m\}$ which satisfies $\Gamma_{m_k} < \Gamma_{m_k+1}$ for each integer $k \geq 1$. Define the sequence $\{\tau(m)\}_{m \geq m_0}$ of integers as follows:*

$$\tau(m) = \max\{k \leq m : \Gamma_k < \Gamma_{k+1}\},$$

where integer $m_0 \geq 1$ such that $\{k \leq m_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following conclusions hold:

(i) $\tau(m_0) \leq \tau(m_0 + 1) \leq \cdots$ *and* $\tau(m) \to \infty$;

(ii) $\Gamma_{\tau(m)} \leq \Gamma_{\tau(m)+1}$ *and* $\Gamma_m \leq \Gamma_{\tau(m)+1}$ $\forall m \geq m_0$.

3. Iterative Algorithms and Convergence Criteria

In this section, let the feasible set C be a nonempty closed convex subset of a real Hilbert space H, and assume always that the following hold:

$T_i : H \to H$ is nonexpansive for $i = 1, ..., N$ and $T : H \to H$ is a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero;

$A : H \to H$ is L-Lipschitz continuous, pseudomonotone on H, and satisfies the condition that for $\{x_n\} \subset C$, $x_n \rightharpoonup z \Rightarrow \|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|$;

$\Omega = \cap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \neq \emptyset$ with $T_0 := T$;

$f : H \to H$ is a contraction with constant $\delta \in [0, 1)$, and $F : H \to H$ is η-strongly monotone and κ-Lipschitzian such that $\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$; $\{\zeta_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$, and $\{\tau_n\} \subset (0, \infty)$ are such that

(i) $\beta_n + \gamma_n < 1$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ and $\tau_n = \circ(\beta_n)$, i.e., $\lim_{n\to\infty} \tau_n/\beta_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ and $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < 1$.

In addition, we write $T_n := T_{n \bmod N}$ for integer $n \geq 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$, i.e., if $n = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_n = T_N$ if $q = 0$ and $T_n = T_q$ if $0 < q < N$.

Algorithm 1. Initialization: Let $\lambda_1 > 0$, $\alpha > 0$, $\mu \in (0, 1)$ and $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\{\alpha, \frac{\tau_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{5}$$

Step 2. Compute $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and $y_n = P_C(w_n - \lambda_n A w_n)$.

Step 3. Construct the half-space $C_n := \{z \in H : \langle w_n - \lambda_n A w_n - y_n, z - y_n \rangle \leq 0\}$, and compute $z_n = P_{C_n}(w_n - \lambda_n A y_n)$.

Step 4. Calculate $v_n = \zeta_n x_n + (1 - \zeta_n) T_n w_n$ and $x_{n+1} = \beta_n f(x_n) + \gamma_n T z_n + ((1 - \gamma_n)I - \beta_n \rho F) v_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A w_n - A y_n, z_n - y_n \rangle}, \lambda_n\} & \text{if } \langle A w_n - A y_n, z_n - y_n \rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{6}$$

Let $n := n+1$ and return to Step 1.

Remark 1. *It is easy to see that, from (5) we get* $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$. *Indeed, we have* $\alpha_n\|x_n - x_{n-1}\| \le \tau_n\ \forall n \ge 1$, *which together with* $\lim_{n\to\infty}\frac{\tau_n}{\beta_n} = 0$ *implies that* $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \le \frac{\tau_n}{\beta_n} \to 0$ *as* $n \to \infty$.

Lemma 8. *Let* $\{\lambda_n\}$ *be generated by (6). Then* $\{\lambda_n\}$ *is a nonincreasing sequence with* $\lambda_n \ge \lambda := \min\{\lambda_1, \frac{\mu}{L}\}\ \forall n \ge 1$, *and* $\lim_{n\to\infty}\lambda_n \ge \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$.

Proof. First, from (6) it is clear that $\lambda_n \ge \lambda_{n+1}\ \forall n \ge 1$. Furthermore, observe that

$$\left.\begin{array}{c}\frac{1}{2}(\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \ge \|w_n - y_n\|\|z_n - y_n\| \\ \langle Aw_n - Ay_n, z_n - y_n\rangle \le L\|w_n - y_n\|\|z_n - y_n\|\end{array}\right\} \Rightarrow \lambda_{n+1} \ge \min\{\lambda_n, \frac{\mu}{L}\}.$$

□

Remark 2. *In terms of Lemmas 2 and 8, we claim that if* $w_n = y_n$ *or* $Ay_n = 0$, *then* y_n *is an element of* $\mathrm{VI}(C, A)$. *Indeed, if* $w_n = y_n$ *or* $Ay_n = 0$, *then* $0 = \|y_n - P_C(y_n - \lambda_n Ay_n)\| \ge \|y_n - P_C(y_n - \lambda Ay_n)\|$. *Thus, the assertion is valid.*

The following lemmas are quite helpful for the convergence analysis of our algorithms.

Lemma 9. *Let* $\{w_n\}, \{y_n\}, \{z_n\}$ *be the sequences generated by Algorithm 1. Then*

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - (1 - \mu\frac{\lambda_n}{\lambda_{n+1}})\|w_n - y_n\|^2 - (1 - \mu\frac{\lambda_n}{\lambda_{n+1}})\|z_n - y_n\|^2\ \ \forall p \in \Omega. \quad (7)$$

Proof. First, by the definition of $\{\lambda_n\}$ we claim that

$$2\langle Aw_n - Ay_n, z_n - y_n\rangle \le \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|^2 + \frac{\mu}{\lambda_{n+1}}\|z_n - y_n\|^2\ \ \forall n \ge 1. \quad (8)$$

Indeed, if $\langle Aw_n - Ay_n, z_n - y_n\rangle \le 0$, then inequality (8) holds. Otherwise, from (6) we get (8). Furthermore, observe that for each $p \in \Omega \subset C \subset C_n$,

$$\begin{aligned}\|z_n - p\|^2 &= \|P_{C_n}(w_n - \lambda_n Ay_n) - P_{C_n}p\|^2 \le \langle z_n - p, w_n - \lambda_n Ay_n - p\rangle \\ &= \tfrac{1}{2}\|z_n - p\|^2 + \tfrac{1}{2}\|w_n - p\|^2 - \tfrac{1}{2}\|z_n - w_n\|^2 - \langle z_n - p, \lambda_n Ay_n\rangle,\end{aligned}$$

which hence yields

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle. \quad (9)$$

From $p \in \mathrm{VI}(C, A)$, we get $\langle Ap, x - p\rangle \ge 0\ \forall x \in C$. By the pseudomonotonicity of A on C we have $\langle Ax, x - p\rangle \ge 0\ \forall x \in C$. Putting $x := y_n \in C$ we get $\langle Ay_n, p - y_n\rangle \le 0$. Thus,

$$\langle Ay_n, p - z_n\rangle = \langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - z_n\rangle \le \langle Ay_n, y_n - z_n\rangle. \quad (10)$$

Substituting (10) for (9), we obtain

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle. \quad (11)$$

Since $z_n = P_{C_n}(w_n - \lambda_n Ay_n)$, we get $z_n \in C_n := \{z \in H : \langle w_n - \lambda_n Aw_n - y_n, z - y_n\rangle \le 0\}$, and hence

$$\begin{aligned}2\langle w_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle &= 2\langle w_n - \lambda_n Aw_n - y_n, z_n - y_n\rangle + 2\lambda_n\langle Aw_n - Ay_n, z_n - y_n\rangle \\ &\le 2\lambda_n\langle Aw_n - Ay_n, z_n - y_n\rangle,\end{aligned}$$

which together with (8), implies that

$$2\langle w_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle \le \mu\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 + \mu\frac{\lambda_n}{\lambda_{n+1}}\|z_n - y_n\|^2.$$

Therefore, substituting the last inequality for (11), we infer that inequality (7) holds. □

Lemma 10. *Suppose that $\{w_n\}, \{x_n\}, \{y_n\}$, and $\{z_n\}$ are bounded sequences generated by Algorithm 1. If $x_n - x_{n+1} \to 0$, $w_n - y_n \to 0$, $w_n - z_n \to 0$, $z_n - T_n z_n \to 0$, and $\exists\{w_{n_k}\} \subset \{w_n\}$ s.t. $w_{n_k} \rightharpoonup z \in H$, then $z \in \Omega$.*

Proof. Utilizing the similar arguments to those in the proof of Lemma 3.3 of [12], we can derive the desired result. □

Lemma 11. *Assume that $\{w_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ are the sequences generated by Algorithm 1. Then they all are bounded.*

Proof. Since $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$ and $0 < \liminf_{n\to\infty}\zeta_n \le \limsup_{n\to\infty}\zeta_n < 1$, we may assume, without loss of generality, that

$$\{\gamma_n\} \subset [a,b] \subset (0,1) \quad \text{and} \quad \{\zeta_n\} \subset [c,d] \subset (0,1). \tag{12}$$

Choose a fixed $p \in \Omega$ arbitrarily. Then we obtain $Tp = p$ and $T_n p = p$ for all $n \ge 1$, and (7) holds. Noticing $\lim_{n\to\infty}(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}) = 1 - \mu > 0$, we might assume that $1 - \mu\frac{\lambda_n}{\lambda_{n+1}} > 0$ for all $n \ge 1$. So it follows from (7) that for all $n \ge 1$,

$$\|z_n - p\| \le \|w_n - p\|. \tag{13}$$

Furthermore, note that

$$\|w_n - p\| \le \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\| = \|x_n - p\| + \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|. \tag{14}$$

In terms of Remark 1, one has $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Hence we deduce that $\exists M_1 > 0$ s.t.

$$M_1 \ge \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \quad \forall n \ge 1. \tag{15}$$

Using (13)–(15), we obtain that for all $n \ge 1$,

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\| + \beta_n M_1. \tag{16}$$

Noticing $\beta_n + \gamma_n < 1\ \forall n \ge 1$, we have $\frac{\beta_n}{1-\gamma_n} < 1$ for all $n \ge 1$. So, using Lemma 6 and (16) we deduce that

$$\begin{aligned}\|v_n - p\| &\le \zeta_n\|x_n - p\| + (1-\zeta_n)\|T_n w_n - p\| \\ &\le \zeta_n\|x_n - p\| + (1-\zeta_n)\|w_n - p\| \\ &\le \zeta_n(\|x_n - p\| + \beta_n M_1) + (1-\zeta_n)(\|x_n - p\| + \beta_n M_1) \\ &= \|x_n - p\| + \beta_n M_1,\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\beta_n f(x_n)+\gamma_n Tz_n+((1-\gamma_n)I-\beta_n\rho F)v_n-p\| \\
&\le \beta_n\|f(x_n)-p\|+\gamma_n\|Tz_n-p\| \\
&\quad +(1-\beta_n-\gamma_n)\|(\tfrac{1-\gamma_n}{1-\beta_n-\gamma_n}I-\tfrac{\beta_n}{1-\beta_n-\gamma_n}\rho F)v_n-p\| \\
&\le \beta_n(\|f(x_n)-f(p)\|+\|f(p)-p\|)+\gamma_n\|z_n-p\| \\
&\quad +(1-\beta_n-\gamma_n)\|(\tfrac{1-\gamma_n}{1-\beta_n-\gamma_n}I-\tfrac{\beta_n}{1-\beta_n-\gamma_n}\rho F)v_n-p\| \\
&\le \beta_n(\delta\|x_n-p\|+\|f(p)-p\|)+\gamma_n\|z_n-p\| \\
&\quad +(1-\gamma_n)\|(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)v_n-(1-\tfrac{\beta_n}{1-\gamma_n})p\| \\
&= \beta_n(\delta\|x_n-p\|+\|f(p)-p\|)+\gamma_n\|z_n-p\| \\
&\quad +(1-\gamma_n)\|(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)v_n-(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)p+\tfrac{\beta_n}{1-\gamma_n}(I-\rho F)p\| \\
&\le \beta_n(\delta\|x_n-p\|+\|f(p)-p\|)+\gamma_n\|z_n-p\| \\
&\quad +(1-\gamma_n)[(1-\tfrac{\beta_n}{1-\gamma_n}\tau)\|v_n-p\|+\tfrac{\beta_n}{1-\gamma_n}\|(I-\rho F)p\|] \\
&= \beta_n(\delta\|x_n-p\|+\|f(p)-p\|)+\gamma_n\|z_n-p\| \\
&\quad +(1-\gamma_n-\beta_n\tau)\|v_n-p\|+\beta_n\|(I-\rho F)p\| \\
&\le \beta_n\delta(\|x_n-p\|+\beta_n M_1)+\beta_n\|f(p)-p\|+\gamma_n(\|x_n-p\|+\beta_n M_1) \\
&\quad +(1-\gamma_n-\beta_n\tau)(\|x_n-p\|+\beta_n M_1)+\beta_n\|(I-\rho F)p\| \\
&\le [1-\beta_n(\tau-\delta)]\|x_n-p\|+\beta_n(M_1+\|f(p)-p\|+\|(I-\rho F)p\|) \\
&= [1-\beta_n(\tau-\delta)]\|x_n-p\|+\beta_n(\tau-\delta)\cdot\tfrac{M_1+\|f(p)-p\|+\|(I-\rho F)p\|}{\tau-\delta} \\
&\le \max\{\|x_n-p\|,\tfrac{M_1+\|f(p)-p\|+\|(I-\rho F)p\|}{\tau-\delta}\}.
\end{aligned}$$

By induction, we obtain $\|x_n-p\|\le\max\{\|x_1-p\|,\frac{M_1+\|f(p)-p\|+\|(I-\rho F)p\|}{\tau-\delta}\}\ \forall n\ge 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{w_n\}, \{y_n\}, \{z_n\}, \{Tz_n\}, \{Fv_n\}, \{T_n w_n\}$. □

Theorem 1. *Let the sequence $\{x_n\}$ be constructed by Algorithm 1. Then $\{x_n\}$ converges strongly to the unique solution $x^*\in\Omega$ of the following VIP:*

$$\langle(\rho F-f)x^*, p-x^*\rangle\ge 0 \quad \forall p\in\Omega.$$

Proof. First, it is not difficult to show that $P_\Omega(f+I-\rho F)$ is a contraction. In fact, by Lemma 6 and the Banach contraction mapping principle, we obtain that $P_\Omega(f+I-\rho F)$ has a unique fixed point. Say $x^*\in H$, i.e., $x^*=P_\Omega(f+I-\rho F)x^*$. Thus, the following VIP has only a solution $x^*\in\Omega$:

$$\langle(\rho F-f)x^*, p-x^*\rangle\ge 0 \quad \forall p\in\Omega. \tag{17}$$

□

We now claim that

$$\gamma_n(1-\mu\frac{\lambda_n}{\lambda_{n+1}})[\|w_n-y_n\|^2+\|z_n-y_n\|^2]\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\beta_n M_4,$$

for some $M_4>0$. In fact, observe that

$$\begin{aligned}
&x_{n+1}-x^*=\beta_n(f(x_n)-x^*)+\gamma_n(Tz_n-x^*)+(1-\beta_n-\gamma_n)\{\tfrac{1-\gamma_n}{1-\beta_n-\gamma_n}[(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)v_n \\
&\quad -(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)x^*]+\tfrac{\beta_n}{1-\beta_n-\gamma_n}(I-\rho F)x^*\} \\
&=\beta_n(f(x_n)-f(x^*))+\gamma_n(Tz_n-x^*)+(1-\gamma_n)[(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)v_n-(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)x^*] \\
&\quad +\beta_n(f-\rho F)x^*.
\end{aligned}$$

Using Lemma 6 and the convexity of the function $h(t)=t^2\ \forall t\in\mathbf{R}$, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\le \|\beta_n(f(x_n)-f(x^*))+\gamma_n(Tz_n-x^*)+(1-\gamma_n)[(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)v_n-(I-\tfrac{\beta_n}{1-\gamma_n}\rho F)x^*]\|^2 \\
&\quad +2\beta_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle \\
&\le \beta_n\delta\|x_n-x^*\|^2+\gamma_n\|z_n-x^*\|^2+(1-\beta_n\tau-\gamma_n)\|v_n-x^*\|^2+\beta_n M_2
\end{aligned} \tag{18}$$

where $M_2 \ge \sup_{n\ge1} 2\|f-\rho F)x^*\|\|x_n-x^*\|$ for some $M_2>0$. From (7) and (17), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 \le \beta_n\delta\|x_n-x^*\|^2+\gamma_n[\|w_n-x^*\|^2-(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\|^2-(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})\|z_n-y_n\|^2] \\
+(1-\beta_n\tau-\gamma_n)[\zeta_n\|x_n-x^*\|^2+(1-\zeta_n)\|w_n-x^*\|^2]+\beta_n M_2.
\end{aligned} \tag{19}$$

Again from (16), we obtain

$$\|w_n-x^*\|^2 \le (\|x_n-x^*\|+\beta_n M_1)^2 \le \|x_n-x^*\|^2+\beta_n M_3, \tag{20}$$

where $M_3 \ge \sup_{n\ge1}(2M_1\|x_n-x^*\|+\beta_n M_1^2)$ for some $M_3>0$. Using (19) and (20), we get

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\le [1-\beta_n(\tau-\delta)](\|x_n-x^*\|^2+\beta_n M_3)-\gamma_n(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})[\|w_n-y_n\|^2+\|z_n-y_n\|^2]+\beta_n M_2 \\
&\le \|x_n-x^*\|^2-\gamma_n(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})[\|w_n-y_n\|^2+\|z_n-y_n\|^2]+\beta_n M_4,
\end{aligned}$$

where $M_4 := M_2+M_3$. Consequently,

$$\gamma_n(1-\mu\frac{\lambda_n}{\lambda_{n+1}})[\|w_n-y_n\|^2+\|z_n-y_n\|^2] \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\beta_n M_4. \tag{21}$$

Next we claim that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 \quad &\le [1-\beta_n(\tau-\delta)]\|x_n-x^*\|^2 \\
&\quad +\beta_n(\tau-\delta)[\tfrac{2}{\tau-\delta}\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle+\tfrac{3M}{\tau-\delta}\cdot\tfrac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|]
\end{aligned}$$

for some $M>0$. In fact, it is easy to see that

$$\|w_n-x^*\|^2 \le \|x_n-x^*\|^2+\alpha_n\|x_n-x_{n-1}\|[2\|x_n-x^*\|+\alpha_n\|x_n-x_{n-1}\|]. \tag{22}$$

Using (16), (18), and (22), we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\delta\|x_n-x^*\|^2+\gamma_n\|w_n-x^*\|^2+(1-\beta_n\tau-\gamma_n)[\zeta_n\|x_n-x^*\|^2 \\
&\quad +(1-\zeta_n)\|w_n-x^*\|^2]+2\beta_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle \\
&\le \beta_n\delta\|x_n-x^*\|^2+\gamma_n[\|x_n-x^*\|^2+\alpha_n\|x_n-x_{n-1}\|(2\|x_n-x^*\|+\alpha_n\|x_n-x_{n-1}\|)] \\
&\quad +(1-\beta_n\tau-\gamma_n)\{\zeta_n\|x_n-x^*\|^2+(1-\zeta_n)[\|x_n-x^*\|^2+\alpha_n\|x_n-x_{n-1}\|(2\|x_n-x^*\| \\
&\quad +\alpha_n\|x_n-x_{n-1}\|)]\}+2\beta_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle \\
&\le [1-\beta_n(\tau-\delta)]\|x_n-x^*\|^2+\beta_n(\tau-\delta)\cdot[\tfrac{2\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle}{\tau-\delta}+\tfrac{3M}{\tau-\delta}\cdot\tfrac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|],
\end{aligned} \tag{23}$$

where $M \ge \sup_{n\ge1}\{\|x_n-x^*\|, \alpha_n\|x_n-x_{n-1}\|\}$ for some $M>0$.

For each $n \ge 0$, we set

$$\begin{aligned}
&\Gamma_n = \|x_n-x^*\|^2, \\
&\varepsilon_n = \beta_n(\tau-\delta), \\
&\vartheta_n = \alpha_n\|x_n-x_{n-1}\|3M+2\beta_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle.
\end{aligned}$$

Then (23) can be rewritten as the following formula:

$$\Gamma_{n+1} \le (1-\varepsilon_n)\Gamma_n+\vartheta_n \quad \forall n \ge 0. \tag{24}$$

We next show the convergence of $\{\Gamma_n\}$ to zero by the following two cases:

Case 1. *Suppose that there exists an integer* $n_0 \geq 1$ *such that* $\{\Gamma_n\}$ *is non-increasing.* Then

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

From (21), we get

$$\gamma_n(1-\mu\frac{\lambda_n}{\lambda_{n+1}})[\|w_n - y_n\|^2 + \|z_n - y_n\|^2] \leq \Gamma_n - \Gamma_{n+1} + \beta_n M_4.$$

Since $\beta_n \to 0$, $\Gamma_n - \Gamma_{n+1} \to 0$, $1-\mu\frac{\lambda_n}{\lambda_{n+1}} \to 1-\mu$ and $\{\gamma_n\} \subset [a,b] \subset (0,1)$, we have

$$\lim_{n\to\infty} \|w_n - y_n\| = \lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{25}$$

Using Lemma 1 (v), we deduce from (16) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\beta_n f(x_n) + \gamma_n T z_n + ((1-\gamma_n)I - \beta_n \rho F)v_n - x^*\|^2 \\
&= \|\beta_n(f(x_n) - \rho F v_n) + \gamma_n(Tz_n - x^*) + (1-\gamma_n)(v_n - x^*)\|^2 \\
&\leq \|\gamma_n(Tz_n - x^*) + (1-\gamma_n)(v_n - x^*)\|^2 + 2\beta_n\langle f(x_n) - \rho F v_n, x_{n+1} - x^*\rangle \\
&= \gamma_n\|Tz_n - x^*\|^2 + (1-\gamma_n)\|v_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|Tz_n - v_n\|^2 \\
&\quad + 2\beta_n\langle f(x_n) - \rho F v_n, x_{n+1} - x^*\rangle \\
&= \gamma_n\|Tz_n - x^*\|^2 + (1-\gamma_n)[\zeta_n\|x_n - x^*\|^2 + (1-\zeta_n)\|T_n w_n - x^*\|^2 - \zeta_n(1-\zeta_n)\|x_n - T_n w_n\|^2] \\
&\quad - \gamma_n(1-\gamma_n)\|Tz_n - v_n\|^2 + 2\beta_n\langle f(x_n) - \rho F v_n, x_{n+1} - x^*\rangle \\
&\leq \gamma_n\|z_n - x^*\|^2 + (1-\gamma_n)[\zeta_n\|x_n - x^*\|^2 + (1-\zeta_n)\|w_n - x^*\|^2 - \zeta_n(1-\zeta_n)\|x_n - T_n w_n\|^2] \\
&\quad - \gamma_n(1-\gamma_n)\|Tz_n - v_n\|^2 + 2\beta_n\langle f(x_n) - \rho F v_n, x_{n+1} - x^*\rangle \\
&\leq \gamma_n(\|x_n - x^*\| + \beta_n M_1)^2 + (1-\gamma_n)(\|x_n - x^*\| + \beta_n M_1)^2 - (1-\gamma_n)\zeta_n(1-\zeta_n)\|x_n - T_n w_n\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|Tz_n - v_n\|^2 + 2\beta_n\|f(x_n) - \rho F v_n\|\|x_{n+1} - x^*\| \\
&= (\|x_n - x^*\| + \beta_n M_1)^2 - (1-\gamma_n)\zeta_n(1-\zeta_n)\|x_n - T_n w_n\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|Tz_n - v_n\|^2 + 2\beta_n\|f(x_n) - \rho F v_n\|\|x_{n+1} - x^*\|,
\end{aligned}$$

which immediately yields

$$\begin{aligned}
&(1-\gamma_n)\zeta_n(1-\zeta_n)\|x_n - T_n w_n\|^2 + \gamma_n(1-\gamma_n)\|Tz_n - v_n\|^2 \\
&\leq (\|x_n - x^*\| + \beta_n M_1)^2 - \|x_{n+1} - x^*\|^2 + 2\beta_n\|f(x_n) - \rho F v_n\|\|x_{n+1} - x^*\| \\
&= \Gamma_n - \Gamma_{n+1} + \beta_n M_1(2\|x_n - x^*\| + \beta_n M_1) + 2\beta_n\|f(x_n) - \rho F v_n\|\|x_{n+1} - x^*\|.
\end{aligned}$$

Since $\beta_n \to 0$, $\Gamma_n - \Gamma_{n+1} \to 0$, $\{\gamma_n\} \subset [a,b] \subset (0,1)$ and $\{\zeta_n\} \subset [c,d] \subset (0,1)$, we have

$$\lim_{n\to\infty} \|x_n - T_n w_n\| = \lim_{n\to\infty} \|Tz_n - v_n\| = 0. \tag{26}$$

Using Lemma 1 (v) again, we have

$$\begin{aligned}
\|Tz_n - v_n\|^2 &= \|\zeta_n(Tz_n - x_n) + (1-\zeta_n)(Tz_n - T_n w_n)\|^2 \\
&= \zeta_n\|Tz_n - x_n\|^2 + (1-\zeta_n)\|Tz_n - T_n w_n\|^2 - \zeta_n(1-\zeta_n)\|T_n w_n - x_n\|^2.
\end{aligned}$$

So it follows from (26) and $\{\zeta_n\} \subset [c,d] \subset (0,1)$ that

$$\lim_{n\to\infty} \|Tz_n - x_n\| = \lim_{n\to\infty} \|Tz_n - T_n w_n\| = 0. \tag{27}$$

Therefore, from (25)–(27), we conclude that

$$\|w_n - z_n\| \leq \|w_n - y_n\| + \|y_n - z_n\| \to 0 \quad (n \to \infty), \tag{28}$$

$$\begin{aligned}
\|z_n - T_n z_n\| &\leq \|z_n - w_n\| + \|w_n - x_n\| + \|x_n - T_n w_n\| + \|T_n w_n - T_n z_n\| \\
&\leq 2\|z_n - w_n\| + \|w_n - x_n\| + \|x_n - T_n w_n\| \to 0 \quad (n \to \infty),
\end{aligned} \tag{29}$$

and

$$
\begin{aligned}
&\|x_{n+1}-x_n\| = \|\beta_n f(x_n)+\gamma_n Tz_n+((1-\gamma_n)I-\beta_n\rho F)v_n-x_n\| \\
&= \|\beta_n(f(x_n)-\rho Fv_n)+\gamma_n(Tz_n-x_n)+(1-\gamma_n)(v_n-x_n)\| \\
&\le \beta_n\|f(x_n)-\rho Fv_n\|+\gamma_n\|Tz_n-x_n\|+(1-\gamma_n)\|v_n-x_n\| \\
&\le \beta_n(\|f(x_n)\|+\|\rho Fv_n\|)+\gamma_n\|Tz_n-x_n\|+(1-\gamma_n)(\|v_n-Tz_n\|+\|Tz_n-x_n\|) \\
&\le \beta_n(\|f(x_n)\|+\|\rho Fv_n\|)+\|Tz_n-x_n\|+\|v_n-Tz_n\| \to 0 \quad (n\to\infty).
\end{aligned} \tag{30}
$$

Next, by the boundedness of $\{x_n\}$, we know that $\exists\{x_{n_k}\}\subset\{x_n\}$ s.t.

$$
\limsup_{n\to\infty}\langle(f-\rho F)x^*, x_n-x^*\rangle = \lim_{k\to\infty}\langle(f-\rho F)x^*, x_{n_k}-x^*\rangle. \tag{31}
$$

Further we might assume that $x_{n_k}\rightharpoonup\hat{x}$. So, from (31) we have

$$
\limsup_{n\to\infty}\langle(f-\rho F)x^*, x_n-x^*\rangle = \langle(f-\rho F)x^*, \hat{x}-x^*\rangle. \tag{32}
$$

Noticing $w_n-x_n\to 0$ and $x_{n_k}\rightharpoonup\hat{x}$, we obtain $w_{n_k}\rightharpoonup\hat{x}$. Since $x_n-x_{n+1}\to 0$, $w_n-y_n\to 0$, $w_n-z_n\to 0$, $z_n-T_nz_n\to 0$ (due to (25) and (28)–(30)) and $w_{n_k}\rightharpoonup\hat{x}$, by Lemma 10 we get $\hat{x}\in\Omega$. So it follows from (17) and (32) that

$$
\limsup_{n\to\infty}\langle(f-\rho F)x^*, x_n-x^*\rangle = \langle(f-\rho F)x^*, \hat{x}-x^*\rangle \le 0, \tag{33}
$$

which hence yields

$$
\begin{aligned}
&\limsup_{n\to\infty}\langle(f-\rho F)x^*, x_{n+1}-x^*\rangle \\
&\le \limsup_{n\to\infty}[\|(f-\rho F)x^*\|\|x_{n+1}-x_n\|+\langle(f-\rho F)x^*, x_n-x^*\rangle] \le 0.
\end{aligned} \tag{34}
$$

Since $\{\beta_n(\tau-\delta)\}\subset[0,1]$, $\sum_{n=1}^{\infty}\beta_n(\tau-\delta)=\infty$, and

$$
\limsup_{n\to\infty}[\frac{2\langle(f-\rho F)x^*, x_{n+1}-x^*\rangle}{\tau-\delta}+\frac{3M}{\tau-\delta}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|] \le 0,
$$

by Lemma 4 we conclude from (23) that $\lim_{n\to 0}\|x_n-x^*\|=0$.

Case 2. *Suppose that* $\exists\{\Gamma_{n_k}\}\subset\{\Gamma_n\}$ *s.t.* $\Gamma_{n_k}<\Gamma_{n_k+1}$ $\forall k\in\mathbf{N}$, *where* $\mathbf{N}$ *is the set of all positive integers.* Define the mapping $\tau:\mathbf{N}\to\mathbf{N}$ by

$$
\tau(n) := \max\{k\le n : \Gamma_k<\Gamma_{k+1}\}.
$$

Using Lemma 7, we have

$$
\Gamma_{\tau(n)}\le\Gamma_{\tau(n)+1} \quad \text{and} \quad \Gamma_n\le\Gamma_{\tau(n)+1}.
$$

Putting $\Gamma_n=\|x_n-x^*\|^2$ $\forall n\in\mathbf{N}$ and using the same inference as in Case 1, we can obtain

$$
\lim_{n\to\infty}\|x_{\tau(n)+1}-x_{\tau(n)}\| = 0 \tag{35}
$$

and

$$
\limsup_{n\to\infty}\langle(f-\rho F)x^*, x_{\tau(n)+1}-x^*\rangle \le 0. \tag{36}
$$

Because of $\Gamma_{\tau(n)}\le\Gamma_{\tau(n)+1}$ and $\beta_{\tau(n)}>0$, we conclude from (23) that

$$
\|x_{\tau(n)}-x^*\|^2 \quad \le \tfrac{2}{\tau-\delta}\langle(f-\rho F)x^*, x_{\tau(n)+1}-x^*\rangle+\tfrac{3M}{\tau-\delta}\cdot\tfrac{\alpha_{\tau(n)}}{\beta_{\tau(n)}}\cdot\|x_{\tau(n)}-x_{\tau(n)-1}\|,
$$

and hence

$$
\limsup_{n\to\infty}\|x_{\tau(n)}-x^*\|^2 \le 0.
$$

Thus, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0.$$

Using (35), we obtain

$$\begin{aligned}
&\|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \\
&= 2\langle x_{\tau(n)+1} - x_{\tau(n)}, x_{\tau(n)} - x^*\rangle + \|x_{\tau(n)+1} - x_{\tau(n)}\|^2 \\
&\le 2\|x_{\tau(n)+1} - x_{\tau(n)}\|\|x_{\tau(n)} - x^*\| + \|x_{\tau(n)+1} - x_{\tau(n)}\|^2 \to 0 \quad (n \to \infty).
\end{aligned}$$

Taking into account $\Gamma_n \le \Gamma_{\tau(n)+1}$, we have

$$\begin{aligned}
&\|x_n - x^*\|^2 \le \|x_{\tau(n)+1} - x^*\|^2 \\
&\le \|x_{\tau(n)} - x^*\|^2 + 2\|x_{\tau(n)+1} - x_{\tau(n)}\|\|x_{\tau(n)} - x^*\| + \|x_{\tau(n)+1} - x_{\tau(n)}\|^2.
\end{aligned}$$

It is easy to see from (35) that $x_n \to x^*$ as $n \to \infty$. This completes the proof.

Next, we introduce another Mann-type inertial subgradient extragradient algorithm.

Algorithm 2. Initialization: Let $\lambda_1 > 0$, $\alpha > 0$, $\mu \in (0,1)$ and $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \ge 1)$, choose α_n such that $0 \le \alpha_n \le \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\{\alpha, \frac{\tau_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{37}$$

Step 2. Compute $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and $y_n = P_C(w_n - \lambda_n A w_n)$.

Step 3. Construct the half-space $C_n := \{z \in H : \langle w_n - \lambda_n A w_n - y_n, z - y_n\rangle \le 0\}$, and compute $z_n = P_{C_n}(w_n - \lambda_n A y_n)$.

Step 4. Calculate $v_n = \zeta_n x_n + (1 - \zeta_n)T z_n$ and $x_{n+1} = \beta_n f(x_n) + \gamma_n T_n w_n + ((1 - \gamma_n)I - \beta_n \rho F)v_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A w_n - A y_n, z_n - y_n\rangle}, \lambda_n\} & \text{if } \langle A w_n - A y_n, z_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{38}$$

Let $n := n + 1$ and return to Step 1.

It is worth pointing out that Lemmas 8–11 are still valid for Algorithm 2.

Theorem 2. *Let the sequence $\{x_n\}$ be constructed by Algorithm 2. Then $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the following VIP:*

$$\langle(\rho F - f)x^*, p - x^*\rangle \ge 0 \quad \forall p \in \Omega.$$

Proof. Utilizing the same arguments as in the proof of Theorem 1, we deduce that there exists a unique solution $x^* \in \Omega = \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ to the VIP (17). □

We now claim that

$$(1 - \beta_n\tau - \gamma_n)(1 - \zeta_n)(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})[\|w_n - y_n\|^2 + \|z_n - y_n\|^2] \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n M_4, \tag{39}$$

for some $M_4 > 0$. In fact, observe that

$$\begin{aligned}
x_{n+1} - x^* &= \beta_n(f(x_n) - f(x^*)) + \gamma_n(T_n w_n - x^*) \\
&\quad + (1 - \gamma_n)[(I - \tfrac{\beta_n}{1-\gamma_n}\rho F)v_n - (I - \tfrac{\beta_n}{1-\gamma_n}\rho F)x^*] + \beta_n(f - \rho F)x^*,
\end{aligned}$$

where $v_n := \zeta_n x_n + (1-\zeta_n)Tz_n$. Using the similar arguments to those of (19) and (20), we have

$$\|x_{n+1}-x^*\|^2 \le \beta_n\delta\|x_n-x^*\|^2 + \gamma_n\|w_n-x^*\|^2 + (1-\beta_n\tau-\gamma_n)\{\zeta_n\|x_n-x^*\|^2 \\ + (1-\zeta_n)[\|w_n-x^*\|^2 - (1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\|^2 - (1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})\|z_n-y_n\|^2]\} + \beta_n M_2.$$

and

$$\|w_n-x^*\|^2 \le (\|x_n-x^*\| + \beta_n M_1)^2 \le \|x_n-x^*\|^2 + \beta_n M_3,$$

where $M_2 \ge \sup_{n\ge1} 2\|(f-\rho F)x^*\|\|x_n-x^*\|$ for some $M_2>0$ and $M_3 \ge \sup_{n\ge1}(2M_1\|x_n-x^*\| + \beta_n M_1^2)$ for some $M_3>0$. Combining the last inequalities, we obtain

$$\begin{aligned} &\|x_{n+1}-x^*\|^2 \\ &\le \beta_n\delta\|x_n-x^*\|^2 + \gamma_n(\|x_n-x^*\|^2+\beta_n M_3) + (1-\beta_n\tau-\gamma_n)(\|x_n-x^*\|^2+\beta_n M_3) \\ &\quad - (1-\beta_n\tau-\gamma_n)(1-\zeta_n)[(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\|^2 + (1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})\|z_n-y_n\|^2] + \beta_n M_2 \\ &\le \|x_n-x^*\|^2 - (1-\beta_n\tau-\gamma_n)(1-\zeta_n)(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})[\|w_n-y_n\|^2 + \|z_n-y_n\|^2] + \beta_n M_4, \end{aligned}$$

where $M_4 := M_2 + M_3$. This ensures that (39) holds.

Next we claim that

$$\begin{aligned} \|x_{n+1}-x^*\|^2 \quad &\le [1-\beta_n(\tau-\delta)]\|x_n-x^*\|^2 \\ &\quad + \beta_n(\tau-\delta)[\tfrac{2}{\tau-\delta}\langle(f-\rho F)x^*, x_{n+1}-x^*\rangle + \tfrac{3M}{\tau-\delta}\cdot\tfrac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|] \end{aligned} \tag{40}$$

for some $M>0$. In fact, using the similar arguments to those of (22) and (23), we have

$$\|w_n-x^*\|^2 \le \|x_n-x^*\|^2 + \alpha_n\|x_n-x_{n-1}\|[2\|x_n-x^*\| + \alpha_n\|x_n-x_{n-1}\|],$$

and

$$\begin{aligned} &\|x_{n+1}-x^*\|^2 \\ &\le \beta_n\delta\|x_n-x^*\|^2 + \gamma_n\|w_n-x^*\|^2 + (1-\beta_n\tau-\gamma_n)[\zeta_n\|x_n-x^*\|^2 + (1-\zeta_n)\|z_n-x^*\|^2] \\ &\quad + 2\beta_n\langle(f-\rho F)x^*, x_{n+1}-x^*\rangle \\ &\le \beta_n\delta\|x_n-x^*\|^2 + (1-\beta_n\tau)[\|x_n-x^*\|^2 + \alpha_n\|x_n-x_{n-1}\|(2\|x_n-x^*\| \\ &\quad + \alpha_n\|x_n-x_{n-1}\|)] + 2\beta_n\langle(f-\rho F)x^*, x_{n+1}-x^*\rangle \\ &\le [1-\beta_n(\tau-\delta)]\|x_n-x^*\|^2 + \alpha_n\|x_n-x_{n-1}\|(2\|x_n-x^*\| + \alpha_n\|x_n-x_{n-1}\|) \\ &\quad + 2\beta_n\langle(f-\rho F)x^*, x_{n+1}-x^*\rangle \\ &\le [1-\beta_n(\tau-\delta)]\|x_n-x^*\|^2 + \beta_n(\tau-\delta)\cdot[\tfrac{2\langle(f-\rho F)x^*, x_{n+1}-x^*\rangle}{\tau-\delta} + \tfrac{3M}{\tau-\delta}\cdot\tfrac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|], \end{aligned} \tag{41}$$

where $M \ge \sup_{n\ge1}\{\|x_n-x^*\|, \alpha_n\|x_n-x_{n-1}\|\}$ for some $M>0$.

For each $n\ge0$, we set

$$\begin{aligned} &\Gamma_n = \|x_n-x^*\|^2, \\ &\varepsilon_n = \beta_n(\tau-\delta), \\ &\vartheta_n = \alpha_n\|x_n-x_{n-1}\|3M + 2\beta_n\langle(f-\rho F)x^*, x_{n+1}-x^*\rangle. \end{aligned}$$

Then (41) can be rewritten as the following formula:

$$\Gamma_{n+1} \le (1-\varepsilon_n)\Gamma_n + \vartheta_n \quad \forall n\ge0. \tag{42}$$

We next show the convergence of $\{\Gamma_n\}$ to zero by the following two cases:

Case 3. *Suppose that there exists an integer $n_0\ge1$ such that $\{\Gamma_n\}$ is non-increasing.* Then

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

Using the similar arguments to those of (25), we have

$$\lim_{n\to\infty}\|w_n-y_n\| = \lim_{n\to\infty}\|z_n-y_n\| = 0. \tag{43}$$

Using Lemma 1 (v), we get

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&=\|\beta_n(f(x_n)-\rho Fv_n)+\gamma_n(T_nw_n-x^*)+(1-\gamma_n)(v_n-x^*)\|^2\\
&\le\|\gamma_n(T_nw_n-x^*)+(1-\gamma_n)(v_n-x^*)\|^2+2\beta_n\langle f(x_n)-\rho Fv_n,x_{n+1}-x^*\rangle\\
&=\gamma_n\|T_nw_n-x^*\|^2+(1-\gamma_n)\|v_n-x^*\|^2-\gamma_n(1-\gamma_n)\|T_nw_n-v_n\|^2\\
&\quad+2\beta_n\langle f(x_n)-\rho Fv_n,x_{n+1}-x^*\rangle\\
&=\gamma_n\|T_nw_n-x^*\|^2+(1-\gamma_n)[\zeta_n\|x_n-x^*\|^2+(1-\zeta_n)\|Tz_n-x^*\|^2-\zeta_n(1-\zeta_n)\|x_n-Tz_n\|^2]\\
&\quad-\gamma_n(1-\gamma_n)\|T_nw_n-v_n\|^2+2\beta_n\langle f(x_n)-\rho Fv_n,x_{n+1}-x^*\rangle\\
&\le\gamma_n\|w_n-x^*\|^2+(1-\gamma_n)[\zeta_n\|x_n-x^*\|^2+(1-\zeta_n)\|z_n-x^*\|^2-\zeta_n(1-\zeta_n)\|x_n-Tz_n\|^2]\\
&\quad-\gamma_n(1-\gamma_n)\|T_nw_n-v_n\|^2+2\beta_n\langle f(x_n)-\rho Fv_n,x_{n+1}-x^*\rangle\\
&\le\gamma_n(\|x_n-x^*\|+\beta_nM_1)^2+(1-\gamma_n)(\|x_n-x^*\|+\beta_nM_1)^2-(1-\gamma_n)\zeta_n(1-\zeta_n)\|x_n-Tz_n\|^2\\
&\quad-\gamma_n(1-\gamma_n)\|T_nw_n-v_n\|^2+2\beta_n\|f(x_n)-\rho Fv_n\|\|x_{n+1}-x^*\|\\
&=(\|x_n-x^*\|+\beta_nM_1)^2-(1-\gamma_n)\zeta_n(1-\zeta_n)\|x_n-Tz_n\|^2\\
&\quad-\gamma_n(1-\gamma_n)\|T_nw_n-v_n\|^2+2\beta_n\|f(x_n)-\rho Fv_n\|\|x_{n+1}-x^*\|,
\end{aligned}
$$

which immediately yields

$$
\begin{aligned}
&(1-\gamma_n)\zeta_n(1-\zeta_n)\|x_n-Tz_n\|^2+\gamma_n(1-\gamma_n)\|T_nw_n-v_n\|^2\\
&\le(\|x_n-x^*\|+\beta_nM_1)^2-\|x_{n+1}-x^*\|^2+2\beta_n\|f(x_n)-\rho Fv_n\|\|x_{n+1}-x^*\|\\
&=\Gamma_n-\Gamma_{n+1}+\beta_nM_1(2\|x_n-x^*\|+\beta_nM_1)+2\beta_n\|f(x_n)-\rho Fv_n\|\|x_{n+1}-x^*\|.
\end{aligned}
$$

Since $\beta_n\to0,\ \Gamma_n-\Gamma_{n+1}\to0,\ \{\gamma_n\}\subset[a,b]\subset(0,1)$ and $\{\zeta_n\}\subset[c,d]\subset(0,1)$, we have

$$\lim_{n\to\infty}\|x_n-Tz_n\|=\lim_{n\to\infty}\|T_nw_n-v_n\|=0. \tag{44}$$

Note that

$$
\begin{aligned}
\|T_nw_n-v_n\|^2&=\|\zeta_n(T_nw_n-x_n)+(1-\zeta_n)(T_nw_n-Tz_n)\|^2\\
&=\zeta_n\|T_nw_n-x_n\|^2+(1-\zeta_n)\|T_nw_n-Tz_n\|^2-\zeta_n(1-\zeta_n)\|Tz_n-x_n\|^2.
\end{aligned}
$$

Hence, from (44) we have

$$\lim_{n\to\infty}\|T_nw_n-x_n\|=\lim_{n\to\infty}\|T_nw_n-Tz_n\|=0. \tag{45}$$

So, from (43)–(45) we infer that

$$\|w_n-z_n\|\le\|w_n-y_n\|+\|y_n-z_n\|\to0\quad(n\to\infty), \tag{46}$$

$$
\begin{aligned}
\|z_n-T_nz_n\|&\le\|z_n-w_n\|+\|w_n-x_n\|+\|x_n-T_nw_n\|+\|T_nw_n-T_nz_n\|\\
&\le2\|z_n-w_n\|+\|w_n-x_n\|+\|x_n-T_nw_n\|\to0\quad(n\to\infty),
\end{aligned} \tag{47}
$$

and

$$
\begin{aligned}
&\|x_{n+1}-x_n\|=\|\beta_nf(x_n)+\gamma_nT_nw_n+((1-\gamma_n)I-\beta_n\rho F)v_n-x_n\|\\
&=\|\beta_n(f(x_n)-\rho Fv_n)+\gamma_n(T_nw_n-x_n)+(1-\gamma_n)(v_n-x_n)\|\\
&\le\beta_n\|f(x_n)-\rho Fv_n\|+\gamma_n\|T_nw_n-x_n\|+(1-\gamma_n)\|v_n-x_n\|\\
&\le\beta_n(\|f(x_n)\|+\|\rho Fv_n\|)+\gamma_n\|T_nw_n-x_n\|+(1-\gamma_n)(\|v_n-T_nw_n\|+\|T_nw_n-x_n\|)\\
&\le\beta_n(\|f(x_n)\|+\|\rho Fv_n\|)+\|T_nw_n-x_n\|+\|v_n-T_nw_n\|\to0\quad(n\to\infty).
\end{aligned} \tag{48}
$$

In addition, using the similar arguments to those of (33) and (34), we have

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_n-x^*\rangle\le0,$$

and hence

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_{n+1}-x^*\rangle\le0.$$

Consequently, applying Lemma 4 to (41), we have $\lim_{n\to0}\|x_n-x^*\|=0$.

Case 4. *Suppose that* $\exists\{\Gamma_{n_k}\} \subset \{\Gamma_n\}$ *s.t.* $\Gamma_{n_k} < \Gamma_{n_k+1}$ $\forall k \in \mathbf{N}$, *where* $\mathbf{N}$ *is the set of all positive integers.* Define the mapping $\tau : \mathbf{N} \to \mathbf{N}$ by $\tau(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}$. In the remainder of the proof, using the same arguments as in Case 2 of the proof of Theorem 1, we obtain the desired assertion. This completes the proof.

It is markable that our results improve and extend the corresponding results of Kraikaew and Saejung [20] and Ceng et al. [11], in the following aspects.

(i) Our problem of finding an element of $\cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ includes as a special case the problem of finding an element of $\mathrm{VI}(C, A)$ in [20], where $T_1, ..., T_N$ are nonexpansive and $T_0 = T$ is quasi-nonexpansive. It is worth mentioning that Halpern's subgradient extragradient method for solving the VIP in [20] is extended to develop our Mann-type inertial subgradient extragradient rule for solving the VIP and CFPP, in which A is L-Lipschitz continuous, pseudomonotone on H, but it is not required to be sequentially weakly continuous on C.

(ii) Our problem of finding an element of $\cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ includes as a special case the problem of finding an element of $\cap_{i=1}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ in [11], where in [11], A is required to be L-Lipschitz continuous, pseudomonotone on H, and sequentially weakly continuous on C. The modified inertial subgradient extragradient method for solving the VIP and CFPP in [11] is extended to develop our Mann-type inertial subgradient extragradient rule for solving the VIP and CFPP, where T_i is nonexpansive for $i = 1, ..., N$ and $T_0 = T$ is quasi-nonexpansive.

4. Applicability and Implementability of Algorithms

In this section, in order to support the applicability and implementability of our Algorithms 1 and 2, we make use of our main results to find a common solution of the VIP and CFPP in two illustrating examples.

Example 2. *Let* $C = [-1, 1]$ *and* $H = \mathbf{R}$ *with the inner product* $\langle a, b\rangle = ab$ *and induced norm* $\|\cdot\| = |\cdot|$. *Let* $x_0, x_1 \in H$ *be arbitrary. Put* $f(x) = F(x) = \frac{1}{2}x$, $\beta_n = \frac{1}{n+1}$, $\tau_n = \beta_n^2$, $\mu = 0.2$, $\alpha = \lambda_1 = 0.1$, $\gamma_n = \zeta_n = \frac{1}{3}$, $\rho = 2$, *and*

$$\alpha_n = \begin{cases} \min\{\frac{\beta_n^2}{\|x_n - x_{n-1}\|}, \alpha\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases}$$

Then we know that $\kappa = \eta = \frac{1}{2}$ *and* $\tau = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} = 1 \in (0, 1]$. *For* $N = 1$, *we now present Lipschitz continuous and pseudomonotone mapping* A, *quasi-nonexpansive mapping* T *and nonexpansive mapping* T_1 *such that* $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A) \neq \emptyset$. *Indeed, let* $A, T, T_1 : H \to H$ *be defined as* $Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$, $T_1 x := \sin x$ *and* $Tx := \frac{x}{2}\sin x$ *for all* $x \in H$. *We first show that* A *is pseudomonotone and* L-*Lipschitz continuous with* $L = 2$. *Indeed, it is easy to see that for all* $x, y \in H$,

$$\begin{aligned} \|Ax - Ay\| &= |\tfrac{1}{1+\|\sin x\|} - \tfrac{1}{1+\|x\|} - \tfrac{1}{1+\|\sin y\|} + \tfrac{1}{1+\|y\|}| \\ &\le |\tfrac{\|y\| - \|x\|}{(1+\|x\|)(1+\|y\|)}| + |\tfrac{\|\sin y\| - \|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}| \\ &\le \tfrac{\|x-y\|}{(1+\|x\|)(1+\|y\|)} + \tfrac{\|\sin x - \sin y\|}{(1+\|\sin x\|)(1+\|\sin y\|)} \\ &\le 2\|x - y\|, \end{aligned}$$

and

$$\langle Ax, y - x\rangle = (\frac{1}{1+|\sin x|} - \frac{1}{1+|x|})(y - x) \ge 0 \Rightarrow \langle Ay, y - x\rangle = (\frac{1}{1+|\sin y|} - \frac{1}{1+|y|})(y - x) \ge 0.$$

Furthermore, it is clear that $\mathrm{Fix}(T) = \{0\}$, T *is quasi-nonexpansive but not nonexpansive. Meantime,* $I - T$ *is demiclosed at* 0 *due to the continuity of* T. *In addition, it is clear that* T_1 *is* *nonexpansive and* $\mathrm{Fix}(T_1) = \{0\}$. *Therefore,* $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. *In this case, Algorithm 1 can be rewritten as follows:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = P_{C_n}(w_n - \lambda_n A y_n), \\ v_n = \frac{1}{3}x_n + \frac{2}{3}T_1 w_n, \\ x_{n+1} = \frac{1}{n+1} \cdot \frac{1}{2}x_n + \frac{1}{3}Tz_n + (\frac{n}{n+1} - \frac{1}{3})v_n \quad \forall n \geq 1, \end{cases} \tag{49}$$

where for each $n \geq 1$, C_n *and* λ_n *are chosen as in Algorithm 1. So, using Theorem 1, we know that* $\{x_n\}$ *converges to* $0 \in \Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. *Meanwhile, Algorithm 2 can be rewritten as follows:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = P_{C_n}(w_n - \lambda_n A y_n), \\ v_n = \frac{1}{3}x_n + \frac{2}{3}Tz_n, \\ x_{n+1} = \frac{1}{n+1} \cdot \frac{1}{2}x_n + \frac{1}{3}T_1 w_n + (\frac{n}{n+1} - \frac{1}{3})v_n \quad \forall n \geq 1, \end{cases} \tag{50}$$

where, for each $n \geq 1$, C_n *and* λ_n *are chosen as in Algorithm 2. So, using Theorem 2, we know that* $\{x_n\}$ *converges to* $0 \in \Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$.

Example 3. *Let* $H = L^2([0, 1])$ *with the inner product and induced norm defined by*

$$\langle x, y \rangle = \int_0^1 x(t)y(t)dt \quad \text{and} \quad \|x\| = (\int_0^1 |x(t)|^2 dt)^{1/2} \quad \forall x, y \in H,$$

respectively. Then $(H, \langle \cdot, \cdot \rangle)$ *is a Hilbert space. Let* $C := \{x \in H : \|x\| \leq 1\}$ *be the unit closed ball of H. It is known that*

$$P_C(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } \|x\| > 1, \\ x & \text{if } \|x\| \leq 1. \end{cases}$$

Let $x_0, x_1 \in H$ *be arbitrary. Put* $f(x) = F(x) = \frac{1}{2}x$, $\beta_n = \frac{1}{n+1}$, $\tau_n = \beta_n^2$, $\mu = 0.2$, $\alpha = \lambda_1 = 0.1$, $\gamma_n = \zeta_n = \frac{1}{3}$, $\rho = 2$, *and*

$$\alpha_n = \begin{cases} \min\{\frac{\beta_n^2}{\|x_n - x_{n-1}\|}, \alpha\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases}$$

Then we know that $\kappa = \eta = \frac{1}{2}$ *and* $\tau = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} = 1 \in (0, 1]$. *For* $N = 1$, *we now present Lipschitz continuous and pseudomonotone mapping A, quasi-nonexpansive mapping T and nonexpansive mapping* T_1 *such that* $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A) \neq \emptyset$. *Indeed, let* $A, T, T_1 : H \to H$ *be defined as* $(Ax)(t) := \max\{0, x(t)\}$, $(T_1 x)(t) := \frac{1}{2}x(t) - \frac{1}{2}\sin x(t)$ *and* $(Tx)(t) := \frac{1}{2}x(t) + \frac{1}{2}\sin x(t)$ *for all* $x \in H$. *It can be easily verified (see, e.g., [8,9]) that A is monotone and L-Lipschitz continuous with* $L = 1$, *and the solution set of the VIP for A is given by*

$$\mathrm{VI}(C, A) = \{0\} \neq \emptyset.$$

We next show that T and T_1 *are nonexpansive and* $\mathrm{Fix}(T) = \mathrm{Fix}(T_1) = \{0\}$. *Indeed, it is easy to see that for all* $x, y \in H$,

$$\begin{aligned} \|Tx - Ty\| &= (\textstyle\int_0^1 |\frac{1}{2}(x(t) - y(t)) + \frac{1}{2}(\sin x(t) - \sin y(t))|^2 dt)^{1/2} \\ &\leq (\textstyle\int_0^1 (\frac{1}{2}|x(t) - y(t)| + \frac{1}{2}|x(t) - y(t)|)^2 dt)^{1/2} \\ &= (\textstyle\int_0^1 |x(t) - y(t)|^2 dt)^{1/2} \\ &= \|x - y\|. \end{aligned}$$

Similarly, we get $\|T_1x - T_1y\| \le \|x - y\|\ \forall x, y \in H$. *Moreover, it is clear that* $\mathrm{Fix}(T) = \mathrm{Fix}(T_1) = \{0\}$. *Therefore,* $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. *In this case, Algorithm 1 can be rewritten as follows:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = P_{C_n}(w_n - \lambda_n A y_n), \\ v_n = \frac{1}{3}x_n + \frac{2}{3}T_1 w_n, \\ x_{n+1} = \frac{1}{n+1} \cdot \frac{1}{2}x_n + \frac{1}{3}Tz_n + (\frac{n}{n+1} - \frac{1}{3})v_n \quad \forall n \ge 1, \end{cases} \tag{51}$$

where for each $n \ge 1$, C_n *and* λ_n *are chosen as in Algorithm 1. So, using Theorem 1, we know that* $\{x_n\}$ *converges strongly to* $0 \in \Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. *Meantime, Algorithm 2 can be rewritten as follows:*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = P_{C_n}(w_n - \lambda_n A y_n), \\ v_n = \frac{1}{3}x_n + \frac{2}{3}Tz_n, \\ x_{n+1} = \frac{1}{n+1} \cdot \frac{1}{2}x_n + \frac{1}{3}T_1 w_n + (\frac{n}{n+1} - \frac{1}{3})v_n \quad \forall n \ge 1, \end{cases} \tag{52}$$

where for each $n \ge 1$, C_n *and* λ_n *are chosen as in Algorithm 2. So, using Theorem 2, we know that* $\{x_n\}$ *converges strongly to* $0 \in \Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$.

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