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Fixed Points Results for Various Types of Tricyclic Contractions

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Abstract: In this paper, we introduce four new types of contractions called in this order Kanan-S-type tricyclic contraction, Chattergea-S-type tricyclic contraction, Riech-S-type tricyclic contraction, Cirić-S-type tricyclic contraction, and we prove the existence and uniqueness for a fixed point for each situation.

Keywords: fixed points; S-type tricyclic contraction; metric spaces

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1. Introduction

It is well known that the Banach contraction principle was published in 1922 by S. Banach as follows:

Theorem 1. *Let (X,d) be a complete metric space and a self mapping $T : X \longrightarrow X$. If there exists $k \in [0,1)$ such that, for all $x,y \in X$, $d(Tx,Ty) \leq kd(x,y)$, then T has a unique fixed point in X.*

The Banach contraction principle has been extensively studied and different generalizations were obtained.

In 1968 [1], Kannan established his famous extension of this contraction.

Theorem 2. *Ref. [1] Let (X,d) be a complete metric space and a self mapping $T : X \longrightarrow X$. If T satisfies the following condition:*

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)] \quad \text{for all} \quad x,y \in X \quad \text{where} \quad 0 < k < \frac{1}{2},$$

then T has a fixed point in X.

A similar contractive condition has been introduced by Chattergea in 1972 [2] as follows:

Theorem 3. *Ref. [2] Let $T : X \longrightarrow X$, where (X,d) is a complete metric space. If there exists $0 < k < \frac{1}{2}$ such that*

$$d(Tx,Ty) \leq k[d(y,Tx) + d(Ty,x)] \quad \text{for all} \quad x,y \in X,$$

then T has a fixed point in X.

We can also find another extension of the Banach contraction principle obtained by S. Reich, Kannan in 1971 [3].

Theorem 4. *Ref. [3] Let $T : X \longrightarrow X$, where (X,d) is a complete metric space. If there exists $0 < k < \frac{1}{3}$ such that*

$$d(Tx,Ty) \leq k[d(x,y) + d(x,Tx) + d(y,Ty)] \quad \text{for all} \quad x,y \in X,$$

then T has a fixed point in X.

In addition, in the same year, Cirić gave the following extension [4].

Theorem 5. *Ref. [4] Let $T : X \longrightarrow X$, where (X, d) a complete metric space. If there exists $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kMax[d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(Ty, x)] \quad \text{for all} \quad x, y \in X,$$

then T has a fixed point in X.

Many authors have investigated these situations and many results were proved (see [5–13]).

In this article, we prove the uniqueness and existence of the fixed points in different types contractions for a self mapping T defined on the union of tree closed subsets of a complete metric space with k in different intervals.

2. Preliminaries

In best approximation theory, the concept of tricyclic mappings extends that of ordinary cyclic mappings. Moreover, in the case where two of the sets, say A and C, coincide, we find a cyclic mapping which is also a self-map, and, hence, a best proximity point result for a tricyclic mappings means also a fixed point and a best proximity point result for a self-map and a cyclic mapping.

Definition 1. *Let A, B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \longrightarrow A \cup B$ is said to be cyclic if :*

$$T(A) \subseteq B, T(B) \subseteq A.$$

In 2003, Kirk et al. [14] proved that, if $T : A \cup B \longrightarrow A \cup B$ is cyclic and, for some $k \in (0, 1)$, $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A, y \in B$, then $A \cap B \neq \emptyset$, and T has a unique fixed point in $A \cap B$.

In 2017, Sabar et al. [15] proved a similar result for tricyclic mappings and introduced the concept of tricyclic contractions.

Theorem 6. *Ref. [15] Let A, B and C be nonempty closed subsets of a complete metric space (X, d), and let a mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$. If $T(A) \subseteq B, T(B) \subseteq C$ and $T(C) \subseteq A$ and there exists $k \in (0, 1)$ such that $D(Tx, Ty, Tz) \leq kD(x, y, z)$ for all $(x, y, z) \in A \times B \times C$, then $A \cap B \cap C$ is nonempty and T has a unique fixed point in $A \cap B \cap C$,*

where $D(x, y, z) = d(x, y) + d(x, z) + d(y, z)$.

Definition 2. *Ref. [15] Let A, B and C be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ is said to be tricyclic contracton if there exists $0 < k < 1$ such that:*

1. *$T(A) \subseteq B, T(B) \subseteq C$ and $T(C) \subseteq A$.*
2. *$D(Tx, Ty, Tz) \leq kD(x, y, z) + (1 - k)\delta(A, B, C)$ for all $(x, y, z) \in A \times B \times C$.*

where $\delta(A, B, C) = \inf\{D(x, y, z) : x \in A, y \in B, z \in C\}$

Very Recently, Sabiri et al. introduced an extension of the aforementioned mappings and called them p-cyclic contractions [16].

3. Main Results

Definition 3. *Let A, B and C be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ is said to be a Kannan-S-type tricyclic contraction, if there exists $k \in \left(0, \frac{1}{3}\right)$ such that*

1. $T(A) \subseteq B, T(B) \subseteq C, T(C) \subseteq A$.
2. $D(Tx, Ty, Tz) \leq k[d(x, Tx) + d(y, Ty) + d(z, Tz)]$ *for all* $(x, y, z) \in A \times B \times C$.

We give an example to show that a map can be a tricyclic contraction but not a Kannan-S-type tricyclic contraction.

Example 1. *Let X be* $\mathbb{R}^2$ *normed by the norm* $\| (x, y) \| = |x| + |y|$, *and* $A = [1, 2] \times \{0\}, B = \{0\} \times [-2, -1], C = [-2, -1] \times \{0\}$, *then*

$$\delta(A, B, C) = D((1, 0), (0, -1), (-1, 0)) = 6.$$

Put $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ *such that*

$$T(x, 0) = \left(0, -\frac{x+2}{3}\right) \quad \text{if } (x, 0) \in A,$$

$$T(0, y) = \left(\frac{y-2}{3}, 0\right) \quad \text{if } (0, y) \in B,$$

$$T(z, 0) = \left(-\frac{z-2}{3}, 0\right) \quad \text{if } (z, 0) \in C,$$

We have $T(A) \subseteq B, T(B) \subseteq C$ *and* $T(C) \subseteq A$, *and*

$$\begin{aligned} D(T(x, 0), T(0, y), T(z, 0)) &= D\left((0, -\frac{x+2}{3}), (\frac{y-2}{3}, 0), (-\frac{z-2}{3}, 0)\right) \\ &= \frac{2}{3}(x - y - z) + 4 \\ &= \frac{1}{3}D((x, 0), (0, y), (z, 0)) + 4 \\ &= \frac{1}{3}D((x, 0), (0, y), (z, 0)) + (1 - \frac{1}{3})\delta(A, B, C) \end{aligned}$$

for all $(x, 0) \in A, (0, y) \in B, (z, 0) \in C$.
On the other hand,

$$D(T(2, 0), T(0, -2), T(-2, 0)) = D\left((0, -\frac{4}{3}), (\frac{-4}{3}, 0), (\frac{4}{3}, 0)\right) = 8$$

and

$$d((2, 0), T(2, 0)) + d((0, -2), T(0, -2)) + d((-2, 0), T(-2, 0)) = 10,$$

which implies that
$D(T(2, 0), T(0, -2), T(-2, 0))$

$$> \frac{1}{3}[d((2, 0), T(2, 0)) + d((0, -2), T(0, -2)) + d((-2, 0), T(-2, 0))]$$

Then, T is tricyclic contraction but not a Kannan-S-type tricyclic contraction.

Now, we give an example for which T is a Kannan-S-type tricyclic contraction but not a tricyclic contraction.

Example 2. *Let* $X = \mathbb{R}$ *with the usual metric. Let* $A = B = C = [0, 1]$, *then* $\delta(A, B, C) = 0$. *Put* $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ *such that*

$$Tx = \frac{1}{6} \text{ if } 0 \leq x < 1, \quad Tx = \frac{1}{4} \text{ if } x = 1$$

For $x = 1, y = 1$ *and* $z = \frac{23}{24}$, *we have*

$$D(T(1), T(1), T(\frac{23}{24})) = D(\frac{1}{4}, \frac{1}{4}, \frac{1}{6}) = 2d(\frac{1}{4}, \frac{1}{6}) = \frac{1}{6}.$$

and

$$D(1, 1, \frac{23}{24}) = 2d(1, \frac{23}{24}) = \frac{1}{12}.$$

Then, T is not tricyclic contraction.
However T is a Kannan-S-type tricyclic contraction. Indeed:

- *If* $x = y = z = 1$, *we have*

$$D(T(1), T(1), T(1)) = 0 \leq \frac{9}{4}k$$

for all $k \geq 0$, *then for* $0 \leq k < \frac{1}{3}$.

- *If* $x \in [0,1)$, $y \in [0,1)$ *and* $z \in [0,1)$, *we have*

$$D(Tx, Ty, Tz) = 0 \leq k(d(x, \frac{1}{6}) + d(y, \frac{1}{6}) + d(z, \frac{1}{6})$$

for all $k \geq 0$, *then for* $0 \leq k < \frac{1}{3}$.

- *If* $x = 1, y \in [0,1)$ *and* $z \in [0,1)$, *we have*

$$D(T_1, Ty, Tz) = D(\frac{1}{4}, \frac{1}{6}, \frac{1}{6}) = \frac{1}{6}$$

and

$$d(1, T(1)) + d(y, Ty) + d(z, Tz) = \frac{3}{4} + d(y, \frac{1}{6}) + d(z, \frac{1}{6}),$$

then, for $k = \frac{2}{9}$, *we have*

$$D(T(1), T(y, Tz) \leq k(d(1, T(1)) + d(y, Ty) + d(z, Tz)).$$

- *If* $x = 1, y = 1$ *and* $z \in [0,1)$, *we have*

$$D(T(1), T(1), Tz) = D(\frac{1}{4}, \frac{1}{4}, \frac{1}{6}) = \frac{1}{6}$$

and

$$d(1, T(1)) + d(1, T(1)) + d(z, Tz) = \frac{3}{2} + d(z, \frac{1}{6}).$$

Then, for $k = \frac{2}{9}$, *we have*

$$D(T(1), T(1), Tz) \leq k(d(1, T(1)) + d(1, T(1)) + d(z, Tz)).$$

Consequently, for $k = \frac{2}{9}$, *we have :*

$$D(Tx, Ty, Tz) \leq k(d(x, Tx) + d(y, Ty) + d(z, Tz)) \textit{ for all } (x, y, z) \in A \times B \times C.$$

Theorem 7. *Let* A, B *and* C *be nonempty closed subsets of a complete metric space* (X, d), *and let* $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ *be a Kannan-S-type tricyclic contraction. Then, T has a unique fixed point in* $A \cap B \cap C$.

Proof. Fix $x \in A$. We have

$$d\left(T^3x, T^2x\right) \leq D\left(T^3x, T^2x, Tx\right) \leq k\left[d\left(T^2x, T^3x\right) + d\left(Tx, T^2x\right) + d(x, Tx)\right].$$

Then,
$$d\left(T^3x, T^2x\right) \le k\left[d\left(T^2x, T^3x\right) + d\left(Tx, T^2x\right) + d(x, Tx)\right],$$
which implies
$$d\left(T^3x, T^2x\right) \le \frac{k}{(1-k)}\left[d\left(Tx, T^2x\right) + d(x, Tx)\right].$$

Similarly, we have
$$d\left(T^2x, Tx\right) \le \frac{k}{(1-k)}\left[d\left(T^3x, T^2x\right) + d(x, Tx)\right]$$
$$d\left(T^2x, Tx\right) \le \frac{k}{(1-k)}\left[\frac{k}{(1-k)}\left[d\left(Tx, T^2x\right) + d(x, Tx)\right] + d(x, Tx)\right]$$
$$\Longrightarrow d\left(T^2x, Tx\right) \le \frac{k}{1-2k}(d(x, Tx)).$$

Then,
$$d\left(T^2x, Tx\right) \le td(x, Tx) \text{ where } t = \frac{k}{1-2k} \text{ and } t \in (0,1),$$
which implies
$$d\left(T^{n+1}x, T^nx\right) \le t^n d(x, Tx), \text{for all } n \ge 1$$
Consequently,
$$\sum_{n=1}^{+\infty} d\left(T^{n+1}x, T^nx\right) \le (\sum_{n=1}^{+\infty} t^n) d(x, Tx) < +\infty$$
implies that $\{T^nx\}$ is a Cauchy sequence in (X, d). Hence, there exists $z \in A \cup B \cup C$ such that $T^nx \longrightarrow z$. Notice that $\{T^{3n}x\}$ is a sequence in A, $\{T^{3n-1}x\}$ is a sequence in C and $\{T^{3n-2}x\}$ is a sequence in B and that both sequences tend to the same limit z. Regarding the fact that A, B and C are closed, we conclude $z \in A \cap B \cap C$, hence $A \cap B \cap C \ne \emptyset$.
To show that z is a fixed point, we must show that $Tz = z$. Observe that
$$\begin{aligned} d(Tz, z) &= \lim d\left(Tz, T^{3n}x\right) \le \lim D\left(T^{3n}x, T^{3n-1}x, Tz\right) \\ &\le \lim k[d\left(T^{3n-1}x, T^{3n}x\right) + d\left(T^{3n-2}x, T^{3n-1}x\right) + d(z, Tz)] \\ &\le kd(Tz, z), \end{aligned}$$
which is equivalent to
$$(1-k)d(Tz, z) = 0.$$

Since $k \in \left(0, \frac{1}{3}\right)$, then $d(Tz, z) = 0$, which implies $Tz = z$.

To prove the uniqueness of z,, assume that there exists $w \in A \cup B \cup C$ such that $w \ne z$ and $Tw = w$. Taking into account that T is tricyclic, we get $w \in A \cap B \cap C$. We have
$$d(z, w) = d(Tz, Tw) \le D(Tz, Tw, Tw) \le k[d(z, Tz) + d(w, Tw) + d(w, Tw)] = 0$$
which implies $d(z, w) = 0$. We get that $z = w$ and hence z is the unique fixed point of T. □

Example 3. *Let X be $\mathbb{R}^2$ normed by the norm $\| (x, y) \| = |x| + |y|$, let $A = \{0\} \times [0, +1]$, $B = [0, +1] \times \{0\}$, $C = \{0\} \times [-1, 0]$ and let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be defined by*
$$T(0, x) = \left(\frac{x}{6}, 0\right) \qquad \text{if } (0, x) \in A\ ,$$
$$T(y, 0) = \left(0, \frac{-y}{6}\right) \qquad \text{if } (y, 0) \in B,$$

$$T(0,z) = \left(0, \frac{-z}{6}\right) \qquad \text{if } (0,z) \in C.$$

We have

$$T(A) \subseteq B, T(B) \subseteq C \text{ and } T(C) \subseteq A$$

In addition, for all $(0,x) \in A$, $(y,0) \in B$, $(0,z) \in C$, *we have*

$$D(T(0,x), T(y,0), T(0,z)) = D\left(\left(\frac{x}{6}, 0\right), \left(0, \frac{-y}{6}\right), \left(0, \frac{-z}{6}\right)\right) = \frac{1}{3}(x+y-z)$$

In addition, we have

$$d((0,x), T(0,x)) + d((y,0), T(y,0)) + d((0,z), T(0,z)) = \frac{7}{6}(x+y-z)$$

This implies

$$D(T(0,x), T(y,0), T(0,z)) = \frac{2}{7}[d((0,x), T(0,x)) + d((y,0), T(y,0)) + d((0,z), T(0,z))].$$

Then, T is a Kannan-S-type tricyclic contraction, and T has a unique fixed point $(0,0)$ in $A \cap B \cap C$.

Corollary 1. *Let (X,d) be a complete metric space and a self mapping $T : X \longrightarrow X$. If there exists $k \in \left(0, \frac{1}{3}\right)$ such that*

$$D(Tx, Ty, Tz) \leq k[d(x,Tx) + d(y,Ty) + d(z,Tz)]$$

for all $(x,y,z) \in X^3$, then T has a unique fixed point.

Now, we shall define another type of a tricyclic contraction.

Definition 4. *Let A, B and C be nonempty subsets of a metric space (X,d). A mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ is said to be a Chattergea-S-type tricyclic contraction if $T(A) \subseteq B, T(B) \subseteq C, T(C) \subseteq A$, and there exist $k \in \left(0, \frac{1}{3}\right)$ such that $D(Tx, Ty, Tz) \leq k[d(y,Tx) + d(z,Ty) + d(x,Tz)]$ for all $(x,y,z) \in A \times B \times C$.*

Theorem 8. *Let A, B and C be nonempty closed subsets of a complete metric space (X,d), and let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a Chattergea-S-type tricyclic contraction. Then, T has a unique fixed point in $A \cap B \cap C$.*

Proof. Fix $x \in A$. We have

$$D\left(Tx, T^2x, T^3x\right) \leq k\left[d(Tx,Tx) + d\left(T^2x, T^2x\right) + d\left(T^3x, x\right)\right]$$

which implies

$$D\left(T^3x, T^2x, Tx\right) \leq kd\left(T^3x, x\right)$$

so

$$d\left(T^3x, T^2x\right) \leq k\left[d\left(T^3x, T^2x\right) + d\left(T^2x, Tx\right) + d(Tx,x)\right] \text{ (by the triangular inequality)}$$

$$\Longrightarrow d\left(T^3x, T^2x\right) \leq \frac{k}{(1-k)}\left[d\left(Tx, T^2x\right) + d(x,Tx)\right]$$

and

$$d\left(T^2x, Tx\right) \leq D\left(T^3x, T^2x, Tx\right) \leq \frac{k}{(1-k)}\left[d\left(T^3x, T^2x\right) + d(x,Tx)\right]$$

$$\Longrightarrow d\left(T^2x, Tx\right) \leq \frac{k}{(1-k)}\left[\frac{k}{(1-k)}\left[d\left(Tx, T^2x\right) + d(x, Tx)\right] + d(x, Tx)\right]$$

$$\Longrightarrow d\left(T^2x, Tx\right) \leq \frac{k}{1-2k}(d(x, Tx))$$

Then,

$$d\left(T^2x, Tx\right) \leq td(x, Tx) \text{ where } t = \frac{k}{1-2k} \text{ and } t \in (0,1),$$

which implies

$$d\left(T^{n+1}x, T^n x\right) \leq t^n d(x, Tx)$$

for all $n \geq 1$. Consequently,

$$\sum_{n=1}^{+\infty} d\left(T^{n+1}x, T^n x\right) \leq (\sum_{n=1}^{+\infty} t^n) d(x, Tx) < +\infty$$

implies that $\{T^n x\}$ is a Cauchy sequence in (X, d). Hence, there exists $z \in A \cup B \cup C$ such that $T^n x \longrightarrow z$. Notice that $\{T^{3n}x\}$ is a sequence in A, $\{T^{3n-1}x\}$ is a sequence in C, and $\{T^{3n-2}x\}$ is a sequence in B and that both sequences tend to the same limit z. Regarding that A, B and C are closed, we conclude $z \in A \cap B \cap C$, hence $A \cap B \cap C \neq \varnothing$.

To show that z is a fixed point, we must show that $Tz = z$. Observe that

$$\begin{aligned} d(Tz, z) &= \lim d\left(Tz, T^{3n}x\right) \leq \lim D\left(Tz, T^{3n}x, T^{3n-1}x\right) \\ &\leq \lim k[d\left(T^{3n-1}x, Tz\right) + \left(T^{3n-2}x, T^{3n}x\right) + d(z, T^{3n-1}x)] \leq kd(Tz, z), \end{aligned}$$

which is equivalent to $(1-k)d(Tz, z) = 0$. Since $k \in \left(1, \frac{1}{3}\right)$, then $d(Tz, z) = 0$, which implies $Tz = z$.

To prove the uniqueness of z, assume that there exists $w \in A \cup B \cup C$ such that $w \neq z$ and $Tw = w$. Taking into account that T is tricyclic, we get $w \in A \cap B \cap C$.

We have

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \leq D(Tz, Tw, Tw) \\ &\leq k[d(Tz, w) + d(Tw, w) + d(Tw, z)] \\ &\leq 2kd(z, w). \end{aligned}$$

Then, $d(z, w) = 0$. We conclude that $z = w$ and hence z is the unique fixed point of T. □

Corollary 2. *Let (X, d) be a complete metric space and a self mapping $T : X \longrightarrow X$. If there exists $k \in \left(0, \frac{1}{3}\right)$ such that*

$$D(Tx, Ty, Tz) \leq k[d(y, Tx) + d(z, Ty) + d(x, Tz)]$$

for all $(x, y, z) \in X^3$, then T has a unique fixed point.

In this step, we define a Reich-S-type tricyclic contraction.

Definition 5. *Let A, B and C be nonempty subsets of a metric space (X, d).*
A mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ is said to be a Reich-S-type tricyclic contraction if there exists $k \in \left(0, \frac{1}{7}\right)$ such that:

1. $T(A) \subseteq B, T(B) \subseteq C, T(C) \subseteq A$.

2. $D(Tx, Ty, Tz) \leq k[D(x, y, z) + d(x, Tx) + d(y, Ty) + d(z, Tz)]$ *for all* $(x, y, z) \in A \times B \times C$.

Theorem 9. *Let A, B and C be nonempty closed subsets of a complete metric space (X, d), and let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a Reich-S-type tricyclic contraction. Then, T has a unique fixed point in $A \cap B \cap C$.*

Proof. Fix $x \in A$. We have

$$\begin{aligned} d\Big(T^2x, T^3x\Big) &\leq D\Big(Tx, T^2x, T^3x\Big) \\ &\leq k\Big[D(x, Tx, T^2x) + d\Big(T^2x, T^3x\Big) + d\Big(Tx, T^2x\Big) + d(x, Tx)\Big] \end{aligned}$$

$$\Longrightarrow d\Big(T^2x, T^3x\Big)(1-k) \leq k[2d\Big(T^2x, Tx\Big) + 2d(x, Tx) + d\Big(T^2x, x\Big)]$$

$$\begin{aligned} \Longrightarrow d\Big(T^2x, T^3x\Big) &\leq \frac{k}{1-k}\Big[2d\Big(T^2x, Tx\Big) + 2d(x, Tx) + d\Big(T^2x, x\Big)\Big] \\ &\leq \frac{k}{1-k}\Big[2d\Big(T^2x, Tx\Big) + 2d(x, Tx) + d\Big(T^2x, Tx\Big) + d(Tx, x)\Big] \\ &\leq \frac{k}{1-k}\Big[3d\Big(T^2x, Tx\Big) + 3d(x, Tx)\Big] \end{aligned}$$

$$\Longrightarrow d\Big(T^2x, T^3x\Big) \leq \frac{3k}{1-k}[d\Big(T^2x, Tx\Big) + d(x, Tx)]$$

and

$$d\Big(T^2x, Tx\Big) \leq D\Big(Tx, T^2x, T^3x\Big) \leq k\Big[D(x, Tx, T^2x) + d\Big(T^2x, T^3x\Big) + d\Big(Tx, T^2x\Big) + d(x, Tx)\Big]$$

$$\Longrightarrow d\Big(T^2x, Tx\Big) \leq k\Big[3d\Big(T^2x, Tx\Big) + 3d(x, Tx) + d\Big(T^2x, T^3x\Big)\Big]$$

$$\Longrightarrow d\Big(T^2x, Tx\Big)(1-3k) \leq k[d\Big(T^2x, T^3x\Big) + 3d(x, Tx)]$$

$$\Longrightarrow d\Big(T^2x, Tx\Big) \leq \frac{k}{1-3k}d\Big(T^2x, T^3x\Big) + \frac{3k}{1-3k}d(x, Tx)$$

$$\Longrightarrow d\Big(T^2x, Tx\Big) \leq \frac{k}{1-3k}\frac{3k}{1-k}[d\Big(T^2x, Tx\Big) + d(x, Tx)] + \frac{3k}{1-3k}d(x, Tx)$$

$$\Longrightarrow d\Big(T^2x, Tx\Big) \leq \frac{3k^2}{(1-3k)(1-k)}d\Big(T^2x, Tx\Big) + (\frac{3k^2}{(1-3k)(1-k)} + \frac{3k}{(1-3k)})d(x, Tx)$$

$$\Longrightarrow d\Big(T^2x, Tx\Big)\left(1 - \frac{3k^2}{(1-3k)(1-k)}\right) \leq \frac{3k^2 + 3k(1-k)}{(1-3k)(1-k)}d(x, Tx)$$

$$\Longrightarrow d\Big(T^2x, Tx\Big)\Big((1-3k)(1-k) - 3k^2\Big) \leq (3k^2 + 3k(1-k))d(x, Tx)$$

$$\Longrightarrow d\Big(T^2x, Tx\Big)(1-4k) \leq 3kd(x, Tx)$$

$$\Longrightarrow d\Big(T^2x, Tx\Big) \leq \frac{3k}{(1-4k)}d(x, Tx).$$

Then,

$$d\Big(T^2x, Tx\Big) \leq td(x, Tx) \text{ where } t = \frac{3k}{(1-4k)} \text{ and } t \in (0, 1),$$

which implies

$$d\Big(T^{n+1}x, T^nx\Big) \leq t^n d(x, Tx),$$

consequently

$$\sum_{n=1}^{+\infty} d\left(T^{n+1}x, T^n x\right) \leq (\sum_{n=1}^{+\infty} t^n) d(x, Tx) < +\infty$$

This implies that $\{T^n x\}$ is a Cauchy sequence in (X, d). Hence, there exists $z \in A \cup B \cup C$ such that $T^n x \longrightarrow z$. Notice that $\{T^{3n}x\}$ is a sequence in $A, \{T^{3n-1}x\}$ is a sequence in C and $\{T^{3n-2}x\}$ is a sequence in B and that both sequences tend to the same limit z. Regarding the fact that A, B and C are closed, we conclude that $z \in A \cap B \cap C$, hence $A \cap B \cap C \neq \varnothing$.

To show that z is a fixed point, we must show that $Tz = z$. Observe that

$$\begin{aligned} d(Tz, z) &= \lim d\left(Tz, T^{3n}x\right) \\ &\leq \lim D\left(T^{3n}x, T^{3n-1}x, Tz\right) \\ &\leq \lim k[d\left(T^{3n-1}x, T^{3n-2}x\right) + d\left(T^{3n-1}x, z\right) + d(T^{3n-2}x, z) \\ &+ d\left(T^{3n-1}x, T^{3n}x\right) + d\left(T^{3n-2}x, T^{3n-1}x\right) + d(z, Tz)] \\ &\leq kd(Tz, z), \end{aligned}$$

which is equivalent to $(1-k)d(Tz, z) = 0$.

Since $k \in \left(0, \frac{1}{7}\right)$, then $d(Tz, z) = 0$, which implies $Tz = z$.

To prove the uniqueness of z, assume that there exists $w \in A \cup B \cup C$ such that $w \neq z$ and $Tw = w$. Taking into account that T is tricyclic, we get $w \in A \cap B \cap C$.

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \\ &\leq D(Tz, Tw, Tw) \\ &\leq k[2d(z, w) + d(w, w) + d(z, Tz) + d(Tw, w) + d(Tw, w)] \\ &\leq 2kd(z, w) \end{aligned}$$

implies $d(z, w) = 0$. We conclude that $z = w$ and hence z is the unique fixed point of T. □

Example 4. *We take the same example 3.*

Let X be $\mathbb{R}^2$ normed by the norm $\| (x, y) \| = |x| + |y|$,

$$A = \{0\} \times [0, +1], B = [0, +1] \times \{0\}, C = \{0\} \times [-1, 0]$$

and let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be defined by

$$T(0, x) = \left(\frac{x}{6}, 0\right) \quad \text{if } (0, x) \in A,$$

$$T(y, 0) = \left(0, \frac{-y}{6}\right) \quad \text{if } (y, 0) \in B,$$

$$T(0, z) = \left(0, \frac{-z}{6}\right) \quad \text{if } (0, z) \in C,$$

We have T is tricyclic and for all $(0, x) \in A, (y, 0) \in B, (0, z) \in C$,

$$\begin{aligned} D(T(0, x), T(y, 0), T(0, z)) &= D\left(\left(\frac{x}{6}, 0\right), \left(0, \frac{-y}{6}\right), \left(0, \frac{-z}{6}\right)\right) \\ &= \frac{1}{3}(x + y - z). \end{aligned}$$

In addition, we have

$$D((0,x),(y,0),(0,z)) + d((0,x),T(0,x)) + d((y,0),T(y,0)) + d((0,z),T(0,z))$$
$$= 2(x+y-z) + \frac{7}{6}(x+y-z) = \frac{19}{6}(x+y-z).$$

Then,

$$\begin{aligned} D(T(0,x),T(y,0),T(0,z)) &= \frac{2}{19}(D((0,x),(y,0),(0,z)) + d((0,x),T(0,x)) \\ &\quad + d((y,0),T(y,0)) + d((0,z),T(0,z))) \\ &\leq \frac{1}{7}(D((0,x),(y,0),(0,z)) + d((0,x),T(0,x)) \\ &\quad + d((y,0),T(y,0)) + d((0,z),T(0,z))) \end{aligned}$$

This implies that T is a Reich-S-type tricyclic contraction, and T has a unique fixed point $(0,0)$ in $A \cap B \cap C$.

Corollary 3. *Let (X,d) a complete metric space and a self mapping $T : X \longrightarrow X$. If there exists $k \in \left(0, \frac{1}{7}\right)$ such that*

$$D(Tx,Ty,Tz) \leq k[D(x,y,z) + d(x,Tx) + d(y,Ty) + d(z,Tz)]$$

for all $(x,y,z) \in X^3$, then T has a unique fixed point in X.

The next tricyclic contraction considered in this section is the Cirić-S-type tricyclic contraction defined below.

Definition 6. *Let A, B and C be nonempty subsets of a metric space (X,d), $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a Cirié-S-type tricyclic contraction, if there exists $k \in (0,1)$ such that*

1. $T(A) \subseteq B, T(B) \subseteq C, T(C) \subseteq A$
2. $D(Tx,Ty,Tz) \leq kM(x,y,z)$ *for all* $(x,y,z) \in A \times B \times C$.

where $M(x,y,z) = \max\{D(x,y,z), d(x,Tx), d(y,Ty), d(z,Tz)\}$

The fixed point theorem of the Cirić-S-type tricyclic contraction reads as follows.

Theorem 10. *Let A, B and C be nonempty closed subsets of a complete metric space (X,d), and let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a Cirić-S- type tricyclic contraction, then T has a unique fixed point in $A \cap B \cap C$.*

Proof. Taking $x \in A$, we have $D(Tx,Ty,Tz) \leq kM(x,y,z)$ for all $(x,y,z) \in A \times B \times C$. If $M(x,y,z) = D(x,y,z)$, Theorem 7 implies the desired result.

Consider the case $M(x,y,z) = d(x,Tx)$. We have:

$$\begin{aligned} D\left(Tx,T^2x,T^3x\right) \leq kd(x,Tx) &\implies d\left(Tx,T^2x\right) \leq kd(x,Tx) \\ &\implies d\left(T^n x,T^{n+1}x\right) \leq k^n d(x,Tx) \end{aligned}$$

Consequently,

$$\sum_{n=1}^{+\infty} d\left(T^{n+1}x,T^n x\right) \leq (\sum_{n=1}^{+\infty} k^n) d(x,Tx) < +\infty$$

which implies that $\{T^n x\}$ is a Cauchy sequence in (X,d). Hence, there exists $z \in A \cup B \cup C$ such that $T^n x \longrightarrow z$. Notice that $\{T^{3n}x\}$ is a sequence in A, $\{T^{3n-1}x\}$ is a sequence in C, and $\{T^{3n-2}x\}$ is a sequence in B and that both sequences tend to the same limit z; regarding the fact that A, B and C are closed, we conclude $z \in A \cap B \cap C$, hence $A \cap B \cap C \neq \varnothing$.

To show that z is a fixed point, we must show that $Tz = z$. Observe that

$$d(Tz,z) = \lim d\left(Tz, T^{3n}x\right) \leq \lim D\left(T^{3n}x, T^{3n-1}x, Tz\right) \leq kd(Tz,z),$$

which is equivalent to $(1-k)d(Tz,z) = 0$. Since $k \in (0,1)$, then $d(Tz,z) = 0$, which implies $Tz = z$.

To prove the uniqueness of z, assume that there exists $w \in A \cup B \cup C$ such that $w \neq z$ and $Tw = w$.

Taking into account that T is tricyclic, we get $w \in A \cap B \cap C$.

$d(z,w) = d(Tz,Tw) \leq D(Tz,Tw,Tw) \leq kd(z,Tz) = 0$ implies $d(z,w) = 0$. We conclude that $z = w$ and hence z is the unique fixed point of T.

Consider the case $M(x,y,z) = d(y,Ty)$. We have :

$$D\left(Tx, T^2x, T^3x\right) \leq kd\left(Tx, T^2x\right) \Longrightarrow d\left(Tx, T^2x\right) \leq kd\left(Tx, T^2x\right) < d\left(Tx, T^2x\right),$$

which is impossible since $k \in (0,1)$

Consider the case $M(x,y,z) = d(z,Tz)$. We have:

$$D\left(Tx, T^2x, T^3x\right) \leq kd\left(T^2x, T^3x\right) \Longrightarrow d\left(T^2x, T^3x\right) \leq kd\left(T^2x, T^3x\right) < d\left(T^2x, T^3x\right),$$

which is impossible since $k \in (0,1)$. □

Corollary 4. *Let A, B and C be a nonempty subset of a complete metric space (X,d) and let a mapping $T : A \cup B \cup C \longrightarrow A \cup B \cup C$. If there exists $k \in (0,1)$ such that*

1. $T(A) \subseteq B, T(B) \subseteq C, T(C) \subseteq A$.
2. $D(Tx,Ty,Tz) \leq k \max\{D(x,y,z), d(x,Tx)\}\ \forall (x,y,z) \in A \times B \times C$.

Then, T has a unique fixed point in $A \cap B \cap C$.

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