

Article

# The $\chi$ -Hessian Quotient for Riemannian Metrics

Pișcoran Laurian-Ioan <sup>1,\*</sup>, Akram Ali <sup>2</sup>, Barbu Cătălin <sup>3</sup> and Ali H. Alkhalidi <sup>2</sup>

<sup>1</sup> North University Center of Baia Mare, Department of Mathematics and Computer Science, Technical University of Cluj Napoca, Victoriei 76, 430122 Baia Mare, Romania

<sup>2</sup> Department of Mathematics, King Khalid University, Abha 9004, Saudi Arabia; akali@kku.edu.sa (A.A.); ahalkhalidi@kku.edu.sa (A.H.A.)

<sup>3</sup> “Vasile Alecsandri” National College, str. Vasile Alecsandri nr. 37, 600011 Bacău, Romania; kafka\_mate@yahoo.com

\* Correspondence: plaurian@yahoo.com

**Abstract:** Pseudo-Riemannian geometry and Hilbert–Schmidt norms are two important fields of research in applied mathematics. One of the main goals of this paper will be to find a link between these two research fields. In this respect, in the present paper, we will introduce and analyze two important quantities in pseudo-Riemannian geometry, namely the H-distorsion and, respectively, the Hessian  $\chi$ -quotient. This second quantity will be investigated using the Frobenius (Hilbert–Schmidt) norm. Some important examples will be also given, which will prove the validity of the developed theory along the paper.

**Keywords:** pseudo Riemannian manifold; Frobenius norm

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## 1. Introduction

The Hessian structural geometry is a fascinating emerging area of research. It is, in particular, related to Kaehlerian geometry, and also with many important pure mathematical fields of research, such as: affine differential geometry, cohomology, and homogeneous spaces. A strong relationship can also be established with the geometry of information in applied mathematics. This systematic introduction to the subject initially develops the foundations of Hessian structures on the basis of a certain pair of a flat connection and a Riemannian metric, and then describes these related fields as theoretical applications.

In Finsler geometry, respectively, in Riemannian geometry are few known invariants. Accordingly, one of the main path of research is to find new invariants and to study their impact in some concrete examples.

As we know,  $B_x^n$  represents, in Finsler geometry, the unit ball in a Finsler space centered at  $p \in M$ , where  $(M, F)$  is a Finsler manifold, i.e.,

$$B_x^n = \{y \in \mathbb{R}^n \mid |y| = F(x, y) < 1\}$$

and  $B^n$  represents the unit ball in the Euclidean space centered at origin:

$$B^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$$

where  $|x| = \sqrt{\delta_{ij}x^i x^j}$ .

Additionally, in Finsler geometry, we know that  $\sigma_F(x)$  is given by:

$$\sigma_F(x) = \frac{\text{volume}(B^n)}{\text{volume}(B_x^n)}.$$

Let now recall one classical definition

**Definition 1** ([1]). For a Finsler space, the distortion  $\tau = \tau(x, y)$  is given by:

$$\tau(x, y) = \ln \left( \frac{\sqrt{\det(g_{ij})}}{\sigma_F(x)} \right).$$

Additionally, according to Shen ([1]), when  $F = \sqrt{g_{ij}(x)y^i y^j}$ , is Riemannian, then

$$\sigma_F(x) = \sqrt{\det(g_{ij}(x))}.$$

**Definition 2** ([2]). A pseudo-Riemannian metric of metric signature  $(p, q)$  on a smooth manifold  $M$  of dimension  $n = p + q$ , is a smooth symmetric differentiable two-form  $g$  on  $M$ , such that, at each point,  $x \in M$ ,  $g_x$  is non-degenerate on  $T_x M$  with the signature  $(p, q)$ . We call  $(M, g)$  a pseudo-Riemannian manifold.

Also a well known results from [2], is the following:

**Theorem 1** ([2]). Given a pseudo-Riemannian manifold  $(M, g)$ , there exists a unique linear connection  $\nabla_g$  on  $M$ , called the Levi-Civita connection of  $g$ , such that:

- (a)  $\nabla_g g = 0$ ,
- (b)  $\nabla_g$  is torsion free, i.e.,  $T = 0$ .

For a coordinate chart  $(U, x^1, \dots, x^n)$ , the Christoffel symbols  $\Gamma_{ij}^k$  of the Levi-Civita connection are related to the components of the metric  $g$  in the following way:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

If  $f : M \rightarrow \mathbb{R}$  is a smooth function, then the second covariant derivative of the function  $f$  is given by:

$$\nabla_g^2 f = \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right) dx^i \otimes dx^j$$

is called the Hessian of the function  $f$ .

In the paper [3], the authors used the following notations, which we will also use in this paper:

$$\begin{aligned} f_{,i} &= \frac{\partial f}{\partial x^i}, \\ f_{,ij} &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k f_{,m}, \\ f_{,ijk} &= \frac{\partial f_{,ij}}{\partial x^k} - \Gamma_{ki}^l f_{,lj} - \Gamma_{kj}^l f_{,li}. \end{aligned}$$

Some important results regarding the “size” of a matrix were recently established in a series of papers. In this respect, please see [4,5].

As we know, the two norm for a matrix  $A$  is given by

$$\|A\|_2 = \max_{\|x\|=1} \|Ax\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

The Frobenius (or the Hilbert–Schmidt norm) of a matrix  $A = (A_{ij})$  is defined, as follows:

$$\|A\|_{HS} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n A_{ij}^2}.$$

The operator norm of a matrix  $A = (A_{ij})$  is given by

$$\|A\|_{op} = \max_{\|x\|=1} \|Ax\|.$$

Next we will recall some properties of these norms. We will be focused on Hilbert–Schmidt norm  $\|\cdot\|_{HS}$ , because we will use it to establish some new main results of this paper:

$$\|A \cdot B\|_{HS} \leq \|A\|_{HS} \cdot \|B\|_{HS} \tag{1}$$

$$\|A \cdot B\|_{op} \leq \|A\|_{op} \cdot \|B\|_{op} \tag{2}$$

$$\|A\|_{HS} \leq \sqrt{n} \|A\|_{op}. \tag{3}$$

In the previous inequality, the equality take place when  $A = I_n$ . Additionally,

$$|\det(A)| \leq \|A\|_{HS}^n \tag{4}$$

$$\|A\|_{HS} \leq \sqrt{r} \|A\|_2 = \frac{\sqrt{r}}{\sigma_{min}(A)}. \tag{5}$$

Here,  $A$  denotes any positive definite symmetric matrix,  $r$  is the rank of  $A$ , and  $\sigma_{min}(A)$  denotes the minimum singular value of  $A$ . Some interesting results regarding the Minkowski norm on Finsler geometry are presented in [6].

### 2. Main Results

Now, using the distortion definition, we will introduce the following definition:

**Definition 3.** For a pseudo-Riemannian manifold, we will denote, by

$$\sigma_{F,\nabla F} = \frac{\sigma_F(x)}{\sigma_{\nabla F}(x)} = \sqrt{\frac{\det(g_{ij})}{\det(\nabla_g^2 f)}} \tag{6}$$

the H-distortion, if and only if  $\sigma_{F,\nabla F}^2(x) = \text{constant}$ .

**Example 1.** We will consider a pseudo-Riemannian manifold,  $\{(x^1, x^2) \mid x^1 \neq -1\} \subset \mathbb{R}^2$ , endowed with the following metric:

$$g_{ij} = \begin{pmatrix} (1+x^1)^2 & x^1+2 \\ x^1 & 1 \end{pmatrix}. \tag{7}$$

After tedious computations, we obtain the following Christoffel symbols for this metric:  $\Gamma_{11}^1 = -1, \Gamma_{11}^2 = x^1 + 1, \Gamma_{22}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{21}^2 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0$ . Now, we get:

$$\begin{aligned}
 f_{,11} &= \frac{\partial^2 f}{\partial x^1 \partial x^1} - \left( \Gamma_{11}^1 f_{,1} + \Gamma_{11}^2 f_{,2} \right) \\
 f_{,12} &= \frac{\partial^2 f}{\partial x^1 \partial x^2} - \left( \Gamma_{12}^1 f_{,1} + \Gamma_{12}^2 f_{,2} \right) \\
 f_{,21} &= \frac{\partial^2 f}{\partial x^2 \partial x^1} - \left( \Gamma_{21}^1 f_{,1} + \Gamma_{21}^2 f_{,2} \right) \\
 f_{,22} &= \frac{\partial^2 f}{\partial x^2 \partial x^2} - \left( \Gamma_{21}^1 f_{,1} + \Gamma_{22}^2 f_{,2} \right).
 \end{aligned}$$

With the above Christoffel symbols, after replacing, we get the following Hessian matrix:

$$\nabla_g^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} + \frac{\partial f}{\partial x^1} - (x^1 + 1) \frac{\partial f}{\partial x^2} & \frac{\partial^2 f}{\partial x^1 \partial x^2} \\ \frac{\partial^2 f}{\partial x^2 \partial x^1} & \frac{\partial^2 f}{\partial x^2 \partial x^2} \end{pmatrix}.$$

We will search a function that respects the condition that the determinant of the Hessian will be a constant. In this respect, let us consider the following function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f(x^1, x^2) = kx^1x^2 + 1$ , where  $k$  is a non-positive real constant. For this function, at a critical point  $x$ , the above Hessian matrix became:

$$\nabla_g^2 f = Hf(x) = \begin{pmatrix} kx^2 + (x^1 + 1)kx^1 & k \\ k & 0 \end{pmatrix}.$$

Subsequently, the determinant of the Hessian will be  $\det(\nabla_g^2 f) = -k^2 = \text{constant}$ . Finally, let us conclude that in this case, the H-distorsion will be:

$$\sigma_{E, \nabla F}^2 = \frac{\det(g_{ij})}{\det(\nabla_g^2 f)} = \frac{-1}{k^2} = \text{constant}.$$

Now, using the theory of Frobenius norms, we will introduce the following quantity:

**Definition 4.** For a pseudo-Riemannian manifold  $(M, g)$ , we will denote, by

$$\chi_H = \frac{\|Hf(x)\|_{HS}}{\|Hf_1(x_1)\|_{HS}}, \tag{8}$$

the Hessian  $\chi$ -quotient for two smooth function  $f, f_1 : M \rightarrow \mathbb{R}$ . Here,  $\|\cdot\|_{HS}$  represent the Frobenius (Hilbert–Schmidt) norm of a Hessian matrix that is attached to the pseudo-Riemannian manifold. Here, the point  $x$  represents the critical point for the first Hessian of the smooth function  $f$  and the point  $x_1$  represents the critical point of the Hessian of the second function  $f_1$ .

**Remark 1.** Because  $\|Hf(x)\|_{HS}$  and  $\|Hf_1(x_1)\|_{HS}$  are two constants, then we can conclude that  $\chi_H$  must be also a constant. Next, we will investigate some of  $\chi_H$  properties.

First, let us recall the following well known properties regarding Frobenius norms: for two matrices  $A = (a_{ij})$ , respectively,  $B = (b_{ij})$ , as we know, the following inequalities hold:

$$\|A \cdot B\|_{HS} \leq \|A\|_{HS} \cdot \|B\|_{HS}$$

respectively

$$\|B^{-1}\|_{HS} \leq \sqrt{r} \|B^{-1}\|_{HS}.$$

Hence, now we can formulate the following:

**Theorem 2.** For the Hessian  $\chi_H$  quotient for two smooth functions  $f, f_1 : M \rightarrow \mathbb{R}$ , which are considered for the same pseudo-Riemannian manifold  $(M, g)$ , the following inequality holds:

$$\chi_H \leq \frac{Q\sqrt{r}}{\sigma_{\min}(Hf_1(x_1))}, \tag{9}$$

where  $Q = \|Hf(x)\|_{HS}$ ,  $x$  is a critical point of the Hessian  $Hf(x)$ ,  $r$  represents the rank of the second Hessian  $Hf_1(x_1)$ ,  $x_1$  is the critical point of the Hessian  $Hf_1(x_1)$ , and  $\sigma_{\min}(Hf_1^{-1}(x))$  is the minimum singular value of the Hessian  $Hf_1^{-1}(x)$ .

**Proof.** Starting with the inequality

$$\|A \cdot B\|_{HS} \leq \|A\|_{HS} \cdot \|B\|_{HS},$$

for  $A = Hf(x)$  and  $B = (Hf_1(x_1))^{-1}$ , one obtains:

$$\chi_H = \|Hf(x) \cdot Hf_1^{-1}(x_1)\|_{HS} \leq \|Hf(x)\|_{HS} \cdot \|Hf_1^{-1}(x_1)\|_{HS}.$$

Using the hypothesis of the theorem, we get, from here, the following inequality:

$$\chi_H \leq Q \cdot \|Hf_1^{-1}(x_1)\|_{HS}.$$

Additionally, we can remark the following fact:

$$\begin{aligned} \|Hf_1^{-1}(x_1)\|_{HS} &= \sqrt{\text{tr}(Hf_1^{-1}(x_1) \times Hf_1^{-1}(x_1))} \\ &= \sum_{i=1}^n \sqrt{\frac{1}{\lambda_i^2}} \leq \sum_{i=1}^n \frac{1}{\lambda_i} = \text{tr}(Hf_1^{-1}(x_1)). \end{aligned}$$

However, using the inequalities between Frobenius norm and of the two-norm, we obtain:

$$\|Hf_1^{-1}(x_1)\| \leq \sqrt{r} \cdot \|Hf_1^{-1}(x_1)\|_2$$

and, since  $\|Hf_1^{-1}(x_1)\|_2 = \frac{1}{\sigma_{\min}(Hf_1^{-1}(x_1))}$ , we find the desired result immediately.  $\square$

**Remark 2.** Let us observe that, because

$$\|Hf_1^{-1}(x_1)\|_{HS} \geq \|Hf_1^{-1}(x_1)\|_{op},$$

where  $\|\cdot\|_{op}$  is the operator norm of the Hessian  $Hf_1^{-1}(x_1)$ , we obtain:

$$\|Hf(x) \cdot Hf_1^{-1}(x_1)\|_{op} \leq \chi_H.$$

Additionally, because the general property

$$\|A\|_2 \leq \|A\|_{HS} \leq \sqrt{r}\|A\|_2$$

holds, we can now give the following:

**Corollary 1.** For the Hessian  $\chi_H$  quotient for two smooth functions  $f, f_1 : M \rightarrow \mathbb{R}$ , considered for the same pseudo-Riemannian manifold  $(M, g)$ , the following inequality holds:

$$\|Hf(x)Hf_1^{-1}(x_1)\|_2 \leq \chi_H \leq \sqrt{r}\|Hf(x)Hf_1^{-1}(x_1)\|_2, \tag{10}$$

where  $x$  is a critical point of the Hessian  $Hf(x)$ ,  $r$  represents the rank of the second Hessian  $Hf_1(x_1)$ ,  $x_1$  is the critical point of the Hessian  $Hf_1(x_1)$ , and  $\sigma_{\min}(Hf_1^{-1}(x))$  is the minimum singular value of the Hessian  $Hf_1^{-1}(x)$ .

Let now recall, from the first example where we get the Hessian for the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x^1, x^2) = kx^1x^2 + 1$ :

$$Hf(x) = \nabla_g^2 f(x) = \begin{pmatrix} (1+x^1)kx^1 + kx^2 & k \\ k & 0 \end{pmatrix},$$

where  $x = (x^1, x^2)$  is one critical point for the Hessian  $Hf(x)$ .

For the same metric as in Example 1,

$$g_{ij} = \begin{pmatrix} (1+x^1)^2 & x^1+2 \\ x^1 & 1 \end{pmatrix},$$

we will construct another function  $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ , such that, in a critical point  $x_1 \in \mathbb{R}^2$ , we could be able to compute the Hessian  $\chi_H$  quotient. First, let us observe that, for the metric  $(g_{ij})$  from the first example, the Hessian of the function  $f(x^1, x^2) = kx^1x^2 + 1$  has the critical point given as a solution of the below equation system:

$$\begin{cases} \frac{\partial f}{\partial x^1} = kx^2 + 1 = 0 \\ \frac{\partial f}{\partial x^2} = kx^1 + 1 = 0. \end{cases}$$

Accordingly, the critical point for this function will be  $(-\frac{1}{k}, -\frac{1}{k})$ . Next, the Hessian that is associated with the metric  $g_{ij}$  for this critical point will be:

$$Hf(x) = \nabla_g^2 f\left(-\frac{1}{k}, -\frac{1}{k}\right) = \begin{pmatrix} -\frac{1}{k} & k \\ k & 0 \end{pmatrix}.$$

Accordingly, the Frobenius (Hilbert–Schmidt) norm in this case will be easily computed, as follows:

$$\|Hf(x)_{HS}\| = \left\| \nabla_g^2 f\left(-\frac{1}{k}, -\frac{1}{k}\right) \right\|_{HS} = \sqrt{2k^2 + \left(\frac{1}{k}\right)^2}.$$

Let us now consider the following example as a continuation of the above results:

**Example 2.** We will consider the following function  $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}, f_1(x^1, x^2) = 2x^1 - (x^1)^2x^2 + x^2$  for the same matrix as in the first example. The critical points for this function  $f_1(x_1)$  are given as a solution of the following equations system

$$\begin{cases} \frac{\partial f_1}{\partial x^1} = 2 - 2x^1x^2 = 0 \\ \frac{\partial f_1}{\partial x^2} = -(x^1)^2 + 1 = 0. \end{cases}$$

Hence, we obtain, for the Hessian of this function  $f_1$ , two critical points  $(1, 1)$ , respectively  $(-1, -1)$ . We will compute the Hessian matrix for this two critical points, as follows:

$$Hf_1(x_1) = \nabla_g^2 f_1 = \begin{pmatrix} -2x^2 + 2 - 2x^1x^2 - (x^1 + 1)(1 - x^1)^2 & -2x^1 \\ -2x^1 & 0 \end{pmatrix}$$

$$Hf_1(-1, -1) = \nabla_g^2 f_1(-1, -1) = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}.$$

Accordingly, we derive  $\|Hf_1(-1, -1)\|_{HS} = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}$ .

In the same way, we point-out, for the another critical point  $(1, 1)$ , the following result for the Hessian matrix:

$$Hf_1(1, 1) = \nabla_g^2 f_1(1, 1) = \begin{pmatrix} -2 & -2 \\ -2 & 0 \end{pmatrix}.$$

Hence, we find  $\|Hf_1(1, 1)\|_{HS} = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}$ , the same value as for the previous critical point. In conclusion, the Hessian  $\chi$ -quotient for the above mentioned functions  $f, f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is given by:

$$\chi_H = \frac{\|Hf(x)\|_{HS}}{\|Hf_1(x_1)\|_{HS}} = \frac{\sqrt{2k^2 + \left(\frac{1}{k}\right)^2}}{2\sqrt{3}} = \text{constant}.$$

Accordingly, we have find an good example with respect to the definition that we have introduced earlier regarding the  $\chi_H$  quotient of some functions  $f, f_1$ .

### 3. Conclusions

In this paper, we have investigated some interesting properties of the two quantities that we have introduced, namely, the H-distorsion and, respectively, the  $\chi_H$  quotient. Finally, we have given some concludent examples regarding this theory.

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### References

1. Shen, Z. *Lectures on Finsler Geometry*; World Scientific: Singapore, 2001.
2. Shima, H. Hessian manifolds of constant Hessian sectional curvature. *J. Math. Soc. Jpn.* **1995**, *47*, 737–753. [[CrossRef](#)]
3. Bercu, G.; Matsuyama, Y.; Postolache, M. *Hessian Matrix and Ricci Solitons*; Fair Partners Publishers: Bucharest, Romania, 2011.
4. Böttcher A.; Wenzel, D. The Frobenius norm and the commutator. *Linear Algebra Appl.* **2008**, *429*, 1864–1885. [[CrossRef](#)]
5. Hu, Y.-Z. *Some Operator Inequalities*; Seminaire de probablilites: Strasbourg, France, 1994; pp. 316–333.
6. Crasmareanu, M. New tools in Finsler geometry: Stretch and Ricci solitons. *Math. Rep. (Bucur.)* **2014**, *16*, 83–93.