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Article

On Factoring Groups into Thin Subsets

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Abstract: A subset X of a group G is called thin if, for every finite subset F of G, there exists a finite subset H of G such that $Fx \cap Fy = \emptyset$, $xF \cap yF = \emptyset$ for all distinct $x, y \in X \setminus H$. We prove that every countable topologizable group G can be factorized $G = AB$ into thin subsets A, B.

Keywords: factorizations of a group; thin subset of a group

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1. Introduction

Let G be a group, and $[G]^{<\omega}$ denote the set of all finite subsets of G. A subset X of is called:

- *left thin* if, for every $F \in [G]^{<\omega}$, there exists $H \in [G]^{<\omega}$ such that $Fx \cap Fy = \emptyset$ for all distinct $x, y \in X \setminus H$;
- *right thin* if, for every $F \in [G]^{<\omega}$, there exists $H \in [G]^{<\omega}$ such that $xF \cap yF = \emptyset$ for all distinct $x, y \in X \setminus H$;
- *thin* if X is left and right thin.

The notion of left thin subsets was introduced in [1]. For motivation to study left thin, right thin and thin subsets and some results and references, see Comments and surveys [2–5]. In *asymptology*, thin subsets play the part of discrete subsets (see Comments 1 and 2).

We recall that the product AB of subsets A, B of a group G is a *factorization* if $G = AB$ and each element $g \in G$ has the unique representation $g = ab$, $a \in A$, $b \in B$ (equivalently, the subsets $\{aB : a \in A\}$ are pairwise disjoint). For factorizations of groups into subsets, see [6].

Our goal is to prove the following theorem. By a countable set, we mean a countably infinite set. The group topology τ is supposed to be Hausdorff.

Theorem 1. *Let (G, τ) be a non-discrete countable topological group. Then G can be factorized $G = AB$ into thin subsets A, B.*

2. Proof

Proof of Theorem 1. Let $G = \{g_n : n < \omega\}$, $g_0 = e$, e is the identity of G, $F_n = \{g_i : i \leq n\}$.

Given two sequences $(a_n)_{n<\omega}$, $(b_n)_{n<\omega}$ in G, we denote

$$A_n = \{a_i, a_i^{-1} : i \leq n\}, \;\; B_n = \{b_i : i \leq n\}, \;\; A = \cup_{n<\omega} A_n, \;\; B = \cup_{n<\omega} B_n.$$

We want to choose $(a_n)_{n<\omega}$, $(b_n)_{n<\omega}$ so that AB is a factorization of G and A, B are thin.

Let X, Y be subsets of G. We say that XY is a *partial factorization* of G if the subsets $\{Xy : y \in Y\}$ are pairwise disjoint (equivalently, the subsets $\{Yx : x \in X\}$ are pairwise disjoint).

We put $a_0 = e$, $b_0 = e$ and suppose that $a_0, \dots, a_n$ and $b_0, \dots, b_n$ have been chosen so that the following conditions are satisfied

(1) A_nB_n is a partial factorization of G and $g_n \in A_nB_n$;

(2) $F_ib_i \cap F_jb_j = \emptyset$, $b_iF_i \cap b_jF_j = \emptyset$ for all distinct $i, j \in \{0, \dots, n\}$;

(3) $F_ia_i \cap F_ja_j = \emptyset$, $a_iF_i \cap a_jF_j = \emptyset$, $F_ia_i^{-1} \cap F_ja_j^{-1} = \emptyset$, $a_i^{-1}F_i \cap a_j^{-1}F_j = \emptyset$ and $F_ia_i^{-1} \cap F_ja_j$, $a_i^{-1}F_i \cap a_jF_j = \emptyset$ for all distinct $i, j \in \{0, \dots, n\}$;

(4) if $a_i \neq a_i^{-1}$ then $F_ia_i \cap F_ia_i^{-1} = \emptyset$, $a_iF_i \cap a_i^{-1}F_i = \emptyset$, $i \in \{0, \dots, n\}$.

We take the first element $g_m \in G \setminus A_nB_n$, put $g = g_m$ and show that there exists a symmetric neighborhood U of e such that

(5) $(A_n \cup \{x, x^{-1}\})(B_n \cup \{xg\})$ is a partial factorization for each $x \in U \setminus \{e\}$.

We choose a symmetric neighborhood V of e such that $(A_n \cup \{x, x^{-1}\})B_n$ is a partial factorization of G for each $x \in V \setminus \{e\}$.

Then we use $A_n = A_n^{-1}$, $g \in G \setminus A_nB_n$ and $e \in A_n \cap B_n$ to choose a symmetric neighborhood U of e such that $U \subset V$ and

$$(A_n \cup \{x, x^{-1}\})B_n \cap (A_n \cup \{x, x^{-1}\})xg = \emptyset,$$

equivalently, $A_nB_n \cap A_nxg = \emptyset$, $A_nB_n \cap \{x, x^{-1}\}xg = \emptyset$, $\{x, x^{-1}\}B_n \cap A_nxg = \emptyset$, $\{x, x^{-1}\}B_n \cap \{x, x^{-1}\}xg = \emptyset$ for each $x \in U \setminus \{e\}$, so we get (5). By the continuity of the group operations, the latter is possible because these 4 equalities hold for $x = e$.

If the set $\{x \in U : x^2 = e\}$ is infinite then we use (5) and choose $a_{n+1} \in U$, $a_{n+1} = a_{n+1}^{-1}$ and $b_{n+1} = a_{n+1}g$ to satisfy (1)–(3) with $n + 1$ in place of n. Otherwise, we choose $a_{n+1} \in U$, $a_{n+1} \neq a_{n+1}^{-1}$ and $b_{n+1} = a_{n+1}g$ to satisfy (1)–(4).

After ω steps, we get the desired factorization $G = AB$. □

3. Comments

1. Given a set X, a family $\mathcal{E}$ of subsets of $X \times X$ is called a *coarse structure* on X if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X := \{(x, x) : x \in X\}$ of X;
- if $E, E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z \ \ ((x, z) \in E, \ (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$.

Elements $E \in \mathcal{E}$ of the coarse structure are called *entourages* on X.

For $x \in X$ and $E \in \mathcal{E}$ the set $E[x] := \{y \in X : (x, y) \in \mathcal{E}\}$ is called the *ball of radius E centered at x*. Since $E = \bigcup_{x \in X}(\{x\} \times E[x])$, the entourage E is uniquely determined by the family of balls $\{E[x] : x \in X\}$. A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* of the coarse structure $\mathcal{E}$ if each set $E \in \mathcal{E}$ is contained in some $E' \in \mathcal{E}'$.

The pair $(X, \mathcal{E})$ is called a *coarse space* [7] or a *ballean* [8,9].

A subset B of X is called *bounded* if $B \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. A subset Y of X is called *discrete* if, for every $E \in \mathcal{E}$, there exists a bounded subset B such that $E[x] \cap E[y] = \emptyset$ for all distinct $x, y \in Y \setminus B$.

2. Formally, coarse spaces can be considered as asymptotic counterparts of uniform topological spaces. However, actually, this notion is rooted in *geometry, geometrical group theory* and *combinatorics* (see [7,8,10,11]).

Given a group G, we denote by $\mathcal{E}_l$ and $\mathcal{E}_r$ the coarse structures on G with the bases

$$\{\{(x, y) : x \in Fy\} : F \in [G]^{<\omega}, e \in F\}, \ \ \{\{(x, y) : x \in yF\} : F \in [G]^{<\omega}, e \in F\}$$

and note that a subset A of G is left (resp. right) thin if and only if A is discrete in the coarse space $(G, \mathcal{E}_l)$ (resp. $(G, \mathcal{E}_r)$).

3. By [12], every countable group G has a thin subset A such that $G = AA^{-1}$. By [13], every countable topological group G has a closed discrete subset A such that $G = AA^{-1}$. For thin subsets of topological groups and factorizations into dense subsets, see [14,15].

4. Can every countable group G be factorized $G = AB$ into infinite subsets A, B? By Theorem 1, an answer to the following question could be negative only in the case of a non-topologizable group G.

On the other hand, analyzing the proof, one can see that Theorem 1 remains true if all mappings $x \longmapsto xg$, $x \longmapsto gx$, $g \in G$, $x \longmapsto x^{-1}$ and $x \longmapsto x^2$ are continuous at e. By [16], every countable group G admits a non-discrete Hausdorff topology in which all shifts and the inversion $x \longmapsto x^{-1}$ are continuous.

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