axioms

MDPI

Article

Parrondo's Paradox for Tent Maps

Jose S. Cánovas

Department of Applied Mathematics and Statistics, Technical University of Cartagena, C/Doctor Flemming sn, 30202 Cartagena, Spain; Jose.Canovas@upct.es

Abstract: In this paper, we study the dynamic Parrondo's paradox for the well-known family of tent maps. We prove that this paradox is impossible when we consider piecewise linear maps with constant slope. In addition, we analyze the paradox "simple + simple = complex" when a tent map with constant slope and a piecewise linear homeomorphism with two different slopes are considered.

Keywords: Parrondo's paradox; tent maps; topological entropy; piecewise linear maps

MSC: 37E05; 26A18

Citation: Cánovas, J.S. Parrondo's Paradox for Tent Maps. *Axioms* **2021**, *10*, 85. https://doi.org/10.3390/axioms10020085

Academic Editor: Ioan-Lucian Popa

Received: 27 March 2021
Accepted: 29 April 2021
Published: 6 May 2021

1. Introduction

Recently, the so-called Parrondo's paradox (see, e.g., [1–3]) has received the attention of many researchers. Although it appears in game theory, the dynamic version can be stated in terms of chaotic and non-chaotic behavior. Briefly (see, e.g., [4]), we consider two discrete dynamical systems given by continuous maps $f_i : X \to X$, $i = 1, 2$, on a metric space X, usually a subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. The Parrondo's paradox appears when the dynamical behavior of both maps is simple (respectively, chaotic) and that of $f_1 \circ f_2$ is chaotic (respectively, simple). Of course, this paradox can involve more maps (see, e.g., [5]). Mathematical examples includes interval dynamics (see [5,6]), dynamics of complex maps (see [7,8]), local stability problems ([9,10]), etc. In addition, we can find applications of this paradox to physics (see [11,12]), biology (see [13–15]) and social sciences (see [16,17]).

In this paper, we study this paradox for interval continuous maps which are piecewise linear. More precisely, we consider continuous interval maps $f : [0,1] \to [0,1]$ for which there is a finite partition P given by $0 = x_0 < x_1 < ... < x_m = 1$ such that the restriction $f|_{[x_i,x_{i+1}]}$ is linear. Note that f is piecewise monotone because there is a subpartition of P such that $f|_{[x_i,x_j]}$, $i < j$, is monotone. We denote by $c(f)$ the number of monotone pieces of f. Then, the topological entropy of f is given by (see, e.g., [18] [Chapter 4])

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log c(f^n),$$

where $f^n = f \circ f^{n-1}$, for $n \in \mathbb{N}$, $f^1 = f$. It is well-known that positive topological entropy implies the existence of some kind of complex behavior. For instance, for positive entropy maps, there exists an uncountable subset $S \subset [0,1]$ such that for each pair of distinct points $x, y \in S$ we have that

$$0 = \liminf_{n\to\infty} |f^n(x) - f^n(y)| < \limsup_{n\to\infty} |f^n(x) - f^n(y)|.$$

This is the well-known definition of chaos in the sense of Li and Yorke (see [19]).

The Parrondo's paradox appears in terms of topological entropy when two zero (respectively, positive) topological entropy maps $f_1, f_2 : [0,1] \to [0,1]$ holds that $h(f_1 \circ f_2) > 0$ (respectively, $h(f_1 \circ f_2) = 0$). This paradox has been shown in several models (see, e.g., [11,20]). Now, we analyze it for continuous piecewise linear maps. The results and necessary background can be find in the next section. A section with conclusions finishes the paper.

2. The Results

By a discrete dynamical system we mean the pair (X, f), where X is a metric space and $f : X \to X$ is continuous. For $x_0 \in X$, its orbit is given by the recursive sequence $x_{n+1} = f(x_n)$. In this paper, $X = [0, 1]$.

Let $f_s : [0, 1] \to [0, 1]$ be the tent map given by

$$f_s(x) = \begin{cases} sx & if \quad x \in [0, 1/2], \\ -sx + s & if \quad x \in [1/2, 1]. \end{cases}$$

The parameter $s \in (0, 2]$ is called the slope of f_s. A piecewise continuous linear map is called of constant slope s if the slope of its linear pieces is either s or $-s$. It is known (see, e.g., [18], Chapter 4) that piecewise continuous linear maps with constant slope $s > 0$ have topological entropy $\max\{0, \log s\}$. Therefore, $h(f_s) = \log s$ if $s > 1$ and zero otherwise.

It is easy to see that $f_{s_1} \circ f_{s_2}$, $s_1, s_2 \in (0, 2]$, is a piecewise linear continuous map with constant slope $s_1 s_2$. Then, $h(f_{s_1} \circ f_{s_2}) = \max\{0, \log(s_1 s_2)\}$. Thus, if both s_1 and s_2 are smaller than or equal to one, $h(f_{s_1} \circ f_{s_2}) = 0$ and if they are greater than one $h(f_{s_1} \circ f_{s_2}) = \log(s_1 s_2) > 0$. The conclusion is that there is no Parrondo's paradox of any type. This is true for any piecewise linear continuous map with constant slope.

Thus, we consider maps with two different slopes. Namely, let $g_{s,p} : I \to I$ be the map

$$g_{s,p}(x) = \begin{cases} sx & if \quad x \in [0, p], \\ \frac{1-sp}{1-p} x + p\frac{s-1}{1-p} & if \quad x \in [p, 1], \end{cases}$$

where $p \in (0, 1)$ and $s \in (0, 1/p)$. This map is strictly increasing, and then $h(g_{s,p}) = 0$ and the property $g_{1,p}(x) = x$ holds. We say that the map $g_{s,p}$ is of Type 1 if $s < 1$ and of Type 2 if $s > 1$. Note that $g_{s,p}(x) < x$ for Type 1 and $g_{s,p}(x) > x$ for Type 2 when $x \in (0, 1)$.

Next, we fix $s_1 \in (0, 2]$, $p \in (0, 1)$ and $s_2 \in (0, 1/p)$ and consider the maps $\varphi_{s_1,s_2,p} = f_{s_1} \circ g_{s_2,p}$ and $\phi_{s_1,s_2,p} = g_{s_2,p} \circ f_{s_1}$. These maps have the same topological entropy (see [21]), and then we can work with both of them producing the same results. Let us analyze if the Parrondo's paradox "simple + simple = complex" happens for these maps. More precisely, we study the relationship between the topological entropies of $\phi_{s_1,s_2,p}$ and f_{s_1}. We start by proving the following easy lemmas.

Lemma 1. *The map $\phi_{s_1,s_2,p}$ has a unique maximum* $1/2$.

Proof. It is straightforward. □

Lemma 2. *Assume that $s_1/2 \le p$. Then, $\phi_{s_1,s_2,p}$ has constant slope $s_1 s_2$.*

Proof. Note that $f_{s_1}([0, 1]) = [0, s_1/2] \subseteq [0, p]$. Then,

$$\phi_{s_1,s_2,p}(x) = \begin{cases} s_1 s_2 x & if \quad x \in [0, 1/2], \\ -s_1 s_2 x + s_1 s_2 & if \quad x \in [1/2, 1], \end{cases}$$

and the proof concludes. □

The above lemma suggests that working with $\phi_{s_1,s_2,p}$ instead of $\varphi_{s_1,s_2,p}$ is a good idea. We can compute its topological entropy in this particular case. On the other hand, the maximum is always $1/2$, and, then, programming the algorithm from Block et al. [22] used for numerical computation of the topological entropy of $\phi_{s_1,s_2,p}$ is slightly simplified.

2.1. Maps $g_{s_2,p}$ of Type 1

First, we consider that $s_2 < 1$. We can prove the following result.

Proposition 1. *Let $s_1 \leq 1$. Then, $h(\phi_{s_1,s_2,p}) = 0$ for all $s_2 \in (0,1)$.*

Proof. If $s_1/2 \leq p$, the result follows by Lemma 2 since $s_1 s_2 < 1$. Thus, let $s_1/2 > p$ and let $x_0 < 1/2$ be such that $s_1 x_0 = p \leq x_0$. Note that $\phi_{s_1,s_2,p}(x_0) = s_2 p < s_2 s_1 x_0 \leq x_0$. On the other hand, $\phi_{s_1,s_2,p}(1/2) = g_{s_2,p}(s_1/2) < s_1/2 \leq 1/2$. Since $\phi_{s_1,s_2,p}$ is piecewise linear, we have that $\phi_{s_1,s_2,p}([0,1]) \subset [0, s_1/2] \subset [0,1/2]$. Since, by [18] [Chapter 4],

$$h(\phi_{s_1,s_2,p}) = h\left(\phi_{s_1,s_2,p}|_{[0,1/2]}\right),$$

and $\phi_{s_1,s_2,p}|_{[0,1/2]}$ is increasing, we have that $h(\phi_{s_1,s_2,p}) = 0$, and the proof concludes. □

The above proposition shows that Parrondo's paradox is not possible for maps $g_{s_2,p}$ of Type 1. It can be generalized for maps $\phi_{s_1,s_2,p}$ with $s_1 > 1$ as follows.

Proposition 2. *Let $s_1 > 1$ and let s_2 and p be such that $\phi_{s_1,s_2,p}(1/2) = 1/2$. Then, $h(\phi_{s_1,s_2,p}) = 0$.*

Proof. It is analogous to that of Proposition 1. □

At this moment, we have obtained that, if the hypothesis of Propositions 1 or 2 are fulfilled, then $0 = h(\phi_{s_1,s_2,p}) \leq h(f_{s_1})$. One might think that this inequality holds for any map f_{s_1}. Figure 1 shows the computation of the topological entropy of $\phi_{s_1,s_2,p}$ with prescribed accuracy using the algorithm from [22]. We use Mathematica, which can work properly with infinite precision with linear maps, and, thus, our computations are not affected by round-off effects. Note that there is not a clear relationship between the topological entropies of the tent and the modified tent maps. In Figure 2, we explore the variation of the parameter p.

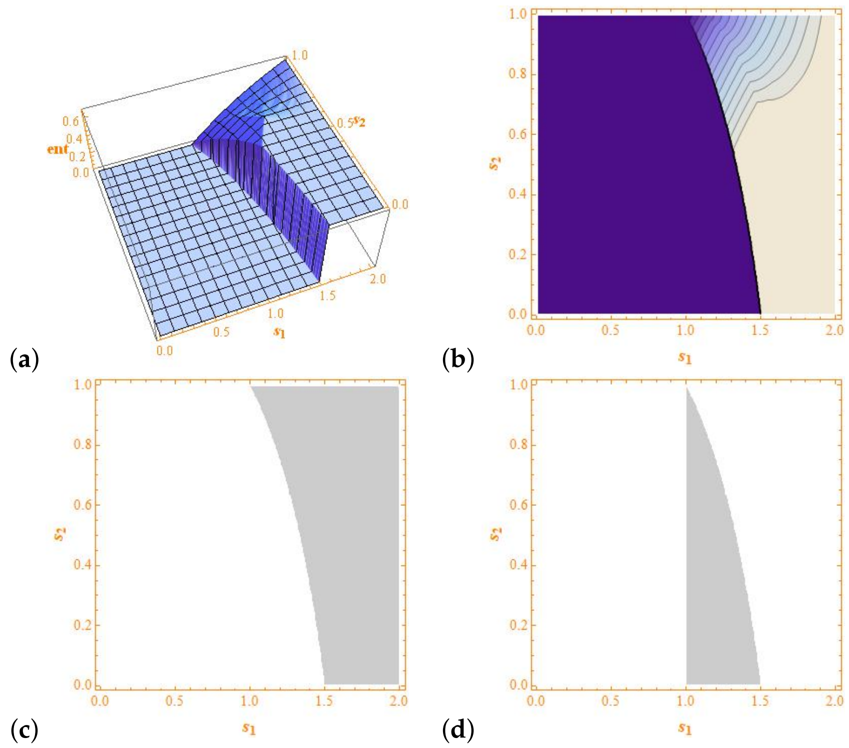

Figure 1. For $p = 1/2$ and $s_2 \in (0,1)$, we depict the graph (**a**) and level curves (**b**) of the topological entropy of $\phi_{s_1,s_2,p}$ with accuracy 10^{-4}. The step size for s_1 and s_2 is 10^{-3}. (**c**) Parameter values for which $h(f_{s_1}) - h(\phi_{s_1,s_2,p}) > 2 \times 10^{-4}$. (**d**) Parameter values for which $h(\phi_{s_1,s_2,p}) - h(f_{s_1}) > 2 \times 10^{-4}$.

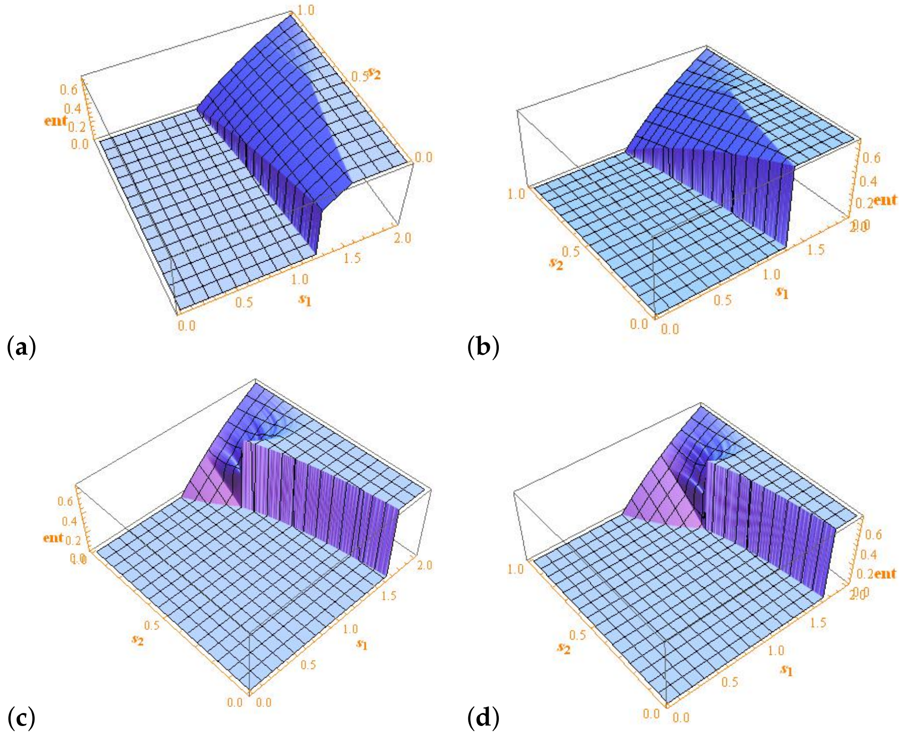

Figure 2. For $s_2 \in (0,1)$, we depict the graph of the topological entropy of $\phi_{s_1,s_2,p}$ with accuracy 10^{-4} for: $p = 1/4$ (**a**); $p = 1/3$ (**b**); $p = 2/3$ (**c**); and $p = 3/4$ (**d**). The step size for s_1 and s_2 is 10^{-3}.

It is important to realize what it means that the algorithm from Block et al. [22] is free of round-off effects. This algorithm depends on the so-called kneading sequences. For a unimodal map f with maximum at c, its kneading sequence is $K_f = (k_1, k_2, ...) \in \{0, 1/2, 1\}^{\mathbb{N}}$ such that, for $i \geq 1$, $k_i = 0$ if $f^i(c) < c$, $k_i = 1/2$ if $f^i(c) = c$ and $k_i = 1$ if $f^i(c) > c$. Since we work without round-off effects, this kneading sequence is free of them. The set of possible kneading sequences can be endowed with order relationship such that, if g is another unimodal map with kneading sequence K_g and $K_g \geq K_f$, then $h(g) \geq h(f)$. Thus, when we are able to state that either $h(f_{s_1}) > h(\phi_{s_1,s_2,p}) + \epsilon$ or $h(\phi_{s_1,s_2,p}) > h(f_{s_1}) + \epsilon$ for some $\epsilon > 0$, as, in Figure 1 c,d, we give a computed-assisted proof of that inequalities.

2.2. Maps $g_{s_2,p}$ of Type 2

Next, we consider the case $s_2 > 1$. We can prove the following result.

Theorem 1. *Let $s_1 \in (0,2]$ be such that $s_1/2 \leq p$. Then, $h(\phi_{s_1,s_2,p}) = \max\{0, \log(s_1 s_2)\}$.*

Proof. By Lemma 2, the map $\phi_{s_1,s_2,p}$ has constant slope $s_1 s_2$. Then, the proof follows easily. □

The above result shows that it is possible to combine two zero topological entropy maps to obtain positive topological entropy, and therefore Parrondo's paradox is possible. Moreover, as $s_2 > 1$, we have that $s_1 s_2 > s_1$ and hence $h(\phi_{s_1,s_2,p}) > h(f_{s_1})$ whenever $s_1 s_2 > 1$. Note that otherwise $h(\phi_{s_1,s_2,p}) = h(f_{s_1}) = 0$. Numerical computations with prescribed accuracy are shown in Figure 3. Note that $h(\phi_{s_1,s_2,p})$ can be greater and lower than $h(f_{s_1})$ when $s_2 > 1$. Note that, as above, our computations are free of round-off errors. In Figure 4, we explore the variation of the parameter p.

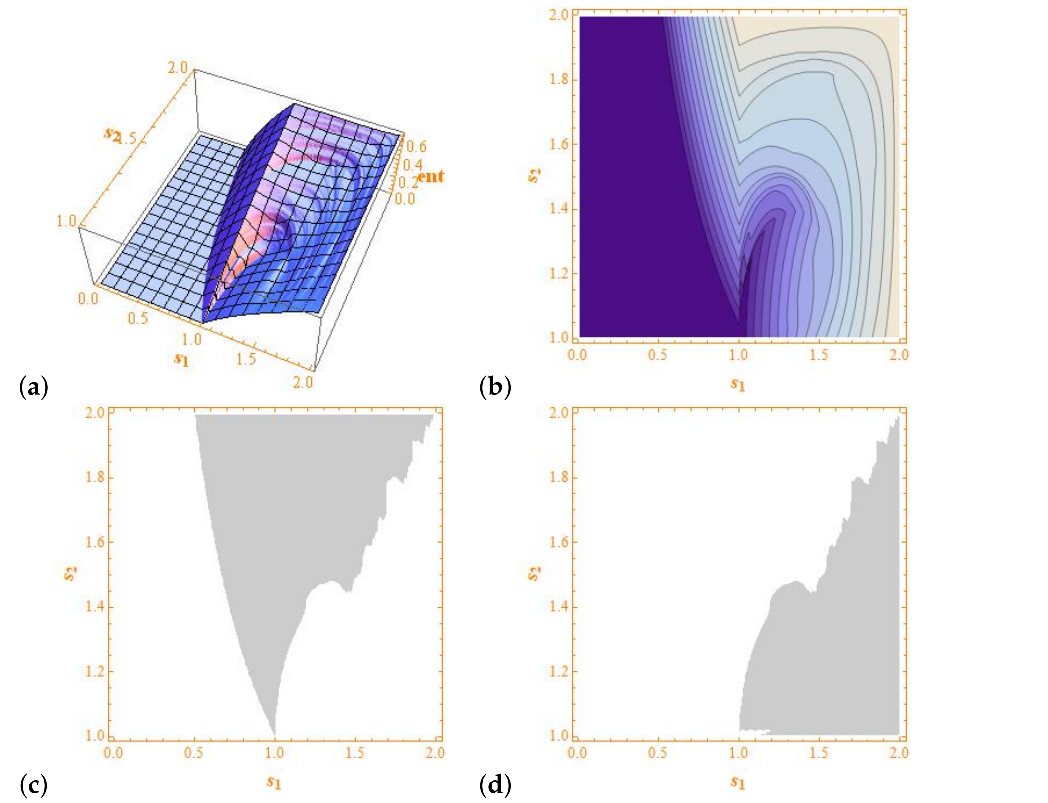

Figure 3. For $p = 1/2$ and $s_2 \in (1, 2]$, we depict the graph (**a**) and level curves (**b**) of the topological entropy of $\phi_{s_1,s_2,p}$ with accuracy 10^{-4}. The step size for s_1 and s_2 is 10^{-3}. (**c**) Parameter values for which $h(f_{s_1}) - h(\phi_{s_1,s_2,p}) > 2 \times 10^{-4}$. (**d**) Parameter values for which $h(\phi_{s_1,s_2,p}) - h(f_{s_1}) > 2 \times 10^{-4}$.

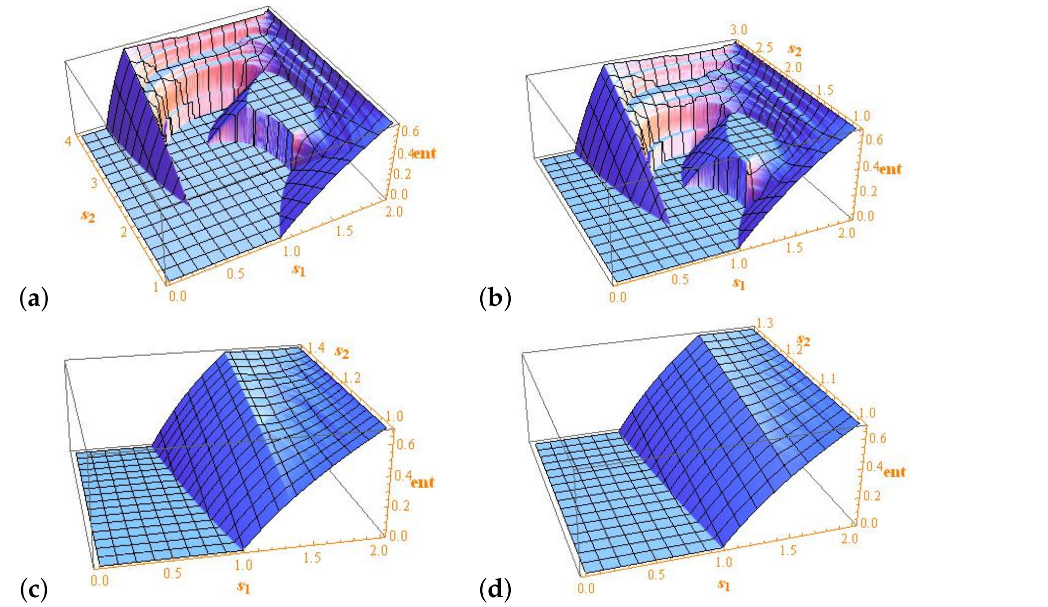

Figure 4. For $1 < s_2 < 1/p$, we depict the graph of the topological entropy of $\phi_{s_1,s_2,p}$ with accuracy 10^{-4} for: $p = 1/4$ (**a**); $p = 1/3$ (**b**); $p = 2/3$ (**c**); and $p = 3/4$ (**d**). The step size for s_1 and s_2 is 10^{-3}.

3. Conclusions

We analyze the Parrondo's paradox for the family of tent maps proving that no paradox is possible if we combine two maps in the family. However, the paradox "simple + simple = complex" is possible when we combine a tent map with a homeomorphism consisting on two linear pieces with different slopes changing the slope at $p \in (0, 1)$. Numerical computations show that there is not a clear relationship between the topological entropies of the tent and modified tent maps. We also find numerical evidence showing that as p increases the topological entropy of the modified tent map decreases. The computations are made using an algorithm depending on kneading sequences, which can be computed without round off effects, and, therefore, they can be taken as a computer-assisted proof of the above-mentioned facts. It is unclear whether, for instance, the boundaries of the regions depicted in Figures 1c,d and 3c,d can be obtained in an analytical way.

Funding: J.S. Cánovas was supported by the project MTM 2017-84079-P Agencia Estatal de Investigación (AEI) y Fondo Europeo de Desarrollo Regional (FEDER).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Harmer, G.P.; Abbott, D. Losing strategies can win by Parrondo's paradox. *Nature* **1999**, *402*, 864. [CrossRef]
2. Harmer, G.P.; Abbott, D. Parrondo's paradox. *Stat. Sci.* **1999**, *14*, 206–213.
3. Parrondo, J.M.R.; Harmer, G.P.; Abbott, D. New paradoxical games based on Brownian ratchets. *Phys. Rev. Lett.* **2000**, *85*, 5226–5229. [CrossRef] [PubMed]
4. Cánovas, J.S.; Linero, A.; Peralta-Salas, D. Dynamic Parrondo's paradox. *Phys. D Nonlinear Phenom.* **2006**, *218*, 177–184. [CrossRef]
5. Cánovas, J.S. Periodic sequences of simple maps can support chaos. *Phys. Stat. Mech. Its Appl.* **2017**, *466*, 153–159. [CrossRef]
6. Cánovas, J.S.; Mu noz, M. Revisiting Parrondo's paradox for the logistic family. *Fluct. Noise Lett.* **2013**, *12*, 1350015. [CrossRef]
7. Blé, G.; Castellanos, V.; Falconi, M. On the coexisting dynamics in the alternate iteration of two logistic maps. *Dyn. Syst.* **2011**, *26*, 189–197. [CrossRef]
8. Blé, G.; Castillo-Santos, F.E.; González, D.; Valdez, R. On a quartic polynomials family of two parameters. *Dyn. Syst.* **2020**, 1–13. [CrossRef]
9. Cima, A.; Gasull, A.; Ma nosa, V.M. Parrondo's dynamic paradox for the stability of non-hyperbolic fixed points. *Discret. Contin. Dyn. Syst.* **2018**, *38*, 889–904. [CrossRef]
10. Gasull, A.; Hernández-Corbato, L.; del Portal, F.R.R. Parrondo's paradox for homeomorphisms. *arXiv* **2020**, arXiv:2010.12893.
11. Cánovas, J.S.; Mu noz,M. On the dynamics of the q-deformed logistic map. *Phys. Lett. A* **2019**, *383*, 1742–1754. [CrossRef]
12. Lai, J.W.; Cheong, K.H. Parrondo's paradox from classical to quantum: A review. *Nonlinear Dyn.* **2020**, *100*, 849–861. [CrossRef]
13. Mendoza, S.A.; Peacock-López, E. Switching induced oscillations in discrete one-dimensional systems. *Chaos Solitons Fractals* **2018**, *115*, 35–44. [CrossRef]
14. Peacock-López, E. Seasonality as a Parrondian game. *Phys. Lett. A* **2011**, *375*, 3124–3129. [CrossRef]
15. Silva, E.; Peacock-López, E. Seasonality and the logisitic map. *Chaos Solitons Fractals* **2017**, *95*, 152–156. [CrossRef]
16. Cheong, K.H.; Wen, T.; Lai, J.W. Relieving Cost of Epidemic by Parrondo's Paradox: A COVID-19 Case Study. *Adv. Sci.* **2020**, *7*, 2002324. [CrossRef] [PubMed]
17. Lai, J.W.; Cheong, K.H. Social dynamics and Parrondo's paradox: A narrative review. *Nonlinear Dyn* **2020**, *101*, 1–20. [CrossRef]
18. Alsedá, L.; Llibre, J.; Misiurewicz, M. *Combinatorial Dynamics and Entropy in Dimension One*; World Scientific Publishing: Singapore, 1993.
19. Li, T.Y.; Yorke, J.A. Period three implies chaos. *Amer. Math. Monthly* **1975**, *82*, 985–992. [CrossRef]
20. Cánovas, J.S. On the Periodic Ricker Equation. In *International Conference in Nonlinear Analysis and Boundary Value Problems*; Springer: Cham, Switzerland, 2018; pp. 121–130.
21. Kolyada, S.; Snoha, L. Topological entropy of nonautononous dynamical systems. *Random Comp. Dyn.* **1996**, *4*, 205–233.
22. Block, L.; Keesling, J.; Li, S.H.; Peterson, K. An improved algorithm for computing topological entropy. *J. Stat. Phys.* **1989**, *55*, 929–939. [CrossRef]