axioms

Article

q-Binomial Convolution and Transformations of q-Appell Polynomials

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Abstract: In this paper, binomial convolution in the frame of quantum calculus is studied for the set $\mathcal{A}_q$ of q-Appell sequences. It has been shown that the set $\mathcal{A}_q$ of q-Appell sequences forms an Abelian group under the operation of binomial convolution. Several properties for this Abelian group structure $\mathcal{A}_q$ have been studied. A new definition of the q-Appell polynomials associated with a random variable is proposed. Scale transformation as well as transformation based on expectation with respect to a random variable is used to present the determinantal form of q-Appell sequences.

Keywords: q-calculus; q-Appell polynomials; binomial convolution; Abelian group; Appell sequences transformation

MSC: 33C45; 33C65; 33C99; 44A35

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Appell polynomials [1] were defined by Appell in 1880. F.A. Costabile and E. Longo studied the Appell polynomial using determinantal approach [2]. Based on the quantum calculus, The family of q-Appell polynomials [3] were introduced by Al-Salam in 1967. Furthermore, M.E. Keleshteri and N.I. Mahmudov studied q-Appell polynomial using determinantal approach [4]. For other literature related to Appell polynomials, one can refer [5–11].

These polynomials have been used in many branches of mathematics including number theory, applied mathematics and theoretical physics. According to the Weierstrass approximation theorem [12], every continuous function can be approximated by polynomials. Thus, polynomials play an important role in approximation theory. For some recent papers related to approximation by polynomials and applications in CAGD, one can refer to [13–19]. Appell and q-Appell polynomial have been studied for interpolation by several authors [20,21]. T. Ernst in [22] introduced the term multiplicative q-Appell polynomial and has shown that the set of q-Appell polynomials forms a commutative ring. Apart from this, convolution plays a very important role in approximation theory, probability, statistics, computer vision, image and signal processing, etc. Motivated by the above facts, we study here various properties of the q-Appell polynomial with the operation of convolution using q-calculus. This paper is organized as follows:

The paper considers the binomial convolution for the set of q-Appel sequences. It is proven that the set of q-Appel sequences equipped with the binomial convolution forms an Abelian group. By using the probabilistic approach to q-Appel polynomials, a new definition of q-Appel polynomials related to a random variable similar to the work done in [21] is discussed. Furthermore, the scale transform and transformations based on expectations are defined and their characteristics discussed.

Let us recall some basics from the quantum calculus (see [23–28]). The quantum or q-analogue $[\mu]_q$ of a number μ is defined by

$$[\mu]_q = \begin{cases} \dfrac{1-q^\mu}{1-q}, & q \neq 1, \\ \mu, & q = 1. \end{cases}$$

The q-factorial $[\mu]_q!$ is defined by

$$[\mu]_q! = \begin{cases} [\mu]_q [\mu-1]_q \cdots [1]_q\,, & \mu \in \mathbb{N}, \\ 1\,, & \mu = 0. \end{cases}$$

The q-binomial coefficient $\begin{bmatrix} \mu \\ s \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} \mu \\ s \end{bmatrix}_q = \frac{[\mu]_q!}{[s]_q!\,[\mu-s]_q!}\,, \quad \mu, s \in \mathbb{N};\ 0 \leq s \leq \mu.$$

The q-analogue of the function $(y+x)^\mu$ are defined by

$$\begin{aligned} (y+x)_q^\mu &= \begin{cases} \prod_{j=0}^{\mu-1} (y \, + \, q^j x), & for \ \ \mu = 1, 2, 3, \cdots \\ 1, & for \ \ \mu = 0 \end{cases} \\ &= \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} y^{\mu-k} x^k. \end{aligned}$$

The q-derivative of a function f is defined by

$$D_q f(y) = \begin{cases} \frac{f(y)\,-\,f(qy)}{(1\,-\,q)\,y}, & y \neq 0 \\ f'(0)\,, & y = 0. \end{cases}$$

Exponential functions based on q-calculus is used in the standard approach as follows:

$$e_q(y) = \sum_{\mu=0}^{\infty} \frac{y^\mu}{[\mu]_q!}\,, \quad 0 <\mid q \mid< 1\,; \quad \mid y \mid< \frac{1}{\mid 1-q \mid}.$$

Let y and x be elements of a commutative multiplicative semigroup. Then, the NWA q-addition is given by [29]

$$(y \oplus_q x)^\mu = \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q y^k x^{\mu-k}.$$

For every power series $f_n(t)$, with $f_n(0) \neq 0$, the q-Appell polynomials of degree μ and order n have the following generating function [29]:

$$f_n(t) e_q(tx) = \sum_{\mu=0}^{\infty} A_{\mu,q}^{(n)}(x) \frac{t^\mu}{[\mu]_q!}\,,$$

Putting $x = 0$, we have:

$$f_n(t) = \sum_{\mu=0}^{\infty} A_{\mu,q}^{(n)} \frac{t^\mu}{[\mu]_q!}\,,$$

where $A_{\mu,q}^{(n)}$ is called a q-Appell number of degree μ and order n

q-Appell polynomials of degree μ and order n satisfy the following q-differential Equation [29]:

$$D_{q,y}A^{(n)}_{\mu,q}(y) = [\mu]_q A^{(n)}_{\mu-1,q}(y), \quad \mu = 1, 2, \cdots . \tag{1}$$

1. Quantum Binomial Convolutions and Generating Functions

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Now, onwards $\mu \in \mathbb{N}_0$, $y \in \mathbb{R}$, and $z \in \mathbb{C}$, satisfying $| z |\leq r$, $r > 0$. Let us denote by $\mathcal{G}_q$ the set of all real sequences $\mathbf{u}_q = (u_{\mu,q})_{\mu\geq 0}$ where $u_{0,q} \neq 0$ and:

$$\sum_{\mu=0}^{\infty} | u_{\mu,q} | \frac{r^{\mu}}{[\mu]_q!} < \infty.$$

If $\mathbf{u}_q \in \mathcal{G}_q$, then its generating function will be denoted by

$$\mathsf{G}(\mathbf{u}_q, z) = \sum_{\mu=0}^{\infty} u_{\mu,q} \frac{z^{\mu}}{[\mu]_q!}.$$

The q-binomial convolution [8] of $\mathbf{u}_q$ and $\mathbf{v}_q$, will be denoted by $\mathbf{u}_q \times_q \mathbf{v}_q = \big((u_q \times_q v_q)_\mu\big)_{\mu\geq 0}$ for $\mathbf{u}_q$ and $\mathbf{v}_q \in \mathcal{G}_q$ is defined as

$$(u_q \times_q v_q)_\mu = \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q u_{k,q} v_{\mu-k,q}. \tag{2}$$

The q-addition is a special case of the q-binomial convolution [29].

The q-multinomial coefficient is given by

$$\begin{bmatrix} \mu \\ j_1, \cdots, j_m \end{bmatrix}_q = \frac{[\mu]_q!}{[j_1]_q! \cdots [j_m]_q!}, \quad j_1 + \cdots + j_m = \mu, \ \ j_i = 0, 1, \cdots, \mu, \ \ i = 0, 1, \cdots, m \ \ and \ \ \mu \in \mathbb{N}_0,$$

Proposition 1. *Let that $\boldsymbol{u}_q^{(k)} = (u_{\mu,q}^{(k)})_{\mu\geq 0} \in \mathcal{G}_q$, $k = 1, 2, \cdots, m$, μ belong to the set of positive integers (see [29]). Then, $\boldsymbol{u}_q^{(1)} \times_q \cdots \times_q \boldsymbol{u}_q^{(m)} \in \mathcal{G}_q$ and:*

$$(\boldsymbol{u}_q^{(1)} \times_q \cdots \times_q \boldsymbol{u}_q^{(m)})_\mu = \sum_{j_1+\cdots+j_m=\mu} \begin{bmatrix} \mu \\ j_1, j_2, \cdots, j_m \end{bmatrix}_q u_{j_1,q}^{(1)} \cdots u_{j_m,q}^{(m)}.$$

In addition:

$$\mathsf{G}(\boldsymbol{u}_q^{(1)} \times_q \cdots \times_q \boldsymbol{u}_q^{(m)}, z) = \mathsf{G}(\boldsymbol{u}_q^{(1)}, z) \cdots \mathsf{G}(\boldsymbol{u}_q^{(m)}, z). \tag{3}$$

Proof. Suppose $\mathbf{u}_q$ and $\mathbf{v}_q \in \mathcal{G}_q$ with r and s as their radii, respectively. Let $t = min(r, s)$. Then, from (2), we have:

$$\begin{aligned}\mathsf{G}(| \ (\mathbf{u}_q \times_q \mathbf{v}_q) \ |, t) &= \sum_{\mu=0}^{\infty} | \ (u_q \times_q v_q)_\mu \ | \frac{t^\mu}{[\mu]_q!} \leq \sum_{\mu=0}^{\infty} \frac{t^\mu}{[\mu]_q!} \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q | \ u_{k,q} \ || \ v_{\mu-k,q} \ | \\ &= \sum_{k=0}^{\infty} | \ u_{k,q} \ | \frac{t^k}{[k]_q!} \sum_{\mu=k}^{\infty} | \ v_{\mu-k,q} \ | \frac{t^{\mu-k}}{[\mu-k]_q!} = \mathsf{G}(| \ \mathbf{u}_q \ |, t)\mathsf{G}(| \ \mathbf{v}_q \ |, t).\end{aligned}$$

□

Then $\mathsf{G}(\mathbf{u}_q \times_q \mathbf{v}_q, z) = \mathsf{G}(\mathbf{u}_q, z)\mathsf{G}(\mathbf{v}_q, z)$, $| z |\leq t$. Thus, by applying the induction on m, result follows.

Corollary 1. *$(\mathcal{G}_q, \times)$ is an Abelian group having an identity element as $e_q = (\delta_{\mu 0})_{\mu\geq 0}$, where $\delta_{00} = 1$ and $\delta_{\mu 0} = 0$ for $\mu \in \mathbb{N}$.*

Proof. Closure: for $\mathbf{u}_q, \mathbf{v}_q \in \mathcal{G}_q$, then $\mathbf{u}_q \times_q \mathbf{v}_q = \big((u_q \times_q v_q)_\mu\big)_{\mu \geq 0} \in \mathcal{G}_q$ as $(u_q \times_q v_q)_\mu = \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q u_{k,q} v_{\mu-k,q} \in \mathbb{R}$.

Associativity: let $\mathbf{u}_q, \mathbf{v}_q, \mathbf{w}_q \in \mathcal{G}_q$ be any elements:
$(\mathbf{u}_q \times_q \mathbf{v}_q) \times_q \mathbf{w}_q = \big(((u_q \times_q v_q) \times_q w_q)_\mu\big)_{\mu \geq 0}$ and $\mathbf{u}_q \times_q (\mathbf{v}_q \times_q \mathbf{w}_q) = \big((u_q \times_q (v_q \times_q w_q)_\mu\big)_{\mu \geq 0}$

$$\begin{aligned}
\big(u_q \times_q (v_q \times_q w_q)\big)_\mu &= \sum_{s=0}^{\mu} \begin{bmatrix} \mu \\ s \end{bmatrix}_q u_{s,q} (v_q \times_q w_q)_{\mu-s} \\
&= \sum_{s=0}^{\mu} \sum_{r=0}^{\mu-s} \begin{bmatrix} \mu \\ s \end{bmatrix}_q \begin{bmatrix} \mu - s \\ r \end{bmatrix}_q u_{s,q} v_{r,q} w_{\mu-s-r,q} \\
&= \sum_{s=0}^{\mu} \sum_{r=s}^{\mu} \begin{bmatrix} \mu \\ s \end{bmatrix}_q \begin{bmatrix} \mu - s \\ r - s \end{bmatrix}_q u_{s,q} v_{r-s,q} w_{\mu-r,q} \\
&= \sum_{r=0}^{\mu} \sum_{s=0}^{r} \begin{bmatrix} \mu \\ r \end{bmatrix}_q \begin{bmatrix} r \\ s \end{bmatrix}_q u_{s,q} v_{r-s,q} w_{\mu-r,q} \\
&= \sum_{r=0}^{\mu} \begin{bmatrix} \mu \\ r \end{bmatrix}_q (u_q \times_q v_q)_r w_{\mu-r,q} \\
&= \big((u_q \times_q v_q) \times_q w_q\big)_\mu
\end{aligned}$$

Existence of identity: it is easy to see that $\mathbf{u}_q \times_q e_q = e_q \times_q \mathbf{u}_q = \mathbf{u}_q$ for all $\mathbf{u}_q \in \mathcal{G}_q$ where $e_q = (\delta_{\mu 0})_{\mu \geq 0}$, Existence of inverse: let $\mathbf{u}_q \in \mathcal{G}_q$. Since $\mathsf{G}(\mathbf{u}_q, 0) = u_{0,q} \neq 0$, then $\mid \mathsf{G}(\mathbf{u}_q, z) \mid > 0, \ \mid z \mid < \lambda$, for some $\lambda > 0$. This implies that $\frac{1}{\mathsf{G}(\mathbf{u}_q, z)}$ is a well-defined function that can be represented via power series due to analyticity as

$$\frac{1}{\mathsf{G}(\mathbf{u}_q, z)} = \sum_{\mu=0}^{\infty} v_{\mu,q} \frac{z^\mu}{[\mu]_q!} =: \mathsf{G}(\mathbf{v}_q, z), \quad \mid z \mid \leq \rho, \tag{4}$$

for some real sequence $\mathbf{v}_q = (v_{\mu,q})_{\mu \geq 0}$ and some $\rho > 0$. Here, one can observe that $v_{0,q} = \frac{1}{u_{0,q}} \neq 0$ by (4), and that $\mathbf{v}_q \in \mathcal{G}_q$. Again, it can be observed from (3) and (4), that $\mathbf{v}_q$ is the inverse of $\mathbf{u}_q$. Thus, $\mathbf{v}_q$ is the unique solution to the systems of equations:

$$(u_q \times_q v_q)_\mu = \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q u_{k,q} v_{\mu-k,q} = \delta_{\mu 0}. \tag{5}$$

Commutative: it is easy to see that $\mathbf{u}_q \times_q \mathbf{v}_q = \mathbf{v}_q \times_q \mathbf{u}_q$ for all $\mathbf{u}_q, \mathbf{v}_q \in \mathcal{G}_q$.
The proof is complete. □

Let $\mathbf{A}_q(y) = \big(A_{\mu,q}(y)\big)_{\mu \geq 0}$ be a sequence of polynomials such that $\mathbf{A}_q(0) = \big(A_{\mu,q}(0)\big)_{\mu \geq 0} \in \mathcal{G}_q$. Recall that $\mathbf{A}_q(y)$ is called a q-Appell sequence if one of the following equivalent conditions is satisfied:

$$\mathsf{D}_{q,y}\big(A_{\mu,q}(y)\big) = [\mu]_q A_{\mu-1,q}(y), \quad \mu \in \mathbb{N}, \tag{6}$$

$$A_{\mu,q}(y) = \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q A_{k,q}(0) y^{\mu-k}, \tag{7}$$

or:

$$\mathsf{G}\big(\mathbf{A}_q(y), z\big) = \mathsf{G}\big(\mathbf{A}_q(0), z\big) e_q^{yz}. \tag{8}$$

The set of all q-Appell sequence will be denoted by $\mathcal{A}_q$. Let $\mathbf{I}_q(y) = (y^\mu)_{\mu\geq 0}$ be the unit q-Appell sequence. Using (2), the condition (7) can be expressed as

$$\mathbf{A}_q(y) = \mathbf{A}_q(0) \times_q \mathbf{I}_q(y). \tag{9}$$

From Proposition (1), $\mathbf{A}_q(y) \in \mathcal{G}_q$, for any $y \in \mathbb{R}$. From the binomial identity, $\mathbf{I}_q(y \oplus_q x) = \mathbf{I}_q(y) \times_q \mathbf{I}_q(x)$, for $x \in \mathbb{R}$. Thus, from Equation (9) and Corollary (1):

$$\mathbf{A}_q(y \oplus_q x) = \mathbf{A}_q(y) \times_q \mathbf{I}_q(x) = \mathbf{A}_q(x) \times_q \mathbf{I}_q(y), \quad x \in \mathbb{R}. \tag{10}$$

2. The Abelian Group Structure of $\mathcal{A}_q$

Let $\mathbf{A}_q(y), \mathbf{C}_q(y) \in \mathcal{A}_q$. The q-binomial convolution of $\mathbf{A}_q(y)$ and $\mathbf{C}_q(y)$, denoted by $(\mathbf{A}_q \times_q \mathbf{C}_q)(y) = \big((A_q \times_q C_q)_\mu(y)\big)_{\mu\geq 0}$ and is defined as

$$(A_q \times_q C_q)(y) = \mathbf{A}_q(y) \times_q \mathbf{C}_q(0) = \mathbf{A}_q(0) \times_q \mathbf{C}_q(y) = \mathbf{A}_q(0) \times_q \mathbf{C}_q(0) \times_q \mathbf{I}_q(y), \tag{11}$$

The last two equalities of (11) can be obtained using (9) and Corollary (1). Equivalently:

$$\begin{aligned}(A_q \times_q C_q)_\mu(y) &= \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q A_{k,q}(0) C_{\mu-k,q}(y) = \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q C_{k,q}(0) A_{\mu-k,q}(y) \\ &= \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q y^{\mu-k} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q A_{j,q}(0) C_{k-j,q}(0) \\ &= \sum_{j_1+j_2+j_3=\mu} \begin{bmatrix} \mu \\ j_1, j_2, j_3 \end{bmatrix}_q A_{j_1,q}(0) C_{j_2,q}(0) y^{j_3}.\end{aligned} \tag{12}$$

Theorem 1. *Let $\boldsymbol{A}_q(y), \boldsymbol{C}_q(y) \in \mathcal{A}_q$. Then, $(\boldsymbol{A}_q \times_q \boldsymbol{C}_q)(y)$ is an q-Appell sequences characterized by its generating function:*

$$\mathsf{G}\big((\boldsymbol{A}_q \times_q \boldsymbol{C}_q)(y), z\big) = \mathsf{G}\big(\boldsymbol{A}_q(0), z\big)\mathsf{G}\big(\boldsymbol{C}_q(0), z\big) e_q^{yz}. \tag{13}$$

As a consequence, $(\mathcal{A}_q, \times_q)$ is an Abelian group with identity element $\mathbf{I}_q(y)$. In addition, we have:

$$(\boldsymbol{A}_q \times_q \boldsymbol{C}_q)(y \oplus_q x) = \boldsymbol{A}_q(y) \times_q \boldsymbol{C}_q(x), \quad x \in \mathbb{R}. \tag{14}$$

In general, for any $\boldsymbol{A}_q^{(i)}(y) \in \mathcal{A}_q$ and $y_i \in \mathbb{R}$, $i = 1, \cdots, m$, with $y_1 \oplus_q \cdots \oplus_q y_m = y$:

$$\boldsymbol{A}_q^{(1)}(y_1) \times_q \cdots \times_q \boldsymbol{A}_q^{(m)}(y_m) = \big(\boldsymbol{A}_q^{(1)} \times_q \cdots \times_q \boldsymbol{A}_q^{(m)}\big)(y). \tag{15}$$

Proof. By (12), $(A_q \times_q C_q)_0(0) = A_{0,q}(0) C_{0,q}(0) \neq 0$. Using (11) and proposition 1, we have:

$$\mathsf{G}\big((\mathbf{A}_q \times_q \mathbf{C}_q)(y), z\big) = \mathsf{G}\big(\mathbf{A}_q(0), z\big)\mathsf{G}\big(\mathbf{C}_q(0), z\big) e_q^{yz} = \mathsf{G}\big((\mathbf{A}_q \times_q \mathbf{C}_q)(0), z\big) e_q^{yz}.$$

Thus, the first statement in Theorem 1 is evident from (8). Similarly, from (13) and Proposition 1, Formula (14) can be obtained. Now, we will show that $\mathbf{C}_q(y) \in \mathcal{A}_q$ will be the inverse of $\mathbf{A}_q(y) \in \mathcal{A}_q$. Similar to the method used in Corollary 1, let $\mathbf{C}_q(0) = (C_{\mu,q}(0))_{\mu\geq 0} \in \mathcal{G}_q$ be the real sequence having a generating function as

$$\mathsf{G}\big(\mathbf{C}_q(0), z\big) = \frac{1}{\mathsf{G}(\mathbf{A}_q(0), z)}. \tag{16}$$

Then, the q-Appell sequences $\mathbf{C}_q(y) = \mathbf{C}_q(0) \times_q \mathbf{I}_q(y)$ will be inverse of $\mathbf{A}_q(y)$. Equivalently, $\mathbf{C}_q(0)$ will be the unique solution to the systems of equations:

$$(A_q \times_q C_q)_\mu(y) = \sum_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q C_{k,q}(0) A_{\mu-k,q}(y) = \mathbf{I}_{\mu,q}(y) = y^\mu, \tag{17}$$

which completes the proof. □

Note that Theorem 1 is equivalent to q-Appell polynomials determinantal approach, now we state the following:

Corollary 2. *(Determinantal form). For the real sequence $\mathbf{A}_q(y) \in \mathcal{A}_q$ and $\mathbf{C}_q(0) \in \mathcal{G}_q$ whose generating function is represented in (16). Then, $A_{0,q}(y) = \frac{1}{C_{0,q}(0)}$ and we have for $\mu \in \mathbb{N}$:*

$$A_{\mu,q}(y) = \frac{(-1)^\mu}{(C_{0,q}(0))^{\mu+1}} \times \begin{vmatrix} 1 & y & y^2 & \cdots & y^{\mu-1} & y^\mu \\ C_{0,q}(0) & C_{1,q}(0) & C_{2,q}(0) & \cdots & C_{\mu-1,q}(0) & C_{\mu,q}(0) \\ 0 & C_{0,q}(0) & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q C_{1,q}(0) & \cdots & \begin{bmatrix} \mu-1 \\ 1 \end{bmatrix}_q C_{\mu-2,q}(0) & \begin{bmatrix} \mu \\ 1 \end{bmatrix}_q C_{\mu-1,q}(0) \\ 0 & 0 & C_{0,q}(0) & \cdots & \begin{bmatrix} \mu-1 \\ 2 \end{bmatrix}_q C_{\mu-3,q}(0) & \begin{bmatrix} \mu \\ 2 \end{bmatrix}_q C_{\mu-2,q}(0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_{0,q}(0) & \begin{bmatrix} \mu \\ \mu-1 \end{bmatrix}_q C_{1,q}(0) \end{vmatrix}$$

Proof. It suffices to put formula (17) in a determinantal form. □

Corollary 2 has applications in Corollary 3.

3. Scale Transformations

Now, we will study scale transformations. For $\alpha \in \mathbb{R}$ and $\mathbf{A}_q(y) \in \mathcal{A}_q$, $T_\alpha A_q(y) = \big(T_\alpha A_{\mu,q}(y)\big)_{\mu \geq 0}$ is defined as

$$\begin{cases} T_\alpha A_{\mu,q}(y) = \alpha^\mu A_{\mu,q}(y/\alpha) = \sum\limits_{k=0}^{\mu} \begin{bmatrix} \mu \\ k \end{bmatrix}_q \alpha^k A_{k,q}(0) y^{\mu-k}, \quad a \neq 0 \\ T_0 A_{\mu,q}(y) = A_{0,q}(0) y^\mu \end{cases} \tag{18}$$

where the last equality of the first equation of (18) is by using (7). These transformations will be characterized next.

Proposition 2. *Let $\alpha, \beta \in \mathbb{R}$, then $T_\alpha \mathbf{A}_q(y)$ is a q-Appell sequence if $\mathbf{A}_q(y), \mathbf{C}_q(y) \in \mathcal{A}_q$ and characterized by generating function:*

$$\mathsf{G}\big(T_\alpha \mathbf{A}_q(y), z\big) = \mathsf{G}\big(\mathbf{A}_q(0), \alpha z\big) e_q^{yz} \tag{19}$$

In addition

$$\mathsf{G}\big((T_\alpha \mathbf{A}_q \times_q T_\beta \mathbf{C}_q)(y), z\big) = \mathsf{G}\big(\mathbf{A}_q(0), \alpha z\big) \mathsf{G}\big(\mathbf{C}_q(0), \beta z\big) e_q^{yz}. \tag{20}$$

As a consequence, the map $T_\alpha : \mathcal{A}_q \longrightarrow \mathcal{A}_q$ is an isomorphism, whenever $a \neq 0$.

Proof. By (18), $T_\alpha A_{0,q}(0) = A_{0,q}(0) \neq 0$. Again by (18) and Proposition 1, we have:

$$\mathsf{G}(T_\alpha \mathbf{A}_q(y), z) = \mathsf{G}(\mathbf{A}_q(0), \alpha z) e_q^{yz} = \mathsf{G}(T_\alpha \mathbf{A}_q(0), z) e_q^{yz}$$

since $T_\alpha A_{0,q}(0) = \alpha^\mu A_{\mu,q}(0)$. Hence, the first statement in Proposition 2 follows from (8). On the other hand, we have from (13) and (19):

$$\mathsf{G}((T_\alpha \mathbf{A}_q \times_q T_\beta \mathbf{C}_q)(y), z) = \mathsf{G}(T_\alpha \mathbf{A}_q(0), z)\mathsf{G}(T_\beta \mathbf{C}_q(0), z) e_q^{yz} = \mathsf{G}(\mathbf{A}_q(0), \alpha z)\mathsf{G}(\mathbf{C}_q(0), \beta z) e_q^{yz},$$

thus showing (20). Moreover, by (13) and (19), we have

$$T_\alpha(\mathbf{A}_q \times_q \mathbf{C}_q)(y) = (T_\alpha \mathbf{A}_q \times_q T_\alpha \mathbf{C}_q)(y) \tag{21}$$

as both sides of (21) have the same generating function.

On the other hand, if $T_\alpha \mathbf{A}_q(y) = T_\alpha \mathbf{C}_q(y)$, then $\mathbf{A}_q(0) = \mathbf{C}_q(0)$, as follows from (19) and we have $\mathbf{A}_q(y) = \mathbf{C}_q(y)$. By (19), $T_\alpha(T_{\alpha^{-1}} \mathbf{A}_q)(y) = \mathbf{A}_q(y)$, thus it shows that T_α is an isomorphism and thus the proof is completed. □

The order m generalized q-Bernoulli polynomials can be expressed in terms of the q-Bernoulli polynomials $\mathbf{B}_q(y)$ as

$$\mathbf{B}_q(\alpha_1, \cdots, \alpha_m; y) = (T_{\alpha_1} \mathbf{B}_q \times_q \cdots \times_q T_{\alpha_m} \mathbf{B}_q)(y). \tag{22}$$

Relation (22) can be obtained using above the table and Proposition 2 as follows:

$$\begin{aligned}
\mathsf{G}(\mathbf{B}_q(\alpha_1, \cdots, \alpha_m; y)\ ,\ z) &= e_q^{yz} \prod_{i=1}^{m} \frac{\alpha_i z}{e_q^{\alpha_i z} - 1} \\
&= e_q^{yz} \prod_{i=1}^{m} \mathsf{G}(\mathbf{B}_q(0)\ ,\ \alpha_i z) \\
&= \mathsf{G}((T_{\alpha_1} \mathbf{B}_q \times_q T_{\alpha_2} \mathbf{B}_q \times_q \cdots \times_q T_{\alpha_m} \mathbf{B}_q)(y)\ ,\ z).
\end{aligned}$$

Similarly, by using the result of relation (20) in Proposition 2, the order m generalized q-Euler polynomials can be expressed by means of the type q-Euler polynomials $\mathbf{E}(y)$ as

$$\mathbf{E}_q(\alpha_1, \cdots, \alpha_m; y) = (T_{\alpha_1} \mathbf{E}_q \times_q \cdots \times_q T_{\alpha_m} \mathbf{E}_q)(y). \tag{23}$$

Finally, we have the relating q-Bernoulli and q-Euler polynomials:

$$(\mathbf{B}_q \times_q \mathbf{E}_q)(y) = T_2 \mathbf{B}_q(y). \tag{24}$$

From Table 1:

$$\begin{aligned}
\sum_{\mu=0}^{\infty} B_{\mu,q}\left(\frac{y}{2}\right) \frac{z^\mu}{[\mu]_q!} &= \frac{z e_q^{yz/2}}{e_q^z - 1} = \frac{z/2}{e_q^{z/2} - 1} \frac{2 e_q^{yz/2}}{e_q^{z/2} + 1} \\
&= \sum_{\mu=0}^{\infty} B_{\mu,q}(0) \frac{(z/2)^\mu}{[\mu]_q!} \sum_{s=0}^{\infty} E_{s,q}(y) \frac{(z/2)^s}{[s]_q!} \\
&= \sum_{\mu=0}^{\infty} \left(2^{-\mu} \sum_{s=0}^{\mu} \begin{bmatrix} \mu \\ s \end{bmatrix}_q B_{\mu-s,q}(0) E_{s,q}(y) \right) \frac{z^\mu}{[\mu]_q!}.
\end{aligned}$$

Table 1. Generating the function of some q-Appell type polynomials.

Notation	Polynomials	Generating Functions
$B_q(y)$	q-Bernoulli polynomials [26]	$\frac{z e_q^{yz}}{e_q^z - 1}$
$B_q(\alpha_1, \cdots, \alpha_m; y)$	Order m generalized q-Bernoulli polynomials [29] (p. 117, 4.125)	$e_q^{yz} \prod_{i=1}^m \frac{\alpha_i z}{e_q^{\alpha_i z} - 1}$
$E_q(y)$	q-Euler polynomials [26]	$\frac{2 e_q^{yz}}{e_q^z + 1}$
$E_q(\alpha_1, \cdots, \alpha_m; y)$	Order m generalized q-Euler polynomials [29] (p. 129, 4.197)	$e_q^{yz} \prod_{i=1}^m \frac{2}{e_q^{\alpha_i z} + 1}$
$G_q(y)$	q-Genocchi polynomials [26]	$\frac{2z e_q^{yz}}{e_q^z + 1}$

Then, we have:

$$B_{\mu,q}\left(\frac{y}{2}\right) = 2^{-\mu} \sum_{s=0}^{\mu} \left[\begin{array}{c} \mu \\ s \end{array} \right]_q B_{s,q}(0) E_{\mu-s,q}(y).$$

Similarly we can get:

$$B_{\mu,q}\left(\frac{y}{2}\right) = 2^{-\mu} \sum_{s=0}^{\mu} \left[\begin{array}{c} \mu \\ s \end{array} \right]_q E_{s,q}(0) B_{\mu-s,q}(y).$$

Then:

$$\begin{aligned}
(B_q \times_q E_q)_\mu(y) &= \sum_{s=0}^{\mu} \left[\begin{array}{c} \mu \\ s \end{array} \right]_q B_{s,q}(0) E_{\mu-s,q}(y) \\
&= \sum_{s=0}^{\mu} \left[\begin{array}{c} \mu \\ s \end{array} \right]_q E_{s,q}(0) B_{\mu-s,q}(y) \\
&= 2^{-\mu} B_{\mu,q}\left(\frac{y}{2}\right) \\
&= T_2 B_{\mu,q}(y).
\end{aligned}$$

4. Transformations Based on Expectations

Let X be a random variables such that:

$$E e_q^{r|X|} < \infty, \text{ for } r > 0.$$

Here, we consider expectations and transformations of q-Appell sequences $\mathbf{A}_q(y)$ by replacing x by X in (10) similar to a classical analogue [7,30]. These transformations are the result due to a probabilistic approach to q-Appell polynomials. For $\mathbf{A}_q(y) \in \mathcal{A}_q$ and a random variable X, we define $R_X \mathbf{A}_q(y) = \left(R_X A_{\mu,q}(y)\right)_{\mu \geq 0}$ as

$$\begin{aligned}
R_X A_{\mu,q}(y) = E A_{\mu,q}(y \oplus_q X) &= \sum_{k=0}^{\mu} \left[\begin{array}{c} \mu \\ k \end{array} \right]_q A_{k,q}(0) E(y \oplus_q X)^{\mu-k} \\
= \sum_{k=0}^{\mu} \left[\begin{array}{c} \mu \\ k \end{array} \right]_q E X^k A_{\mu-k,q}(y) &= \sum_{k=0}^{\mu} \left[\begin{array}{c} \mu \\ k \end{array} \right]_q E A_{k,q}(X) y^{\mu-k},
\end{aligned} \tag{25}$$

follow from (7) and (10). Notice that whenever $X = 0$, $R_X \mathbf{A}_q(y) = \mathbf{A}_q(y)$. In addition:

$$R_X \mathbf{I}_q(y) = \left(E(y \oplus_q X)^\mu\right)_{\mu \geq 0}. \tag{26}$$

Identity $\mathbf{I}_q(y)$ transformation plays an important role as which is evident from equality in (25), we obtain:

$$R_X \mathbf{A}_q(y) = (\mathbf{A}_q \times_q R_X \mathbf{I}_q)(y). \tag{27}$$

Now, we will study some results following the characterization of the transformations based on expectations and will present some of their properties.

Proposition 3. *Let Y and X denote two independent random variables and $\boldsymbol{A}_q(y), \boldsymbol{C}_q(y) \in \mathcal{A}_q$. Then, $R_X \boldsymbol{A}_q(y)$ is a q-Appell sequence represented in terms of generating a function as*

$$\mathsf{G}\big(R_X \boldsymbol{A}_q(y), z\big) = \mathsf{G}\big(\boldsymbol{A}_q(0), z\big) E e_q^{z(y \oplus_q X)}. \tag{28}$$

Moreover, the following properties are true:
(a) $R_X(\boldsymbol{A}_q \times_q \boldsymbol{C}_q)(y) = (R_X \boldsymbol{A}_q \times_q \boldsymbol{C}_q)(y) = (\boldsymbol{A}_q \times_q R_X \boldsymbol{C}_q)(y).$
(b) $(R_Y \boldsymbol{A}_q \times_q R_X \boldsymbol{C}_q)(y) = R_{Y \oplus_q X}(\boldsymbol{A}_q \times_q \boldsymbol{C}_q)(y).$
(c) $R_Y R_X \boldsymbol{A}_q(y) = R_{Y \oplus_q X} \boldsymbol{A}_q(y).$

Proof. Using (25), $R_X \mathbf{A}_{0,q}(0) = \mathbf{A}_{0,q}(0) \neq 0$. By interchanging sum with expectation, from (26) we have:

$$\mathsf{G}\big(R_X \mathbf{I}_q(y), z\big) = \sum_{\mu=0}^{\infty} E(y \oplus_q X)^{\mu} \frac{z^{\mu}}{[\mu]_q!} = E e_q^{z(y \oplus_q X)}. \tag{29}$$

□

Thus, from (13), (27)–(29) can be obtained. Formula (28) implies that:

$$\mathsf{G}\big(R_X \mathbf{A}_q(y), z\big) = \mathsf{G}\big(R_X \mathbf{A}_q(0), z\big) e_q^{yz}.$$

Combining it with (8), will prove the first statement in Proposition 3. From Proposition 1 and (27), we obtain:

$$R_X(\mathbf{A}_q \times_q \mathbf{C}_q)(y) = (\mathbf{A}_q \times_q \mathbf{C}_q \times_q R_X \mathbf{I}_q)(y) = (R_X \mathbf{A}_q \times_q \mathbf{C}_q)(y) = (\mathbf{A}_q \times_q R_X \mathbf{C}_q)(y),$$

hence, we obtain (a). Notice that:

$$(R_Y \mathbf{I}_q \times_q R_X \mathbf{I}_q)(y) = R_{Y \oplus_q X} \mathbf{I}_q(y),$$

as it is clear from (13), (29), and the independence between the variables Y and X. From (27) and Proposition 1, we obtain:

$$(R_Y \mathbf{A}_q \times_q R_X \mathbf{C}_q)(y) = (\mathbf{A}_q \times_q R_Y \mathbf{I}_q \times_q \mathbf{C}_q \times_q R_X \mathbf{I}_q)(y) = R_{Y \oplus_q X}(\mathbf{A}_q \times_q \mathbf{C}_q)(y),$$

which justifies (b). Similarly, one can prove (c).

It is to note that the map $R_X : \mathcal{A}_q \longrightarrow \mathcal{A}_q$ is not a homomorphism, which follows from Proposition 3(a).

5. The Subset $\mathcal{R}_q$

Let us consider the exponential moments (finite) $M_X(t) := E(e_q^{tX})$ for a random variable X s.t. $\mid t \mid < \rho$ similar to its classical case defined in [30]. Then, $E(X^{\mu}) < \infty$ for all $\mu = 1, 2, \cdots$, and:

$$E(e_q^{tX}) = \sum_{\mu=0}^{\infty} \frac{t^{\mu}}{[\mu]_q!} E(X^{\mu}), \quad \mid t \mid < \rho.$$

For $t \in \mathbb{C}$, the right hand side of (5) will be a complex analytic function $z \mapsto \varphi(z)$, $\mid z \mid < \rho$, $z \in \mathbb{C}$. As $M_X(0) = 1$, $\mid \varphi(z) \mid > 0$ for $\mid z \mid < \rho$ due to the continuity of φ. Thus,

$z \mapsto \frac{1}{\varphi(z)}$ is a well-defined analytic function and can be expressed in the form of power series:

$$\frac{1}{\varphi(z)} = \sum_{\mu=0}^{\infty} c_{\mu,q} z^{\mu}, \quad | z |< \lambda,$$

where $\lambda > 0$ denotes the radius of convergence. For $z = t \in \mathbb{R}$ such that $| z |< \lambda$ we have:

$$\frac{1}{E(e_q^{tX})} = \sum_{\mu=0}^{\infty} c_{\mu,q} t^{\mu} = \sum_{\mu=0}^{\infty} \hat{c}_{\mu,q} \frac{t^{\mu}}{[\mu]_q!},$$

where $\hat{c}_{\mu,q} = c_{\mu,q}[\mu]_q!$. For $y \in \mathbb{R}$ and $| t |< \lambda$ it holds:

$$\begin{aligned} \frac{e_q^{ty}}{E(e_q^{tX})} &= \sum_{s=0}^{\infty} \frac{t^s}{[s]_q!} y^s \sum_{m=0}^{\infty} \frac{t^m}{[m]_q!} \hat{c}_{m,q} \\ &= \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^{s+m}}{[s]_q![m]_q!} \hat{c}_{m,q} y^s \\ &= \sum_{\mu=0}^{\infty} \frac{t^{\mu}}{[\mu]_q!} \sum_{m=0}^{\infty} \begin{bmatrix} \mu \\ m \end{bmatrix}_q \hat{c}_m y^{\mu-m}. \end{aligned} \tag{30}$$

As both series are absolutely convergent in the neighborhood of the origin, the sum will therefore not depend on the order of summation. Motivated by (30), we now present the definition of q-Appell polynomials related to a random variable X.

Definition 1. *Let X be a random variable having some exponential moments. The polynomials $A_{\mu,q}(y), \ \mu = 0, 1, 2, \cdots$, satisfying:*

$$\sum_{\mu=0}^{\infty} \frac{t^{\mu}}{[\mu]_q!} A_{\mu,q}(y) = \frac{e_q^{ty}}{E(e_q^{tX})} \quad \textit{for all } y \in \mathbb{R} \tag{31}$$

where $A_{\mu,q}(y)$ are called as q-Appell polynomials of order μ associated with random variable X.

Now, we denote by $\mathcal{R}_q$ the set of Appell sequences $\mathbf{A}_q(y)$ such that:

$$\mathsf{G}(\mathbf{A}_q(y), z) = \frac{e_q^{yz}}{E e_q^{zX}}, \tag{32}$$

for a random variable X. For another random variable Y satisfying (32), due to the uniqueness theorem for characteristic functions, Y and X will follow same law. Thus, $\mathbf{A}_q(y)$ has associated the random variable X. Notice that for $X = 0$, $\mathbf{I}_q(y) \in \mathcal{R}_q$. Then, we present a preposition for the construction of other q-Appell polynomials.

Proposition 4. *For associated independent random variable Y and X and let: $\boldsymbol{A}_q(y), \boldsymbol{C}_q(y) \in \mathcal{R}_q$, respectively, where $\alpha, \beta \in \mathbb{R}$. Then, $(T_\alpha \boldsymbol{A}_q \times_q T_\beta \boldsymbol{C}_q)(y)$ belong to a $\mathcal{R}_q$ with the associated random variable $\alpha Y \oplus_q \beta X$.*

In particular, if $\boldsymbol{A}_q(y), \boldsymbol{C}_q(y) \in \mathcal{R}_q$, then $(\boldsymbol{A}_q \times_q \boldsymbol{C}_q)(y)$ belongs to $\mathcal{R}_q$ with associated random variable $Y \oplus_q X$.

Proof. From Equations (13), (19), and (32) and using the property that Y and X are independent:

$$
\begin{aligned}
\mathsf{G}\big((T_\alpha \mathbf{A}_q \times_q T_\beta \mathbf{C}_q)(y), z\big) &= \mathsf{G}\big(T_\alpha \mathbf{A}_q(0), z\big)\mathsf{G}\big(T_\beta \mathbf{C}_q(0), z\big)e_q^{yz} \\
&= \mathsf{G}\big(\mathbf{A}_q(0), \alpha z\big)\mathsf{G}\big(\mathbf{C}_q(0), \beta z\big)e_q^{yz} \\
&= \frac{e_q^{yz}}{Ee_q^{z(\alpha Y \oplus_q \beta X)}}
\end{aligned}
$$

where we get the required result. □

The q-Appell sequence in $\mathcal{R}_q$ is characterized as follows.

Theorem 2. *The following statements are equivalent:*
(a) $\boldsymbol{A}_q(y) \in \mathcal{R}_q$ *with the associated random variable* X.
(b) $\mathsf{G}\big(\boldsymbol{A}_q(0), z\big) = (Ee_q^{zX})^{-1}$.
(c) $R_X\boldsymbol{A}_q(y) = \boldsymbol{I}_q(y)$.
(d) The inverse element of $\boldsymbol{A}_q(y)$ *is* $R_X\boldsymbol{I}_q(y)$.

Proof. From (32), equivalence of (a) and (b) can be obtained. Similarly, (27) gives equivalence between (c) and (d). If (b) is true, then from (28), we have:

$$
\mathsf{G}\big(R_X\mathbf{A}_q(y), z\big) = \mathsf{G}\big(\mathbf{A}_q(0), z\big)Ee_q^{z(y\oplus_q X)} = e_q^{yz} = \mathsf{G}\big(\mathbf{I}_q(y), z\big),
$$

which together with (8), shows (c). Finally, if (d) holds, we see from (13) and (28):

$$
e_q^{yz} = \mathsf{G}\big((R_X\mathbf{I}_q \times_q \mathbf{A})(y), z\big) = \mathsf{G}\big(R_X\mathbf{I}_q(0), z\big)\mathsf{G}\big(\mathbf{A}_q(0), z\big)e_q^{yz} = Ee_q^{zX}\mathsf{G}\big(\mathbf{A}_q(0), z\big)e_q^{yz}
$$

thus showing (b) which completes the proof. □

For any q-Appell sequence, its determinantal form $\mathbf{A}_q(y) \in \mathcal{R}_q$ can be expressed in terms of the moments of its associated random variable X.

Corollary 3. *Let* $\boldsymbol{A}_q(y) \in \mathcal{R}_q$ *with associated random variable* X. *Denote* $R_X I_{\mu,q}(0) = EX^\mu =: T_{\mu,q}$. *Then,* $\boldsymbol{A}_{0,q}(y) = \frac{1}{T_{0,q}} = 1$ *and:*

$$
A_{\mu,q}(y) = (-1)^\mu \begin{vmatrix}
1 & y & y^2 & \cdots & y^{\mu-1} & y^\mu \\
1 & T_{1,q} & T_{2,q} & \cdots & T_{\mu-1,q} & T_{\mu,q} \\
0 & 1 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q T_{1,q} & \cdots & \begin{bmatrix} \mu-1 \\ 1 \end{bmatrix}_q T_{\mu-2,q} & \begin{bmatrix} \mu \\ 1 \end{bmatrix}_q T_{\mu-1,q} \\
0 & 0 & 1 & \cdots & \begin{bmatrix} \mu-1 \\ 2 \end{bmatrix}_q T_{\mu-3,q} & \begin{bmatrix} \mu \\ 2 \end{bmatrix}_q T_{\mu-2,q} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix} \mu \\ \mu-1 \end{bmatrix}_q T_{1,q}
\end{vmatrix}, \quad \mu \in \mathbb{N} \tag{33}
$$

Proof. From Corollary 2 and Theorem 2(d) proof follows. □

Another important result from Theorem 2 is as follows:

Corollary 4. *Let* $\boldsymbol{A}_q(y) \in \mathcal{R}_q$ *with associated random variable* X. *For any* $\boldsymbol{C}_q(y) \in \mathcal{A}_q$, *we have:*

$$
\boldsymbol{C}_q(y) = E(\boldsymbol{C}_q \times_q \boldsymbol{A}_q)(y \oplus_q X). \tag{34}
$$

As a consequence, we have for any $x \in \mathbb{R}$:

$$C_{\mu,q}(y \oplus_q x) = E(C_q \times_q A_q)_\mu (y \oplus_q x \oplus_q X) = \sum_{k=0}^{\mu} \left[\begin{array}{c} \mu \\ k \end{array} \right]_q EC_{k,q}(x \oplus_q X) A_{\mu-k,q}(y). \quad (35)$$

In particular:

$$y^\mu = EA_{\mu,q}(y \oplus_q X) = \sum_{k=0}^{\mu} \left[\begin{array}{c} \mu \\ k \end{array} \right]_q T_{k,q} A_{\mu-k,q}(y). \quad (36)$$

Proof. By Proposition 3(a) and Theorem 2(c), we have:

$$\mathbf{C}_q(y) = (\mathbf{I}_q \times_q \mathbf{C}_q)(y) = (R_X \mathbf{A}_q \times_q \mathbf{C}_q)(y) = R_X(\mathbf{A}_q \times_q \mathbf{C}_q)(y) = E(\mathbf{A}_q \times_q \mathbf{C}_q)(y \oplus_q X),$$

which shows (34). Formula (35) follows by replacing y by $y \oplus_q x$ in (34) and then applying (14). Identity (36) follows by setting $\mathbf{C}_q(y) = \mathbf{I}_q(y)$ and $x = 0$ in (35). Thus, the proof is completed. □

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