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Logic-Sensitivity of Aristotelian Diagrams in Non-Normal Modal Logics

Lorenz Demey [1,2]

[1] Center for Logic and Philosophy of Science, KU Leuven, 3000 Leuven, Belgium; lorenz.demey@kuleuven.be
[2] KU Leuven Institute for Artificial Intelligence, KU Leuven, 3000 Leuven, Belgium

Abstract: Aristotelian diagrams, such as the square of opposition, are well-known in the context of normal modal logics (i.e., systems of modal logic which can be given a relational semantics in terms of Kripke models). This paper studies Aristotelian diagrams for non-normal systems of modal logic (based on neighborhood semantics, a topologically inspired generalization of relational semantics). In particular, we investigate the phenomenon of logic-sensitivity of Aristotelian diagrams. We distinguish between four different types of logic-sensitivity, viz. with respect to (i) Aristotelian families, (ii) logical equivalence of formulas, (iii) contingency of formulas, and (iv) Boolean subfamilies of a given Aristotelian family. We provide concrete examples of Aristotelian diagrams that illustrate these four types of logic-sensitivity in the realm of normal modal logic. Next, we discuss more subtle examples of Aristotelian diagrams, which are not sensitive with respect to normal modal logics, but which nevertheless turn out to be highly logic-sensitive once we turn to non-normal systems of modal logic.

Keywords: Aristotelian diagram; non-normal modal logic; square of opposition; logical geometry; neighborhood semantics; bitstring semantics

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1. Introduction

Aristotelian diagrams, such as the so-called square of opposition, visualize a number of formulas from some logical system, as well as certain logical relations holding between them. These diagrams have a rich history in philosophy and logic [1–3], and today they are also widely used in artificial intelligence, to study and compare knowledge representation formalisms such as the rough set theory [4–6], formal concept analysis and possibility theory [7–9], formal argumentation theory [10–13], fuzzy set theory [14–17], logical theories of analogical and proportional reasoning [18–23], probabilistic logic [24–26] and multiple-criterion decision-making [27–29]. Without a doubt, the oldest and most widely used Aristotelian diagram is the square of opposition for the categorical statements from syllogistics, such as 'all Greeks are mortal' and 'some Greeks are mortal'. The theoretical roots of this square can be traced back to the logical works of Aristotle, but the diagram was actually drawn only from the second century CE onwards [30–32]. It is equally without any doubt that the second oldest and most widely used Aristotelian diagram is the modal square of opposition, for statements such as 'it is necessary that p' and 'it is possible that p'. This diagram, too, has its theoretical roots in Aristotle, and was explicitly drawn from the 12th century onwards [33–36]. The modal square of opposition continues to be used quite frequently in contemporary mathematical work on modal logic, and can be found in well-known textbooks [37–39], as well as research papers [40–44]. Furthermore, in recent decades, many other, more complex Aristotelian diagrams for modal logic have been developed, including hexagons, octagons and even three-dimensional diagrams, such as rhombic dodecahedra [45–50].

In the research program of logical geometry, we study Aristotelian diagrams as objects of independent mathematical interest, regardless of the specific details of their concrete applications in philosophy, artificial intelligence and elsewhere [51–60]. One of the main insights from logical geometry is that Aristotelian diagrams are highly sensitive to the details of the logical system with respect to which they are defined [52]. This phenomenon is usually studied using the technique of bitstring semantics [57], and informally means that it is possible for one and the same set of formulas to give rise to two completely different Aristotelian diagrams, when we work in two different logical systems. This logic-sensitivity of Aristotelian diagrams is also well-known with respect to modal logic. In particular, it is easy to show that the set $\{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$ gives rise to a *classical* square of opposition in the modal logic KD, but to a *degenerate* square of opposition in the modal logic K (cf. Figure 1).

(**a**) Classical square of opposition in the modal logic KD.

(**b**) Degenerate square of opposition in the modal logic K.

Figure 1. Aristotelian diagrams for $\{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$ with respect to two systems of modal logic. Full, dashed, and dotted lines visualize contradiction, contrariety, and subcontrariety, respectively; arrows visualize subalternations.

All existing research on modal Aristotelian diagrams generally, and on their logic-sensitivity more specifically, has focused exclusively on *normal* modal logics, such as K, KD, KT, KD4, S4 and S5. These are logics which can be given a relational semantics in terms of Kripke models, and are therefore most well-known in the literature. Pizzi [44] is very explicit about this, for example in his discussion of the logic-sensitivity of the modal square:

> The standard modal square [. . .] is valid with respect to any modal system at least as strong as the deontic system KD, but invalid in any *normal* system strictly weaker than KD ([44], p. 313, emphasis added).

However, there does not seem to be any deep philosophical or mathematical reason for restricting ourselves to *normal* modal logics when studying modal Aristotelian diagrams and their logic-sensitivity. The goal of this paper will therefore be to present some examples of Aristotelian diagrams that naturally arise in *non-normal* systems of modal logic, and to study various aspects of their logic-sensitivity, which we will show can get quite intricate. In particular, we will distinguish between four different types of logic-sensitivity, viz. with respect to (i) Aristotelian families, (ii) logical equivalence of formulas, (iii) contingency of formulas, and (iv) Boolean subfamilies of a given Aristotelian family.

The paper is organized as follows. Section 2 provides some technical background on modal logic, logical geometry and bitstring semantics, in order to keep the paper relatively self-contained. Sections 3–6 consecutively discuss the four types of logic-sensitivity that were mentioned above. These four sections share roughly the same structure: first, we introduce the relevant type of logic-sensitivity and provide a concrete example from the realm of normal modal logic; next, we provide more subtle examples of Aristotelian diagrams, which are not sensitive with respect to normal modal logics, but which nevertheless do turn out to display the relevant type of logic-sensitivity once we turn to non-normal systems of modal logic. Finally, Section 7 wraps things up, and mentions some questions for further research.

2. Technical Background

2.1. Modal Logic

Although we assume that the reader has a basic familiarity with modal logic, its basic tenets are summarized here for the sake of reference. For more details and proofs, see [38,39]. Starting from a countably infinite set **Prop** of atomic propositions, the modal language $\mathcal{L}$ is defined by means of the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

(where $p \in \mathbf{Prop}$). The other connectives and modal operators can be defined in the usual way; in particular, we put $\Diamond\varphi := \neg\Box\neg\varphi$, $\bot := p \wedge \neg p$ and $\top := \neg\bot$. This language is usually given a relational semantics, that is, it is interpreted on Kripke models:

Definition 1. *A* Kripke frame $\mathbb{F}$ *is a tuple* $\langle W, R\rangle$*, where* W *is a non-empty set (the elements of which are called 'states'), and* R *is a binary relation over* W*, that is,* $R \subseteq W \times W$*. Similarly, a* Kripke model *is a tuple* $\langle \mathbb{F}, V\rangle$*, where* $\mathbb{F} = \langle W, R\rangle$ *is a Kripke frame, and* $V\colon \textbf{\textit{Prop}} \to \wp(W)$ *is a valuation function.*

The class of all Kripke frames will be denoted K. We will often also be interested in special subclasses of frames. In particular, KD is the class of all *serial* Kripke frames (i.e., frames $\langle W, R\rangle$ such that for all $w \in W$ there exists at least one $v \in W$ such that wRv), while KF is the class of all *partially functional* Kripke frames (i.e., frames $\langle W, R\rangle$ such that for all $w, v, u \in W$, if wRv and wRu then $v = u$). A final class that will be of interest is $\mathsf{KDF} := \mathsf{KD} \cap \mathsf{KF}$.

Definition 2. *Let* $\mathbb{F} = \langle W, R\rangle$ *be a Kripke frame,* $\mathbb{M} = \langle \mathbb{F}, V\rangle$ *a Kripke model, and* $w \in W$*. The semantics for the modal language* $\mathcal{L}$ *is inductively defined as follows:*

$$\begin{array}{lll}
\mathbb{M}, w \models p & \textit{iff} & w \in V(p) \\
\mathbb{M}, w \models \neg\varphi & \textit{iff} & \mathbb{M}, w \not\models \varphi \\
\mathbb{M}, w \models \varphi \wedge \psi & \textit{iff} & \mathbb{M}, w \models \varphi \textit{ and } \mathbb{M}, w \models \psi \\
\mathbb{M}, w \models \Box\varphi & \textit{iff} & \textit{for all } v \in W\textit{: if } wRv \textit{ then } \mathbb{M}, v \models \varphi.
\end{array}$$

Given a Kripke frame $\mathbb{F} = \langle W, R\rangle$*, we write* $\mathbb{F} \models \varphi$ *if* $\langle \mathbb{F}, V\rangle, w \models \varphi$ *for all valuations* $V\colon \textbf{\textit{Prop}} \to \wp(W)$ *and states* $w \in W$*. Finally, given a class* C *of Kripke frames, we write* $\models_{\mathsf{C}} \varphi$ *if* $\mathbb{F} \models \varphi$ *for all* $\mathbb{F} \in \mathsf{C}$*.*

We now switch from the semantic to the syntactic perspective:

Definition 3. *A normal modal logic is a set* $\mathsf{S} \subseteq \mathcal{L}$ *that (i) contains all propositional tautologies, (ii) contains* $\Box(p \to q) \to (\Box p \to \Box q)$*, (iii) is closed under modus ponens: if* $\varphi \to \psi \in \mathsf{S}$ *and* $\varphi \in \mathsf{S}$ *then* $\psi \in \mathsf{S}$*, (iv) is closed under necessitation: if* $\varphi \in \mathsf{S}$ *then* $\Box\varphi \in \mathsf{S}$*, (vi) is closed under uniform substitution: if* $\varphi \in \mathsf{S}$ *then* $\varphi(\alpha_1/p_1, \ldots, \alpha_n/p_n) \in \mathsf{S}$*. We usually write* $\vdash_{\mathsf{S}} \varphi$ *instead of* $\varphi \in \mathsf{S}$*. The smallest normal modal logic is called* K*. If a normal modal logic contains some further axioms* $\mathsf{A}_1, \ldots, \mathsf{A}_n \in \mathcal{L}\backslash\mathsf{K}$*, then it will be called* $\mathsf{KA}_1 \ldots \mathsf{A}_n$*.*

In Sections 3–6 we will encounter examples from the normal modal logics K, KD, KF and KDF, which are defined using the axioms D: $\Box p \to \Diamond p$ and F: $\Diamond p \to \Box p$. Note that expressions such as K and KD are used to denote logics (i.e., sets of formulas) as well as classes of Kripke frames. This slight abuse of notation is justified by standard soundness and completeness results:

$$\begin{array}{lllllll}
\vdash_{\mathsf{K}} \varphi & \text{iff} & \models_{\mathsf{K}} \varphi & \qquad & \vdash_{\mathsf{KF}} \varphi & \text{iff} & \models_{\mathsf{KF}} \varphi \\
\vdash_{\mathsf{KD}} \varphi & \text{iff} & \models_{\mathsf{KD}} \varphi & \qquad & \vdash_{\mathsf{KDF}} \varphi & \text{iff} & \models_{\mathsf{KDF}} \varphi
\end{array}$$

For certain applications, normal modal logics and their accompanying relational semantics are considered to be 'too strong'. For example, $\Box(p \to q) \to (\Box p \to \Box q)$ belongs to every normal modal logic, even though this principle might be undesirable in certain contexts (cf. [61], p. 12ff. for details). This has led to the development of non-normal modal logics, which do not have a relational semantics in terms of Kripke models, but rather a topologically inspired semantics in terms of neighborhood models. Once again, for more details and proofs, see [61,62] and [63], Chapters 7–9.

Definition 4. *A* neighborhood frame $\mathbb{F}$ *is a tuple* $\langle W, N\rangle$*, where* W *is as in Definition 1, and* $N\colon W \to \wp(\wp(W))$ *is a neighborhood function. Similarly, a* neighborhood model *is a tuple* $\langle \mathbb{F}, V\rangle$*, where* $\mathbb{F} = \langle W, N\rangle$ *is a neighborhood frame, and* $V\colon \textbf{\textit{Prop}} \to \wp(W)$ *is a valuation function.*

The class of all neighborhood frames will be denoted E. We will also often be interested in special subclasses of frames. In particular, EM is the class of all *monotone* neighborhood frames (i.e., frames $\langle W, N\rangle$ such that for all $w \in W$ and $X \subseteq Y \subseteq W$: if $X \in N(w)$ then $Y \in N(w)$); EC is the class of all neighborhood frames that are *closed under intersection* (i.e., frames $\langle W, N\rangle$ such that for all $w \in W$ and $X, Y \subseteq W$: if $X, Y \in N(w)$ then $X \cap Y \in N(w)$); EN is the class of all neighborhood frames that *contain the unit* (i.e., frames $\langle W, N\rangle$ such that for all $w \in W$ we have $W \in N(w)$). Two further classes are $\mathsf{EMC} := \mathsf{EM} \cap \mathsf{EC}$ and $\mathsf{EMN} := \mathsf{EM} \cap \mathsf{EN}$. Finally, EMNC is the class of all *augmented* neighborhood frames (i.e., frames $\langle W, N\rangle$ such that for all $w \in W$ and $X \subseteq W$: $X \in N(w)$ iff $\bigcap N(w) \subseteq X$).

Definition 5. *Let* $\mathbb{F} = \langle W, N\rangle$ *be a neighborhood frame,* $\mathbb{M} = \langle \mathbb{F}, V\rangle$ *a neighborhood model, and* $w \in W$*. The semantics for the modal language* $\mathcal{L}$ *is inductively defined. The cases for* p*,* $\neg$ *and* $\wedge$ *are identical to those of Definition 2, while the case for* $\Box$ *looks as follows:*

$$\mathbb{M}, w \models \Box\varphi \quad \textit{iff} \quad \{v \in W \mid \mathbb{M}, v \models \varphi\} \in N(w).$$

Given a neighborhood frame $\mathbb{F} = \langle W, N\rangle$*, we write* $\mathbb{F} \models \varphi$ *if* $\langle \mathbb{F}, V\rangle, w \models \varphi$ *for all valuations* $V\colon \textbf{\textit{Prop}} \to \wp(W)$ *and states* $w \in W$*. Finally, given a class* C *of neighborhood frames, we write* $\models_{\mathsf{C}} \varphi$ *if* $\mathbb{F} \models \varphi$ *for all* $\mathbb{F} \in \mathsf{C}$*.*

There is a precise sense in which neighborhood semantics is a generalization of relational semantics: the class of Kripke frames corresponds to a subclass of neighborhood frames, viz., those that are augmented:

- For every augmented neighborhood frame $\mathbb{A} = \langle W, N\rangle$, there exists a modally equivalent Kripke frame $\mathbb{A}^k$, that is, for all valuations $V\colon \textbf{Prop} \to \wp(W)$, states $w \in W$ and formulas $\varphi \in \mathcal{L}$ we have $\langle \mathbb{A}, V\rangle, w \models \varphi$ if $\langle \mathbb{A}^k, V\rangle, w \models \varphi$;
- For every Kripke frame $\mathbb{K} = \langle W, R\rangle$, there exists a modally equivalent augmented neighborhood frame $\mathbb{K}^n$, that is, for all valuations $V\colon \textbf{Prop} \to \wp(W)$, states $w \in W$ and formulas $\varphi \in \mathcal{L}$ we have $\langle \mathbb{K}^n, V\rangle, w \models \varphi$ if $\langle \mathbb{K}, V\rangle, w \models \varphi$.

Furthermore, $(\cdot)^k$ and $(\cdot)^n$ are each other's inverses, that is, $(\mathbb{A}^k)^n = \mathbb{A}$ and $(\mathbb{K}^n)^k = \mathbb{K}$, for every augmented neighborhood frame $\mathbb{A}$ and Kripke frame $\mathbb{K}$. We now switch again from the semantic to the syntactic perspective:

Definition 6. *A* classical modal logic *is a set* $\mathsf{S} \subseteq \mathcal{L}$ *that (i) contains all propositional tautologies, (ii) is closed under modus ponens: if* $\varphi \to \psi \in \mathsf{S}$ *and* $\varphi \in \mathsf{S}$ *then* $\psi \in \mathsf{S}$*, (iii) is closed under uniform substitution: if* $\varphi \in \mathsf{S}$ *then* $\varphi(\alpha_1/p_1, \ldots, \alpha_n/p_n) \in \mathsf{S}$*, (iv) is closed under the so-called RE-rule: if* $\varphi \leftrightarrow \psi \in \mathsf{S}$ *then* $\Box\varphi \leftrightarrow \Box\psi$*. We usually write* $\vdash_{\mathsf{S}} \varphi$ *instead of* $\varphi \in \mathsf{S}$*. The smallest classical modal logic is called* E*. If a classical modal logic contains some further axioms* $\mathsf{A}_1, \ldots, \mathsf{A}_n \in \mathcal{L} \backslash \mathsf{E}$*, then it will be called* $\mathsf{EA}_1 \ldots \mathsf{A}_n$*.*

Upon comparison with Definition 3, it is clear that classical modal logics are a strict generalization of normal modal logics, since they are not required to be closed under necessitation, while the requirement of containing $\Box(p \to q) \to (\Box p \to \Box q)$ is weakened to being closed under RE. Every normal modal logic is thus a classical modal logic, but not vice versa. *Non-normal modal logics* are precisely those logics that are classical, but not normal.

In Sections 3–6 we will encounter examples from the non-normal modal logics E, EM, EC, EMC, EMN and EMNC, which are defined using the further axioms M: $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$, C: $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ and N: $\Box\top$. Once again, our use of expressions such as E and EMN to denote logics as well as classes of neighborhood frames is justified by standard soundness and completeness results:

$$\begin{array}{lllllll} \vdash_{\mathsf{E}} \varphi & \text{iff} & \models_{\mathsf{E}} \varphi & \qquad & \vdash_{\mathsf{EMC}} \varphi & \text{iff} & \models_{\mathsf{EMC}} \varphi \\ \vdash_{\mathsf{EM}} \varphi & \text{iff} & \models_{\mathsf{EM}} \varphi & & \vdash_{\mathsf{EMN}} \varphi & \text{iff} & \models_{\mathsf{EMN}} \varphi \\ \vdash_{\mathsf{EC}} \varphi & \text{iff} & \models_{\mathsf{EC}} \varphi & & \vdash_{\mathsf{EMNC}} \varphi & \text{iff} & \models_{\mathsf{EMNC}} \varphi \end{array}$$

Given the correspondence between augmented neighborhood frames and Kripke frames, it should not be surprising that the classical modal logic EMNC coincides with the smallest normal modal logic K: $\vdash_{\mathsf{EMNC}} \varphi$ iff $\models_{\mathsf{EMNC}} \varphi$ iff $\models_{\mathsf{K}} \varphi$ iff $\vdash_{\mathsf{K}} \varphi$.

Finally, generalizing terminology from Segerberg [62], a classical modal logic S will be called *absurd* iff $\vdash_{\mathsf{S}} \Box p$ or $\vdash_{\mathsf{S}} \neg\Box p$. It is easy to check that all of the modal logics that we will be interested in here (normal and non-normal alike) are not absurd.

2.2. Logical Geometry

We begin by defining the basic ingredients of logical geometry, that is, Aristotelian relations and diagrams. For more details and philosophical discussion, see [51,56].

Definition 7. *Let* S *be a logical system with Boolean connectives and a model-theoretic semantics* $\models_{\mathsf{S}}$. *The* Aristotelian relations for S *are defined as follows: two formulas* $\varphi, \psi \in \mathcal{L}_{\mathsf{S}}$ *are*

$$\begin{array}{lllll} \mathsf{S}\text{-}\textit{contradictory} & \textit{iff} & \models_{\mathsf{S}} \neg(\varphi \wedge \psi) & \textit{and} & \models_{\mathsf{S}} \varphi \vee \psi \\ \mathsf{S}\text{-}\textit{contrary} & \textit{iff} & \models_{\mathsf{S}} \neg(\varphi \wedge \psi) & \textit{and} & \not\models_{\mathsf{S}} \varphi \vee \psi \\ \mathsf{S}\text{-}\textit{subcontrary} & \textit{iff} & \not\models_{\mathsf{S}} \neg(\varphi \wedge \psi) & \textit{and} & \models_{\mathsf{S}} \varphi \vee \psi \\ \textit{in } \mathsf{S}\text{-}\textit{subalternation} & \textit{iff} & \models_{\mathsf{S}} \varphi \rightarrow \psi & \textit{and} & \not\models_{\mathsf{S}} \psi \rightarrow \varphi \end{array}$$

Furthermore, φ *and* ψ *are said to be* S-unconnected *iff (i)* $\not\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$*, (ii)* $\not\models_{\mathsf{S}} \varphi \vee \psi$*, (iii)* $\not\models_{\mathsf{S}} \varphi \rightarrow \psi$ *and (iv)* $\not\models_{\mathsf{S}} \psi \rightarrow \varphi$*.*

Note that unconnectedness is itself not considered to be an Aristotelian relation, but it can be viewed as the *absence* of any Aristotelian relations between two formulas φ and ψ. For example, condition (i) entails that φ and ψ are not contradictory or contrary, while conditions (iii) and (iv) entail that there is no subalternation from φ to ψ or vice versa.

Definition 8. *Let* S *be a logical system as in Definition 7, and consider a finite fragment* $\mathcal{F} \subseteq \mathcal{L}_{\mathsf{S}}$*. An* Aristotelian diagram for $(\mathcal{F}, \mathsf{S})$ *is a vertex- and edge-labeled graph: its vertices are labeled by the formulas from* $\mathcal{F}$*, while its edges are labeled by the Aristotelian relations. Specifically, if vertices* v *and* w *are labeled by resp.* φ *and* ψ*, and relative to* S *these formulas* φ *and* ψ *stand in the Aristotelian relation* R*, then the edge from* v *to* w *is labeled by* R*. Finally, it is required that (i)* S*-equivalent formulas from* $\mathcal{F}$ *label the same vertex in the diagram, and (ii) only* S*-contingent formulas from* $\mathcal{F}$ *occur as vertex labels in the diagram.*

The labeling of edges by means of Aristotelian relations is usually in accordance with the convention described in the caption of Figure 1, that is, contrariety edges are visualized as dashed lines, and so forth. This same figure shows Aristotelian diagrams for $(\mathcal{F}, \mathsf{KD})$ and $(\mathcal{F}, \mathsf{K})$, where $\mathcal{F} = \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$. On a terminological level, note that instead of 'an Aristotelian diagram for $(\mathcal{F}, \mathsf{S})$', we often say 'an Aristotelian diagram for $\mathcal{F}$ in S'.

The requirements regarding (i) S-equivalence and (ii) S-contingency in Definition 8 are an empirical generalization (the vast majority of Aristotelian diagrams found in the literature indeed satisfy these requirements), but they also have extensive theoretical and cognitive motivations [51]. For example, it is easy to show that if φ and ψ are both contingent, they can stand in at most one Aristotelian relation [48]; however, if φ and/or ψ is not contingent, then they stand in multiple Aristotelian relations to each other, thus yielding more cumbersome diagrams that are harder to process (this will be illustrated later in the paper). Furthermore, while Definition 8 specifies that Aristotelian diagrams simply do not contain any non-contingent formulas at all, there also exists an alternative perspective, based on sound geometrical reasons [53,55,64], which holds that an Aristotelian diagram can contain non-contingent formulas after all, but that they are (invisibly) collapsed in the diagram's center of symmetry. We will revisit this alternative perspective later in the paper.

Aristotelian diagrams are sometimes defined exclusively for those pairs $(\mathcal{F}, \mathsf{S})$ such that all $\mathcal{F}$-formulas are S-contingent and pairwise non-S-equivalent to begin with. Under this approach, the two conditions (i) and (ii) in Definition 8 are vacuously fulfilled, and can even be left out of the definition altogether. For our present purposes, however, it will be better to take the more liberal approach of Definition 8, that is, we allow $\mathcal{F}$ to contain non-S-contingent and/or S-equivalent formulas, and then explicitly say how such cases are to be dealt with in Aristotelian diagrams for $(\mathcal{F}, \mathsf{S})$.

Definition 9. *For $i = 1, 2$, let S_i and $\mathcal{F}_i$ be a logical system and a fragment of formulas as in Definition 8; furthermore, let $\mathcal{C}_i$ be the set of S_i-contingent formulas. An* Aristotelian isomorphism *$f\colon (\mathcal{F}_1, \mathsf{S}_1) \to (\mathcal{F}_2, \mathsf{S}_2)$ is a surjective function $f\colon \mathcal{F}_1 \cap \mathcal{C}_1 \to \mathcal{F}_2 \cap \mathcal{C}_2$ such that for all $\varphi, \psi \in \mathcal{F}_1 \cap \mathcal{C}_1$, we have:*

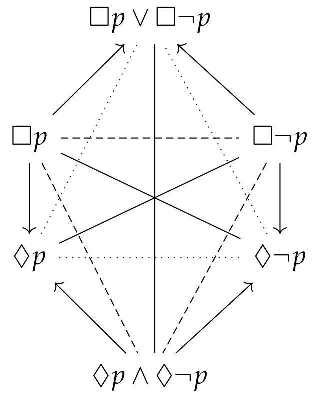
- *$R_{\mathsf{S}_1}(\varphi, \psi)$ iff $R_{\mathsf{S}_2}(f(\varphi), f(\psi))$, for all Aristotelian relations R,*
- *$\varphi \equiv_{\mathsf{S}_1} \psi$ iff $f(\varphi) \equiv_{\mathsf{S}_2} f(\psi)$.*

Finally, an Aristotelian family *is a maximal collection $\mathcal{A}$ of Aristotelian diagrams that is closed under Aristotelian isomorphism: (i) all $(\mathcal{F}_1, \mathsf{S}_1)$ and $(\mathcal{F}_2, \mathsf{S}_2)$ in $\mathcal{A}$ are Aristotelian isomorphic to each other, and vice versa, (ii) if $(\mathcal{F}_1, \mathsf{S}_1) \in \mathcal{A}$ is Aristotelian isomorphic to $(\mathcal{F}_2, \mathsf{S}_2)$, then $(\mathcal{F}_2, \mathsf{S}_2) \in \mathcal{A}$ as well.*

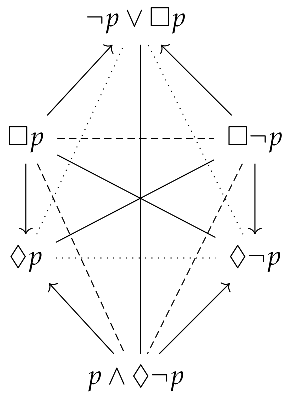
Note that Aristotelian isomorphisms are injective (and thus, also bijective) up to logical equivalence: if $f(\varphi) = f(\psi)$, then maybe not $\varphi = \psi$, but at least $\varphi \equiv_{\mathsf{S}_1} \psi$. Furthermore, note that Aristotelian isomorphisms also preserve and reflect unconnectedness: φ and ψ are S_1-unconnected iff $f(\varphi)$ and $f(\psi)$ are S_2-unconnected. Finally, recall that Aristotelian diagrams are sometimes defined exclusively for those $(\mathcal{F}, \mathsf{S})$ such that all $\mathcal{F}$-formulas are S-contingent and pairwise non-S-equivalent; on that approach, Definition 9 can be simplified significantly: $\mathcal{F}_i \cap \mathcal{C}_i$ will simply be equal to $\mathcal{F}_i$, while the condition regarding $\equiv_{\mathsf{S}_1}$ and $\equiv_{\mathsf{S}_2}$ will be vacuously fulfilled and can thus be left out (however, the requirement of surjectivity will have to be strengthened to bijectivity) [57].

One of the main ongoing research efforts in logical geometry is to develop a systematic typology of Aristotelian families [54,65]. Several hundreds of Aristotelian families are known to exist; however, only a few dozen of these occur frequently in the literature (cf. Section 1) and have been studied in depth. In this paper, we will restrict ourselves to just four families: the so-called PCDs, the classical squares of opposition, the degenerate squares of opposition, and the JSB hexagons. Examples of the families of classical and degenerate squares were already given in Figure 1. The term PCD, or 'pair of contradictory formulas', is self-explanatory: PCDs consist of just two formulas which are contradictory to each other. PCDs constitute the simplest Aristotelian family; a concrete example is shown in Figure 2a. Finally, the family of JSB hexagons is named after Jacoby [66], Sesmat [67], and Blanché [68], who were the first to study diagrams belonging to this family; two concrete examples are shown in Figure 2b,c.

(**a**) PCD for $\{p, \neg p\}$ in K.



(**b**) Strong JSB hexagon for $\{\Box p, \Box\neg p, \Diamond p \wedge \Diamond\neg p, \Box p \vee \Box\neg p, \Diamond p, \Diamond\neg p\}$ in KD.



(**c**) Weak JSB hexagon for $\{\Box p, \Box\neg p, p \wedge \Diamond\neg p, \neg p \vee \Box p, \Diamond p, \Diamond\neg p\}$ in KD.

Figure 2. Three further examples of Aristotelian diagrams.

Definition 10. *For $i = 1, 2$, let S_i, $\mathcal{F}_i$ and $\mathcal{C}_i$ be as in Definition 9. The* Boolean closure of $\mathcal{F}_i$ in S_i*, written as $\mathbb{B}_{\mathsf{S}_i}(\mathcal{F}_i)$, is the smallest Boolean subalgebra $\mathbb{B}$ of the Lindenbaum–Tarski algebra of S_i, such that $\mathcal{F}_i \subseteq \mathbb{B}$. A* Boolean isomorphism $f\colon (\mathcal{F}_1, \mathsf{S}_1) \to (\mathcal{F}_2, \mathsf{S}_2)$ *is a surjective function $f\colon \mathcal{F}_1 \cap \mathcal{C}_i \to \mathcal{F}_2 \cap \mathcal{C}_2$ such that there exists a Boolean algebra isomorphism $\varphi\colon \mathbb{B}_{\mathsf{S}_1}(\mathcal{F}_1) \to \mathbb{B}_{\mathsf{S}_2}(\mathcal{F}_2)$ and $f = \varphi \restriction \mathcal{F}_1$.*

Informally, the Boolean closure of $\mathcal{F}$ in S consists of all Boolean combinations of formulas from $\mathcal{F}$, considered up to S-equivalence [54]. A Boolean isomorphism between two Aristotelian diagrams is a surjective function between those diagrams that can be extended to a Boolean algebra isomorphism between their respective Boolean closures. It is easy to show that every Boolean isomorphism is also an Aristotelian isomorphism; however, there exist Aristotelian isomorphisms that are not Boolean isomorphisms [57].

From a typological perspective, this means that a given Aristotelian family can have multiple *Boolean subfamilies*. Diagrams belonging to different Boolean subfamilies of a given Aristotelian family are not Boolean isomorphic (since they belong to different Boolean subfamilies), but they are Aristotelian isomorphic to each other (since they belong to the same Aristotelian family). For example, it is well-known that the Aristotelian family of JSB hexagons has two Boolean subfamilies, which are usually called the *strong* and the *weak* JSB hexagons [69]. In a strong JSB hexagon, the disjunction of the three pairwise contrary formulas is a tautology, whereas in a weak JSB hexagon, this disjunction is not a tautology. For example, Figure 2b shows a strong JSB hexagon (since $\models_{\mathsf{KD}} \Box p \vee \Box\neg p \vee (\Diamond p \wedge \Diamond\neg p)$), while Figure 2c shows a weak JSB hexagon (since $\not\models_{\mathsf{KD}} \Box p \vee \Box\neg p \vee (p \wedge \Diamond\neg p)$).

2.3. Bitstring Semantics

As a final prerequisite, we introduce the technique of bitstring semantics. This technique was initially developed within the specific context of logical geometry [70], but in recent years it has evolved into a powerful tool with applications in logic [65], philosophy [71], psychology [72], business intelligence [73], and so forth. In general, bitstring semantics allows us to systematically compute combinatorial representations of a system of concepts or propositions, thus providing a concrete grip on their logical behavior (including their Aristotelian relations, as well as their Boolean structure). We will now summarize the main tenets of bitstring semantics; for more details and examples, see [57].

Given a logical system S and fragment $\mathcal{F}$ as in Definition 8, the *partition induced by $\mathcal{F}$ in* S is defined as follows:

$$\Pi_{\mathsf{S}}(\mathcal{F}) := \{ \bigwedge_{\varphi \in \mathcal{F}} \pm\varphi \mid \bigwedge_{\varphi \in \mathcal{F}} \pm\varphi \text{ is S-consistent} \},$$

where $+\varphi = \varphi$ and $-\varphi = \neg\varphi$. The elements of $\Pi_{\mathsf{S}}(\mathcal{F})$ are called *anchor formulas*. The set $\Pi_{\mathsf{S}}(\mathcal{F})$ is said to be a partition, because the anchor formulas are (i) jointly exhaustive, that is, $\models_{\mathsf{S}} \bigvee \Pi_{\mathsf{S}}(\mathcal{F})$, and (ii) mutually exclusive, that is, $\models_{\mathsf{S}} \neg(\alpha \wedge \beta)$ for distinct $\alpha, \beta \in \Pi_{\mathsf{S}}(\mathcal{F})$. For some simple examples from classical propositional logic (CPL), note that $\Pi_{\mathsf{CPL}}(\{p, q\}) = \{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$ and that $\Pi_{\mathsf{CPL}}(\{p \wedge q, p\}) = \{p \wedge q, p \wedge \neg q, \neg p\}$.

It will sometimes be useful to compare different partitions in terms of their 'resolution' or 'granularity'. In particular, given two partitions Π_1 and Π_2, we say that Π_1 is a *coarsening* of Π_2 (or equivalently, that Π_2 is a *refinement* of Π_1) iff, for all anchor formulas $\alpha_2 \in \Pi_2$, there exists an anchor formula $\alpha_1 \in \Pi_1$ such that $\models_{\mathsf{S}} \alpha_2 \rightarrow \alpha_1$. Returning to our example from CPL, note that $\{p \wedge q, p \wedge \neg q, \neg p\}$ is a coarsening of $\{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$, since the last two anchor formulas of the latter partition, that is, $\neg p \wedge q$ and $\neg p \wedge \neg q$, have collapsed into a single anchor formula in the former partition, that is, $\neg p$. Formally: $\neg p \equiv_{\mathsf{CPL}} (\neg p \wedge q) \vee (\neg p \wedge \neg q)$.

It can be shown that every proposition in the Boolean closure of $\mathcal{F}$ in S is logically equivalent to a disjunction of anchor formulas: for every $\varphi \in \mathbb{B}_{\mathsf{S}}(\mathcal{F})$ we have $\varphi \equiv_{\mathsf{S}} \bigvee\{\alpha \in \Pi_{\mathsf{S}}(\mathcal{F}) \mid \models_{\mathsf{S}} \alpha \rightarrow \varphi\}$. The bitstring semantics $\beta^{\mathcal{F}}_{\mathsf{S}} \colon \mathbb{B}_{\mathsf{S}}(\mathcal{F}) \rightarrow \{0,1\}^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$ maps every formula $\varphi \in \mathbb{B}_{\mathsf{S}}(\mathcal{F})$ onto its bitstring representation $\beta^{\mathcal{F}}_{\mathsf{S}}(\varphi)$, which is a sequence of $|\Pi_{\mathsf{S}}(\mathcal{F})|$ bits that keeps track of which anchor formulas enter into this disjunction. For example, if $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\varphi \equiv_{\mathsf{S}} \alpha_1 \vee \alpha_3 \vee \alpha_4$, then φ is represented by the bitstring 1011. It can be shown that $\beta^{\mathcal{F}}_{\mathsf{S}}$ is a Boolean (and thus also an Aristotelian) isomorphism between $\mathbb{B}_{\mathsf{S}}(\mathcal{F})$ and $\{0,1\}^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$. Consequently, bitstring length can be used to uniquely characterize the different Boolean subfamilies of a given Aristotelian family. For example, strong JSB hexagons are represented by bitstrings of length 3, while weak JSB hexagons are represented by bitstrings of length 4 [57].

3. Logic-Sensitivity and Aristotelian Families

3.1. Introduction

We now begin our exploration of the logic-sensitivity of Aristotelian diagrams with respect to Aristotelian families. Our starting point is the observation that the Aristotelian relations themselves are sensitive to the details of the underlying logical system (cf. Definition 7). For example, consider the formulas $\Box p$ and $\Box\neg p$: in the modal logic KD, these two formulas are contrary to each other (since $\models_{\mathsf{KD}} \neg(\Box p \wedge \Box\neg p)$ but $\not\models_{\mathsf{KD}} \Box p \vee \Box\neg p$), but in the modal logic KDF, these same two formulas are contradictory to each other (since $\models_{\mathsf{KDF}} \neg(\Box p \wedge \Box\neg p)$ and also $\models_{\mathsf{KDF}} \Box p \vee \Box\neg p$). Or to give an example from non-normal modal logic: it is straightforward to check that the formulas $\Box(p \wedge q)$ and $\Diamond\neg p \vee \Diamond\neg q$ are contraries in EM, but subcontraries in EC.

This observation generalizes from pairs of formulas to arbitrary sets of formulas. Given a fragment $\mathcal{F}$ of formulas and two logical systems S_1 and S_2, the configuration of Aristotelian relations among the formulas of $\mathcal{F}$ with respect to S_1 might be completely different from the configuration of Aristotelian relations among these same formulas with respect to S_2. The two Aristotelian diagrams for the same set $\mathcal{F}$ with respect to the two logical systems S_1 and S_2 might thus belong to two different Aristotelian families. Put more precisely: for some fragments $\mathcal{F}$ and logical systems S_1 and S_2, there does not exist an Aristotelian isomorphism between $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$. This is the standard way of formally capturing the logic-sensitivity of Aristotelian diagrams [57]. However, we will soon show that this approach is slightly too narrow in scope, since it does not cover certain cases in which the two Aristotelian diagrams for $\mathcal{F}$ with respect to S_1 and S_2 belong to one and the same Aristotelian family, but which we nevertheless do want to count as genuine examples of logic-sensitivity. A more adequate formulation of logic-sensitivity is therefore that for some fragments $\mathcal{F}$ and logical systems S_1 and S_2, the identity function $id_{\mathcal{F}} \colon (\mathcal{F}, \mathsf{S}_1) \rightarrow (\mathcal{F}, \mathsf{S}_2)$ is not an Aristotelian isomorphism. Unlike the previous one, this formulation also covers cases of logic-sensitivity in which there does exist an Aristotelian isomorphism between $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$, as long as this isomorphism is not the identity function $id_{\mathcal{F}}$.

We are now ready to study some detailed examples of this type of logic-sensitivity. We will first consider a cluster of examples from normal modal logic, and then one from non-normal modal logic.

3.2. Examples from Normal Modal Logic

For our first series of examples, we revisit and extend the motivating example that was already presented in Section 1. For future reference, we define the fragment $\mathcal{F}_{1a}$:

$$\mathcal{F}_{1a} := \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}.$$

We first study this fragment relative to K. Recall that K is the *smallest* normal modal logic, and is thus quite important for theoretical reasons (e.g., it is sound and complete with respect to the class of *all* Kripke frames). It is easy to check that $\Box p$ and $\Diamond\neg p$ are K-contradictories, since $\models_{\mathsf{K}} \neg(\Box p \wedge \Diamond\neg p)$ and $\models_{\mathsf{K}} \Box p \vee \Diamond\neg p$. Similarly, $\Box\neg p$ and $\Diamond p$ are K-contradictories. However, relative to K, the formulas of $\mathcal{F}_{1a}$ do not enter into any other Aristotelian relations with each other (and are thus pairwise K-unconnected). For example, there is no K-subalternation from $\Box p$ to $\Diamond p$, since $\not\models_{\mathsf{K}} \Box p \rightarrow \Diamond p$. To summarize, the Aristotelian diagram for $(\mathcal{F}_{1a}, \mathsf{K})$ is a degenerate square of opposition, which was already shown in Figure 1b, and is repeated here in Figure 3a, for the sake of reference. An easy computation yields the partition that is induced by $\mathcal{F}_{1a}$ in K:

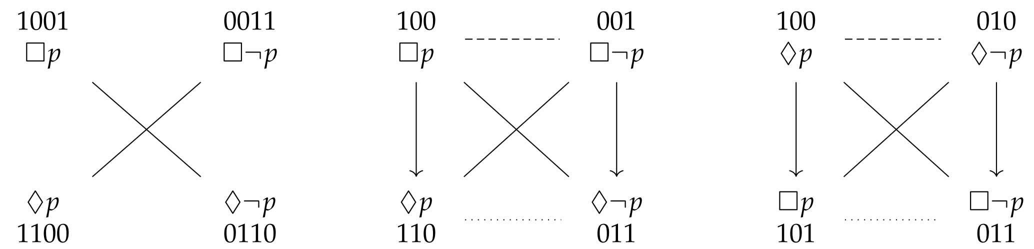
$$\Pi_{\mathsf{K}}(\mathcal{F}_{1a}) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \Box p \wedge \Box\neg p\}.$$

This partition yields bitstrings of length 4. For example, $\Box p$ is represented as 1001, since $\Box p \equiv_{\mathsf{K}} (\Box p \wedge \Diamond p) \vee (\Box p \wedge \Box\neg p)$. The bitstring representations of all formulas in $\mathcal{F}_{1a}$ can be found in Figure 3a.

We now turn to KD. The practical importance of this logic cannot be underestimated, since the D-axiom $\Box p \rightarrow \Diamond p$ holds for many important interpretations of the $\Box$- and $\Diamond$-operators, such as alethic, epistemic and deontic modalities. Since KD is strictly stronger than K, the two pairs of K-contradictories are also KD-contradictories. Furthermore, the formulas of $\mathcal{F}_{1a}$ enter into four additional Aristotelian relations relative to KD. For example, $\Box p$ and $\Box\neg p$ are KD-contraries, since $\models_{\mathsf{KD}} \neg(\Box p \wedge \Box\neg p)$ but $\not\models_{\mathsf{KD}} \Box p \vee \Box\neg p$. To summarize, the Aristotelian diagram for $(\mathcal{F}_{1a}, \mathsf{KD})$ is a classical square of opposition, which was already shown in Figure 1a, and is repeated here in Figure 3b, for the sake of reference. It is straightforward to check that there does not exist an Aristotelian isomorphism between $(\mathcal{F}_{1a}, \mathsf{K})$ and $(\mathcal{F}_{1a}, \mathsf{KD})$, which means that we have obtained our first concrete example of the logic-sensitivity of Aristotelian diagrams with respect to Aristotelian families. Furthermore, an easy computation yields the partition that is induced by $\mathcal{F}_{1a}$ in KD:

$$\Pi_{\mathsf{KD}}(\mathcal{F}_{1a}) = \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\}.$$

In comparison with $\Pi_{\mathsf{K}}(\mathcal{F}_{1a})$, we find that the first and third anchor formulas have been simplified and, most importantly, that the fourth anchor formula, that is, $\Box p \wedge \Box\neg p$, has been dropped (since it is K-consistent but KD-inconsistent). The partition $\Pi_{\mathsf{KD}}(\mathcal{F}_{1a})$ thus yields bitstrings of length 3, which are obtained by deleting the fourth bit from the $\Pi_{\mathsf{K}}(\mathcal{F}_{1a})$-bitstrings. For example, $\Diamond p$ is represented as 110, since $\Diamond p \equiv_{\mathsf{KD}} (\Diamond p \wedge \Diamond\neg p) \vee \Box\neg p$. The bitstring representations of all formulas in $\mathcal{F}_{1a}$ can be found in Figure 3b.



(**a**) Degenerate square of opposition for $\mathcal{F}_{1a}$ in K.

(**b**) Classical square of opposition for $\mathcal{F}_{1a}$ in KD.

(**c**) Classical square of opposition for $\mathcal{F}_{1a}$ in KF.

Figure 3. Aristotelian diagrams for $\mathcal{F}_{1a} := \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$ in three normal modal logics.

To conclude this cluster of examples, we turn to KF. This logic is less familiar than KD, but the F-axiom $\Diamond p \to \Box p$ is quite important in studying dynamic modalities [48]. It is once again easy to check that the Aristotelian diagram for $(\mathcal{F}_{1a}, \mathsf{KF})$ is a classical square of opposition, which is shown in Figure 3c. There again does not exist an Aristotelian isomorphism between $(\mathcal{F}_{1a}, \mathsf{K})$ and $(\mathcal{F}_{1a}, \mathsf{KF})$, and thus we have obtained another basic example of logic-sensitivity with respect to Aristotelian families. However, the situation gets more subtle when we compare KD and KF: there does exist an Aristotelian isomorphism $f\colon (\mathcal{F}_{1a}, \mathsf{KD}) \to (\mathcal{F}_{1a}, \mathsf{KF})$, which maps $\Box p$ onto $\Diamond p$, $\Box\neg p$ onto $\Diamond\neg p$, and so forth—just compare Figure 3b,c. It is easy to check that this is indeed an Aristotelian isomorphism; for example, $\Box p$ and $\Box\neg p$ are KD-contraries, and $f(\Box p)$ and $f(\Box\neg p)$ are KF-contraries. Nevertheless, the identity function $id_{\mathcal{F}_{1a}}$ is not an Aristotelian isomorphism between $(\mathcal{F}_{1a}, \mathsf{KD})$ and $(\mathcal{F}_{1a}, \mathsf{KF})$ (e.g., $\Diamond p$ and $\Diamond\neg p$ are KD-subcontrary but KF-contrary), and hence, we have obtained a third, more subtle example of logic-sensitivity with respect to Aristotelian families. Furthermore, an easy computation yields the partition that is induced by $\mathcal{F}_{1a}$ in KF:

$$\Pi_{\mathsf{KF}}(\mathcal{F}_{1a}) = \{\Diamond p, \Diamond\neg p, \Box p \wedge \Box\neg p\}.$$

In comparison with $\Pi_{\mathsf{K}}(\mathcal{F}_{1a})$, we find that the first and third anchor formulas have been simplified and, most importantly, that the second anchor formula, that is, $\Diamond p \wedge \Diamond\neg p$, has been dropped (since it is K-consistent but KF-inconsistent). The partition $\Pi_{\mathsf{KF}}(\mathcal{F}_{1a})$ thus yields bitstrings of length 3, which are obtained by deleting the second bit from the $\Pi_{\mathsf{K}}(\mathcal{F}_{1a})$-bitstrings. For example, $\Box p$ is represented as 101, since $\Box p \equiv_{\mathsf{KF}} \Diamond p \vee (\Box p \vee \Box\neg p)$. The bitstring representations of all formulas in $\mathcal{F}_{1a}$ can be found in Figure 3c.

3.3. Examples from Non-Normal Modal Logic

For the cluster of examples from non-normal modal logic, we will consider the fragment $\mathcal{F}_{1b}$:

$$\mathcal{F}_{1b} := \{\Box(p \wedge q), \Box p \wedge \Box q, \Diamond\neg p \vee \Diamond\neg q, \Diamond(\neg p \vee \neg q)\}.$$

Let's first briefly consider $\mathcal{F}_{1b}$ relative to normal modal logic. In any normal modal logic $\mathsf{K}\alpha$, we have $\Box(p \wedge q) \equiv_{\mathsf{K}\alpha} \Box p \wedge \Box q$ and $\Diamond(\neg p \vee \neg q) \equiv_{\mathsf{K}\alpha} \Diamond\neg p \vee \Diamond\neg q$, and the first two formulas are $\mathsf{K}\alpha$-contradictory to the last two. The Aristotelian diagram for $(\mathcal{F}_{1b}, \mathsf{K}\alpha)$ is thus a PCD, as shown in Figure 4a. More specifically, for any normal modal logics $\mathsf{K}\alpha$ and $\mathsf{K}\beta$, the identity function $id_{\mathcal{F}_{1b}}$ is an Aristotelian isomorphism between $(\mathcal{F}_{1b}, \mathsf{K}\alpha)$ and $(\mathcal{F}_{1b}, \mathsf{K}\beta)$. Furthermore, it is easy to compute the partition that is induced by $\mathcal{F}_{1b}$ in any normal modal logic $\mathsf{K}\alpha$:

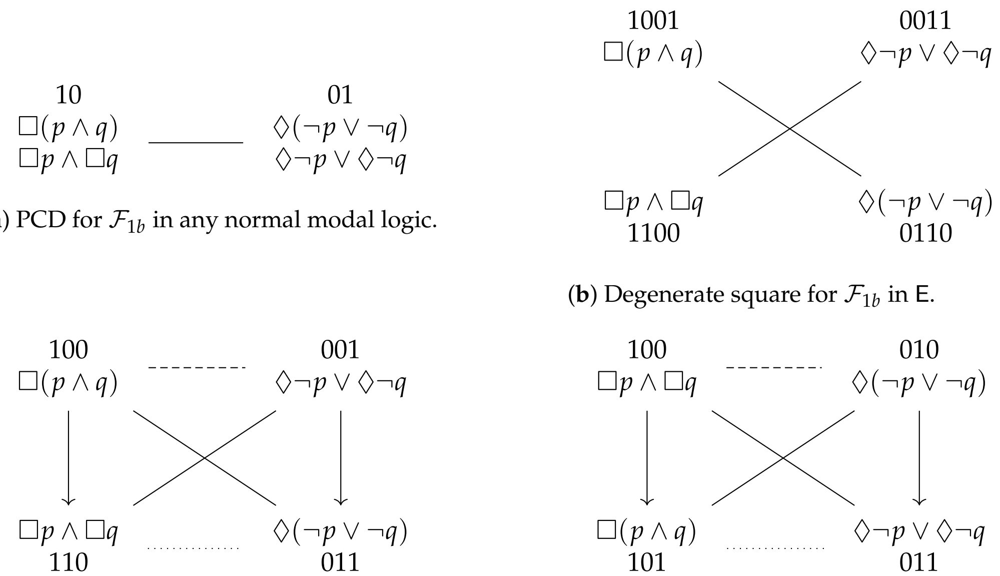
$$\Pi_{\mathsf{K}\alpha}(\mathcal{F}_{1b}) = \{\Box(p \wedge q), \Diamond(\neg p \vee \neg q)\}.$$

This partition yields bitstrings of length 2, which are given in Figure 4a. Within the realm of *normal* modal logics, the fragment $\mathcal{F}_{1b}$ is thus not at all logic-sensitive: for any normal modal logic, it gives rise to the same Aristotelian diagram and the same partition. However, this changes drastically as soon as we move to *non-normal* modal logics.

We first consider E. Recall that E is the *smallest* non-normal modal logic, and is thus quite important for theoretical reasons (e.g., it is sound and complete with respect to the class of *all* neighborhood frames). It is easy to check that $\Box(p \wedge q)$ and $\Diamond(\neg p \vee \neg q)$ are E-contradictories, and similar for $\Box p \wedge \Box q$ and $\Diamond\neg p \vee \Diamond\neg q$. However, relative to E, the formulas of $\mathcal{F}_{1b}$ do not enter into any other Aristotelian relations with each other (and are thus pairwise E-unconnected). The Aristotelian diagram for $(\mathcal{F}_{1b}, \mathsf{E})$ is thus a degenerate square of opposition, as shown in Figure 4b. The partition that is induced by $\mathcal{F}_{1b}$ in E looks as follows:

$$\Pi_{\mathsf{E}}(\mathcal{F}_{1b}) = \{\Box(p \wedge q) \wedge \Box p \wedge \Box q,\ \Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q),\ (\Diamond\neg p \vee \Diamond\neg q) \wedge \Diamond(\neg p \vee \neg q),\ \Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q)\}.$$

This partition yields bitstrings of length 4, as indicated in Figure 4b.



(**a**) PCD for $\mathcal{F}_{1b}$ in any normal modal logic.

(**b**) Degenerate square for $\mathcal{F}_{1b}$ in E.

(**c**) Classical square for $\mathcal{F}_{1b}$ in EM.

(**d**) Classical square for $\mathcal{F}_{1b}$ in EC.

Figure 4. Aristotelian diagrams for $\mathcal{F}_{1b} := \{\Box(p \wedge q), \Box p \wedge \Box q, \Diamond\neg p \vee \Diamond\neg q, \Diamond(\neg p \vee \neg q)\}$ in any normal modal logic and in three non-normal modal logics.

We now turn to EM. The M-axiom $\Box(p \wedge q) \to (\Box p \wedge \Box q)$ yields some of the most important applications of non-normal modal logics, for example in game logic and coalition logic [74]. Since EM is strictly stronger than E, the two pairs of E-contradictories are also EM-contradictories. Furthermore, the formulas of $\mathcal{F}_{1b}$ enter into four additional Aristotelian relations relative to EM, and hence, the Aristotelian diagram for $(\mathcal{F}_{1b}, \mathsf{EM})$ is a classical square of opposition, as shown in Figure 4c. It is straightforward to check that there does not exist an Aristotelian isomorphism between $(\mathcal{F}_{1b}, \mathsf{E})$ and $(\mathcal{F}_{1b}, \mathsf{EM})$, so this constitutes our first example of logic-sensitivity with respect to Aristotelian families in the realm of non-normal modal logics. Furthermore, an easy computation yields the partition that is induced by $\mathcal{F}_{1b}$ in EM:

$$\Pi_{\mathsf{EM}}(\mathcal{F}_{1b}) = \{\Box(p \wedge q), \Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q), \Diamond\neg p \vee \Diamond\neg q\}.$$

In comparison with $\Pi_{\mathsf{E}}(\mathcal{F}_{1b})$, we find that the first and third anchor formulas have been simplified and, most importantly, that the fourth anchor formula, that is, $\Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q)$, has been dropped (since it is E-consistent but EM-inconsistent). The partition $\Pi_{\mathsf{EM}}(\mathcal{F}_{1b})$ thus yields bitstrings of length 3, which are obtained by systematically deleting the fourth bit from the $\Pi_{\mathsf{E}}(\mathcal{F}_{1b})$-bitstrings; cf. Figure 4c.

Finally, we turn to EC. It is once again easy to check that the Aristotelian diagram for $(\mathcal{F}_{1b}, \mathsf{EC})$ is a classical square of opposition, which is shown in Figure 3d. There does not exist an Aristotelian isomorphism between $(\mathcal{F}_{1b}, \mathsf{E})$ and $(\mathcal{F}_{1b}, \mathsf{EC})$, and thus we have another easy example of logic-sensitivity with respect to Aristotelian families. However, the situation again gets more subtle when we compare EM and EC: there does exist an Aristotelian isomorphism $f\colon (\mathcal{F}_{1b}, \mathsf{EM}) \to (\mathcal{F}_{1b}, \mathsf{EC})$, which maps $\Box(p \wedge q)$ onto $\Box p \wedge \Box q$, $\Diamond\neg p \vee \Diamond\neg q$ onto $\Diamond(\neg p \vee \neg q)$, and so forth—just compare Figure 4c,d. Nevertheless, the identity function $id_{\mathcal{F}_{1b}}$ is not an Aristotelian isomorphism between $(\mathcal{F}_{1b}, \mathsf{EM})$ and $(\mathcal{F}_{1b}, \mathsf{EC})$ (e.g., $\Box(p \wedge q)$ and $\Diamond\neg p \wedge \Diamond\neg q$ are EM-contrary but EC-subcontrary), and hence we have obtained one more example of logic-sensitivity with respect to Aristotelian families. Finally, an easy computation yields the partition that is induced by $\mathcal{F}_{1b}$ in EC:

$$\Pi_{\mathsf{EC}}(\mathcal{F}_{1b}) = \{\Box p \wedge \Box q, \Diamond(\neg p \vee \neg q), \Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q)\}.$$

In comparison with $\Pi_{\mathsf{E}}(\mathcal{F}_{1b})$, we find that the first and third anchor formulas have been simplified and, most importantly, that the second anchor formula, that is, $\Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q)$, has been dropped (since it is E-consistent but EC-inconsistent). The partition $\Pi_{\mathsf{EC}}(\mathcal{F}_{1b})$ thus yields bitstrings of length 3, which are obtained by systematically deleting the second bit from the $\Pi_{\mathsf{E}}(\mathcal{F}_{1b})$-bitstrings; cf. Figure 4d.

4. Logic-Sensitivity and Logical Equivalence of Formulas

4.1. Introduction

In the previous section, we focused on an aspect of logic-sensitivity that is primarily due to the Aristotelian *relations* themselves (cf. Definition 7). However, Aristotelian *diagrams* are defined up to logical equivalence and should only contain contingent formulas (cf. Definition 8). These diagrammatic requirements are two further sources of logic-sensitivity. In this section, we will focus on the logical equivalence requirement, and in Section 5 on the contingency requirement.

Definition 8 specifies that different vertices of an Aristotelian diagram can only contain formulas that are not logically equivalent to each other. However, of course, two formulas might be logically equivalent relative to a logical system S_1, but not relative to another logical system S_2. Consequently, if $\mathcal{F}$ is a fragment containing these formulas, then the Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_1)$ will have fewer distinct vertices than the diagram for $(\mathcal{F}, \mathsf{S}_2)$, and a fortiori, the two diagrams will not be Aristotelian isomorphic to each other. This aspect of logic-sensitivity can thus be viewed as a special case of the general logic-sensitivity that was discussed in Section 3. More concretely, in the general case we were dealing with Aristotelian diagrams for $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$ that are not Aristotelian isomorphic but that still have the same number of vertices (e.g., a degenerate *square* and a classical *square* both have *four* distinct vertices; cf. Section 3), whereas the cases that we are going to study now involve Aristotelian diagrams for $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$ that do not even have the same number of vertices.

4.2. Examples from Normal Modal Logic

We return to the fragment $\mathcal{F}_{1a} = \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$. We already saw in Section 3.2 that the Aristotelian diagram for $(\mathcal{F}_{1a}, \mathsf{K})$ is a degenerate square of opposition, which is repeated here as in Figure 5a, together with the bitstrings corresponding to the induced partition

$$\Pi_{\mathsf{K}}(\mathcal{F}_{1a}) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \Box p \wedge \Box\neg p\}.$$

We also saw that adding either the D- or the F-axiom yielded a classical square of opposition. A natural next step is now to investigate the effect of adding both axioms simultaneously, and thus to move from K to KDF. It is trivial that $\Box p \equiv_{\mathsf{KDF}} \Diamond p$ and $\Box\neg p \equiv_{\mathsf{KDF}} \Diamond\neg p$. Furthermore, each of the first two formulas is KDF-contradictory to each of the last two formulas. To summarize, the Aristotelian diagram for $(\mathcal{F}_{1a}, \mathsf{KDF})$ is not a square (be it degenerate or classical), but rather collapses into a horizontal PCD, as shown in Figure 5b. It is easy to see that there does not exist an Aristotelian isomorphism $f\colon (\mathcal{F}_{1a}, \mathsf{K}) \to (\mathcal{F}_{1a}, \mathsf{KDF})$. After all, if such an isomorphism f would exist, then the fact that $\Box p$ and $\Diamond p$ are K-unconnected would entail that $f(\Box p)$ and $f(\Diamond p)$ are KDF-unconnected as well, but $(\mathcal{F}_{1a}, \mathsf{KDF})$ does not contain any pairs of unconnected formulas. The PCD for $(\mathcal{F}_{1a}, \mathsf{KDF})$ is thus not Aristotelian isomorphic to the degenerate square for $(\mathcal{F}_{1a}, \mathsf{K})$; completely analogously, this PCD is not isomorphic to the classical squares for $(\mathcal{F}_{1a}, \mathsf{KD})$ and $(\mathcal{F}_{1a}, \mathsf{KF})$ either. Finally, an easy computation yields the partition that is induced by $\mathcal{F}_{1a}$ in KDF:

$$\Pi_{\mathsf{KDF}}(\mathcal{F}_{1a}) = \{\Box p, \Box\neg p\}.$$

In comparison with $\Pi_{\mathsf{K}}(\mathcal{F}_{1a})$, we find that the first and third anchor formulas have been simplified and, most importantly, that the second and fourth anchor formulas, that is, $\Diamond p \wedge \Diamond\neg p$ and $\Box p \wedge \Box\neg p$, have been dropped (since they are K-consistent but KDF-inconsistent). The partition $\Pi_{\mathsf{KDF}}(\mathcal{F}_{1a})$ thus yields bitstrings of length 2, which are obtained by systematically deleting the second and fourth bits from the $\Pi_{\mathsf{K}}(\mathcal{F}_{1a})$-bitstrings; cf. Figure 5b.

(**a**) Degenerate square for $\mathcal{F}_{1a}$ in K.

(**b**) PCD for $\mathcal{F}_{1a}$ in KDF.

Figure 5. Aristotelian diagrams for $\mathcal{F}_{1a} = \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$ in two normal modal logics.

4.3. Examples from Non-Normal Modal Logic

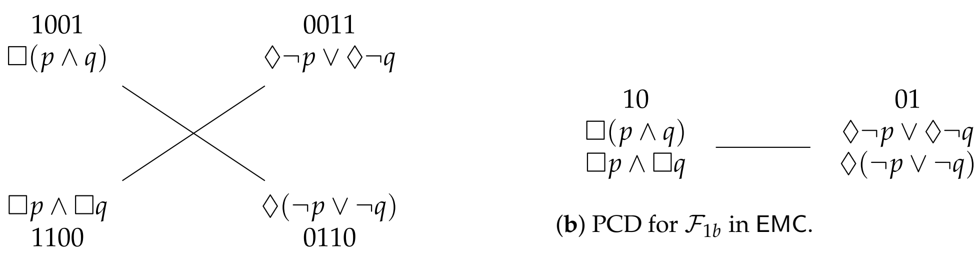
For an analogous example from non-normal modal logic, we return to the fragment $\mathcal{F}_{1b} = \{\Box(p \wedge q), \Box p \wedge \Box q, \Diamond\neg p \vee \Diamond\neg q, \Diamond(\neg p \vee \neg q)\}$. We already saw in Section 3.3 that this fragment does not exhibit any logic-sensitivity within the realm of *normal* modal logics, but that it does so once we move to *non-normal* modal logics. In particular, we saw that the Aristotelian diagram for $(\mathcal{F}_{1b}, \mathsf{E})$ is a degenerate square of opposition, which is repeated here as in Figure 6a, together with the bitstrings corresponding to the induced partition

$$\Pi_{\mathsf{E}}(\mathcal{F}_{1b}) = \{\Box(p \wedge q) \wedge \Box p \wedge \Box q,\ \ \Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q),$$
$$(\Diamond\neg p \vee \Diamond\neg q) \wedge \Diamond(\neg p \vee \neg q),\ \ \Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q)\}.$$

Furthermore, we also saw that adding the M- or the C-axiom yielded a classical square of opposition. Once again, a natural next step is to investigate the effect of adding both axioms simultaneously, and thus to move from E to EMC. We again find that the Aristotelian diagram for $(\mathcal{F}_{1b}, \mathsf{EMC})$ is not a square (be it degenerate or classical), but rather collapses into a horizontal PCD, as shown in Figure 6b. Just as before, the PCD for $(\mathcal{F}_{1b}, \mathsf{EMC})$ is again easily seen not to be Aristotelian isomorphic to either the degenerate square for $(\mathcal{F}_{1b}, \mathsf{E})$ or the classical squares for $(\mathcal{F}_{1b}, \mathsf{EM})$ and $(\mathcal{F}_{1b}, \mathsf{EC})$, thus yielding more examples of logic-sensitivity with respect to logical equivalence of formulas. Furthermore, the partition that is induced by $\mathcal{F}_{1b}$ in EMC looks as follows:

$$\Pi_{\mathsf{EMC}}(\mathcal{F}_{1b}) = \{\Box(p \wedge q), \Diamond(\neg p \vee \neg q)\}.$$

In comparison with $\Pi_{\mathsf{E}}(\mathcal{F}_{1b})$, we find that the first and third anchor formulas have been simplified and, most importantly, that the second and fourth anchor formulas, that is, $\Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q)$ and $\Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q)$, have been dropped (since they are E-consistent but EMC-inconsistent). The partition $\Pi_{\mathsf{EMC}}(\mathcal{F}_{1b})$ thus yields bitstrings of length 2, which are obtained by deleting the second and fourth bits from the $\Pi_{\mathsf{E}}(\mathcal{F}_{1b})$-bitstrings; cf. Figure 6b.



(**b**) PCD for $\mathcal{F}_{1b}$ in EMC.

(**a**) Degenerate square for $\mathcal{F}_{1b}$ in E.

Figure 6. Aristotelian diagrams for $\mathcal{F}_{1b} = \{\Box(p \wedge q), \Box p \wedge \Box q, \Diamond\neg p \vee \Diamond\neg q, \Diamond(\neg p \vee \neg q)\}$ in two non-normal modal logics.

4.4. Theory and Further Examples

We now take a more theoretical perspective, and show that logic-sensitivity with respect to logical equivalence (as studied in this section) can be systematically linked to logic-sensitivity with respect to Aristotelian families (as studied in Section 3). Specifically, Theorem 1 shows how the logical equivalence or non-equivalence of certain formulas is related to the Aristotelian families of classical and degenerate squares of oppositions. This theorem can be proved from first principles (i.e., without appealing to bitstring semantics), but we present a bitstring-based proof that is easier to follow and is more insightful.

Theorem 1. *Consider the function $f_1\colon \wp(\mathcal{L}) \to \wp(\mathcal{L})$, which maps $\mathcal{F} = \{\alpha, \beta, \neg\alpha, \neg\beta\}$ onto*

$$f_1(\mathcal{F}) := \{\alpha \wedge \beta, \alpha, \neg\alpha, \neg\alpha \vee \neg\beta\}.$$

For any logical system S *as in Definition 7, the following hold:*

1. *If the Aristotelian diagram for $(\mathcal{F}, \mathsf{S})$ is a degenerate square, then the Aristotelian diagram for $(f_1(\mathcal{F}), \mathsf{S})$ is a classical square (with an* S*-subalternation from $\alpha \wedge \beta$ to α);*
2. *If the Aristotelian diagram for $(\mathcal{F}, \mathsf{S})$ is a classical square (with an* S*-subalternation from α to β), then the Aristotelian diagram for $(f_1(\mathcal{F}), \mathsf{S})$ is a PCD (with $\alpha \wedge \beta \equiv_{\mathsf{S}} \alpha$).*

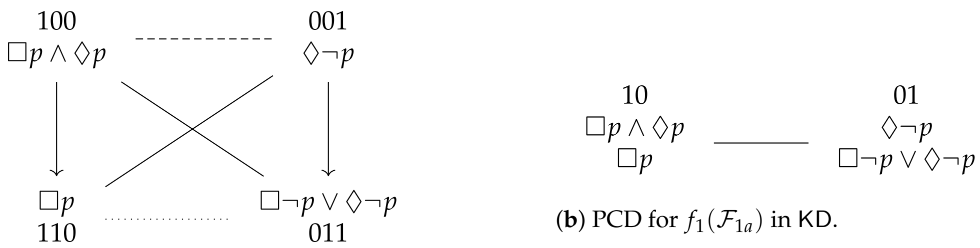
Furthermore, in both cases, $\Pi_{\mathsf{S}}(f_1(\mathcal{F}))$ is a coarsening of $\Pi_{\mathsf{S}}(\mathcal{F})$.

Proof. 1. If the Aristotelian diagram for $(\mathcal{F}, \mathsf{S})$ is a degenerate square, then $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha \wedge \beta, \alpha \wedge \neg\beta, \neg\alpha \wedge \beta, \neg\alpha \wedge \neg\beta\}$. The bitstring semantics $\beta^{\mathcal{F}}_{\mathsf{S}}$ corresponding to this partition maps $\alpha \wedge \beta$ to 1000, α to 1100, $\neg\alpha$ to 0011 and $\neg\alpha \vee \neg\beta$ to 0111. These four bitstrings constitute a classical square (with a subalternation from 1000 to 1100). Since $\beta^{\mathcal{F}}_{\mathsf{S}}$ is an Aristotelian isomorphism, it follows that the four $f_1(\mathcal{F})$-formulas also constitute a classical square (with a subalternation from $\alpha \wedge \beta$ to α) in S. Finally, note that $\Pi_{\mathsf{S}}(f_1(\mathcal{F})) = \{\alpha \wedge \beta, \alpha \wedge \neg\beta, \neg\alpha\}$ is indeed a coarsening of $\Pi_{\mathsf{S}}(\mathcal{F})$, since the two anchor formulas $\neg\alpha \wedge \beta$ and $\neg\alpha \wedge \neg\beta$ from $\Pi_{\mathsf{S}}(\mathcal{F})$ have collapsed into a single one, $\neg\alpha$, in $\Pi_{\mathsf{S}}(f_1(\mathcal{F}))$.

2. If the Aristotelian diagram for $(\mathcal{F}, \mathsf{S})$ is a classical square (with an S-subalternation from α to β), then $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha, \neg\alpha \wedge \beta, \neg\beta\}$. The bitstring semantics $\beta^{\mathcal{F}}_{\mathsf{S}}$ corresponding to this partition maps $\alpha \wedge \beta$ and α to 100, and $\neg\alpha$ and $\neg\alpha \vee \neg\beta$ to 011. These two bitstrings constitute a PCD. Since $\beta^{\mathcal{F}}_{\mathsf{S}}$ is an Aristotelian isomorphism, it follows that the four $f_1(\mathcal{F})$-formulas also constitute a PCD (with $\alpha \wedge \beta \equiv_{\mathsf{S}} \alpha$) in S. Finally, note that $\Pi_{\mathsf{S}}(f_1(\mathcal{F})) = \{\alpha, \neg\alpha\}$ is indeed a coarsening of $\Pi_{\mathsf{S}}(\mathcal{F})$, since the two anchor formulas $\neg\alpha \wedge \beta$ and $\neg\beta$ from $\Pi_{\mathsf{S}}(\mathcal{F})$ have collapsed into a single one, $\neg\alpha$, in $\Pi_{\mathsf{S}}(f_1(\mathcal{F}))$. □

This theorem can straightforwardly be applied to yield many more examples of logic-sensitivity with respect to logical equivalence. For example, consider once again

$\mathcal{F}_{1a} = \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$; applying f_1 yields the new fragment $f_1(\mathcal{F}_{1a}) = \{\Box p \wedge \Diamond p, \Box p, \Diamond\neg p, \Box\neg p \vee \Diamond\neg p\}$. We already saw in Section 3.2 that $(\mathcal{F}_{1a}, \mathsf{K})$ yields a degenerate square, while $(\mathcal{F}_{1a}, \mathsf{KD})$ yields a classical square. By Theorem 1, it follows immediately that $(f_1(\mathcal{F}_{1a}), \mathsf{K})$ yields a classical square, which is shown in Figure 7a, while $(f_1(\mathcal{F}_{1a}), \mathsf{KD})$ yields a PCD, which is shown in Figure 7b. Furthermore, easy computations yield $\Pi_{\mathsf{K}}(f_1(\mathcal{F}_{1a})) = \{\Box p \wedge \Diamond p, \Box p \wedge \Box\neg p, \Diamond\neg p\}$ and $\Pi_{\mathsf{KD}}(f_1(\mathcal{F}_{1a})) = \{\Box p, \Diamond\neg p\}$; the corresponding bitstrings of length 3 and 2 are shown in Figure 7a,b, respectively.

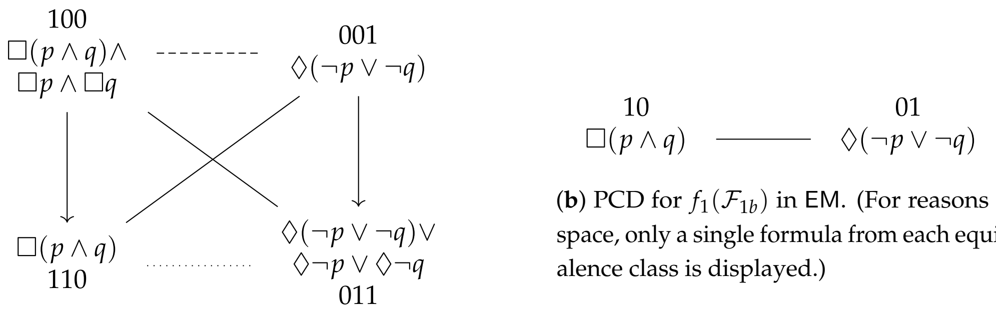


(**b**) PCD for $f_1(\mathcal{F}_{1a})$ in KD.

(**a**) Classical square for $f_1(\mathcal{F}_{1a})$ in K.

Figure 7. Aristotelian diagrams for $f_1(\mathcal{F}_{1a}) = \{\Box p \wedge \Diamond p, \Box p, \Diamond\neg p, \Box\neg p \vee \Diamond\neg p\}$ in two normal modal logics.

Theorem 1 applies equally straightforwardly in the realm of non-normal modal logics. For example, consider once again $\mathcal{F}_{1b} = \{\Box(p \wedge q), \Box p \wedge \Box q, \Diamond\neg p \vee \Diamond\neg q, \Diamond(\neg p \vee \neg q)\}$; we already saw in Section 3.3 that $(\mathcal{F}_{1b}, \mathsf{E})$ yields a degenerate square, while $(\mathcal{F}_{1b}, \mathsf{EM})$ yields a classical square. By Theorem 1 it follows immediately that $(f_1(\mathcal{F}_{1b}), \mathsf{E})$ yields a classical square, which is shown in Figure 8a, while $(f_1(\mathcal{F}_{1b}), \mathsf{EM})$ yields a PCD, which is shown in Figure 8b. Two further easy computations yield $\Pi_{\mathsf{E}}(f_1(\mathcal{F}_{1b})) = \{\Box(p \wedge q) \wedge \Box p \wedge \Box q, \Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q), \Diamond(\neg p \vee \neg q)\}$ and $\Pi_{\mathsf{EM}}(f_1(\mathcal{F}_{1b})) = \{\Box(p \wedge q), \Diamond(\neg p \vee \neg q)\}$; the corresponding bitstrings of length 3 and 2 are again shown in Figure 8a,b, respectively.



(**b**) PCD for $f_1(\mathcal{F}_{1b})$ in EM. (For reasons of space, only a single formula from each equivalence class is displayed.)

(**a**) Classical square for $f_1(\mathcal{F}_{1b})$ in E.

Figure 8. Aristotelian diagrams for $f_1(\mathcal{F}_{1b}) = \{\Box(p \wedge q) \wedge \Box p \wedge \Box q, \Box(p \wedge q), \Diamond(\neg p \vee \neg q), \Diamond(\neg p \vee \neg q) \vee \Diamond\neg p \vee \Diamond\neg q\}$ in two non-normal modal logics.

5. Logic-Sensitivity and Contingency of Formulas

5.1. Introduction

In this section, we continue our exploration of logic-sensitivity due to diagrammatic requirements. Definition 8 specifies that only contingent formulas can appear in an Aristotelian diagram. Of course, a formula might be contingent relative to a logical system S_1, but not relative to another logical system S_2. Consequently, if $\mathcal{F}$ is a fragment containing such a formula, then the Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_2)$ will have fewer vertices than the diagram for $(\mathcal{F}, \mathsf{S}_1)$, and a fortiori, the two diagrams will not be Aristotelian isomorphic to each other. This aspect of logic-sensitivity, too, can be viewed as a special case of the general logic-sensitivity that was discussed in Section 3. More concretely, in the general case, we deal with Aristotelian diagrams for $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$ that are not Aristotelian isomorphic but that still have the same number of vertices, whereas now we are going to deal with Aristotelian diagrams for $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$ that do not even have the same number of vertices.

It is worth pointing out that logic-sensitivity with respect to contingency can also be viewed as a special case of logic-sensitivity with respect to logical equivalence (which was discussed in the previous section). After all, recall from Section 2 that the standard perspective holds that Aristotelian diagrams simply do not contain any non-contingent formulas, but that there is also an alternative perspective, which holds that an Aristotelian diagram can contain non-contingent formulas after all, but that they are (invisibly) collapsed in the diagram's center of symmetry. Now suppose that $\varphi \in \mathcal{F}$ is S_1-contingent but not S_2-contingent. According to the standard perspective, going from $(\mathcal{F}, \mathsf{S}_1)$ to $(\mathcal{F}, \mathsf{S}_2)$ means that φ 'drops out' of the Aristotelian diagram altogether. According to the alternative perspective, however, it means that φ becomes equivalent to $\top$ (or to $\bot$), and thus 'collapses' with $\top/\bot$ in the diagram's center of symmetry. This alternative approach clearly merits further investigation, but in the concrete examples in the remainder of this section we will stick to the standard approach, and thus draw a clear distinction between equivalence-based and contingency-based aspects of logic-sensitivity.

5.2. Examples from Normal Modal Logic

Our main example from normal modal logic is based on the fragment $\mathcal{F}_{2a}$:

$$\mathcal{F}_{2a} := \{\Diamond p, \Diamond\top, \Box\bot, \Box\neg p\}.$$

It is easy to check that each of these formulas is K-contingent, and the Aristotelian diagram for $(\mathcal{F}_{2a}, \mathsf{K})$ is a classical square of opposition, as shown in Figure 9a. Furthermore, the partition that is induced by $\mathcal{F}_{2a}$ in K looks as follows:

$$\Pi_{\mathsf{K}}(\mathcal{F}_{2a}) = \{\Diamond p, \Diamond\top \wedge \Box\neg p, \Box\bot\}.$$

This partition yields bitstrings of length 3. For example, $\Box\neg p$ is represented as 011, since $\Box\neg p \equiv_{\mathsf{K}} (\Diamond\top \wedge \Box\neg p) \vee \Box\bot$. All bitstrings for $\mathcal{F}_{2a}$ can be found in Figure 9a.

We now turn to KD. The formulas $\Diamond p$ and $\Box\neg p$ are still KD-contingent. By contrast, $\Diamond\top$ and $\Box\bot$ go from being K-contingent to not being KD-contingent, since $\models_{\mathsf{KD}} \Diamond\top$ and $\models_{\mathsf{KD}} \neg\Box\bot$. Consequently, these last two formulas are not included in the Aristotelian diagram for $(\mathcal{F}_{2a}, \mathsf{KD})$, which is thus a PCD, as shown in Figure 9b. It is easy to see that there does not exist an Aristotelian isomorphism $f\colon (\mathcal{F}_{2a}, \mathsf{K}) \to (\mathcal{F}_{2a}, \mathsf{KD})$. After all, if such an isomorphism f would exist, then the fact that the K-contingent formulas $\Diamond p$ and $\Box\bot$ are K-contrary would entail that $f(\Diamond p)$ and $f(\Box\bot)$ are KD-contrary as well, but $(\mathcal{F}_{2a}, \mathsf{KD})$ does not contain any pairs of contraries. The PCD for $(\mathcal{F}_{2a}, \mathsf{KD})$ is thus not Aristotelian isomorphic to the classical square for $(\mathcal{F}_{2a}, \mathsf{K})$. The partition that is induced by $\mathcal{F}_{2a}$ in KD looks as follows:

$$\Pi_{\mathsf{KD}}(\mathcal{F}_{2a}) = \{\Diamond p, \Box\neg p\}.$$

In comparison with $\Pi_{\mathsf{K}}(\mathcal{F}_{2a})$, we find that the second anchor formula has been simplified and the third one, that is, $\Box\bot$, has been dropped (since it is K-consistent but KD-inconsistent). The partition $\Pi_{\mathsf{KD}}(\mathcal{F}_{2a})$ thus yields bitstrings of length 2, which are obtained by systematically deleting the third bit from the $\Pi_{\mathsf{K}}(\mathcal{F}_{2a})$-bitstrings; cf. Figure 9b.

(**a**) Classical square for $\mathcal{F}_{2a}$ in K. (**b**) PCD for $\mathcal{F}_{2a}$ in KD.

Figure 9. Aristotelian diagrams for $\mathcal{F}_{2a} = \{\Diamond p, \Diamond\top, \Box\bot, \Box\neg p\}$ in two normal modal logics.

To conclude this subsection, note that the Aristotelian diagrams for $(\mathcal{F}_{1a}, \mathsf{KDF})$ and $(\mathcal{F}_{2a}, \mathsf{KD})$ are both PCDs—compare Figures 5b and 9b—, and thus belong to the same Aristotelian family. After all, it is easy to check that the function which maps $\Box p$ and $\Diamond p$ onto $\Diamond p$, and $\Box\neg p$ and $\Diamond\neg p$ onto $\Box\neg p$, is an Aristotelian isomorphism from $(\mathcal{F}_{1a}, \mathsf{KDF})$ to $(\mathcal{F}_{2a}, \mathsf{KD})$. However, these two PCDs arise in two completely different ways. On the one hand, the Aristotelian diagram for $(\mathcal{F}_{1a}, \mathsf{KDF})$ contains four KDF-contingent formulas (which are two by two KDF-equivalent), and can thus be viewed as the result of 'collapsing' the degenerate square for $(\mathcal{F}_{1a}, \mathsf{K})$ to a single, horizontal PCD. On the other hand, the Aristotelian diagram for $(\mathcal{F}_{2a}, \mathsf{KD})$ contains only two KD-contingent formulas (and no KD-equivalences), and can thus be viewed as the result of 'deleting' half of the classical square for $(\mathcal{F}_{2a}, \mathsf{K})$ to obtain a single, diagonal PCD.

To illustrate this point more vividly, Figure 10 shows two diagrams that 'approximate' these two PCDs (together with their $\Pi_{\mathsf{KDF}}(\mathcal{F}_{1a})$- and $\Pi_{\mathsf{KD}}(\mathcal{F}_{2a})$-bitstrings). These are not valid Aristotelian diagrams, since they do not satisfy the requirements from Definition 8: Figure 10a puts logically equivalent formulas on distinct vertices, while Figure 10b contains non-contingent formulas. The strong differences between these two diagrams clearly illustrate how the PCDs for $(\mathcal{F}_{1a}, \mathsf{KDF})$ and $(\mathcal{F}_{2a}, \mathsf{KD})$ come about in two completely different ways. Finally, note that these two diagrams also illustrate the reasons for imposing these two requirements in the first place: the presence of equivalent formulas on distinct vertices duplicates the contradiction relations, thus making Figure 10a quite redundant, while the presence of non-contingent formulas leads to four additional Aristotelian relations, thus making Figure 10b harder to process.

(**a**) Invalid diagram for $(\mathcal{F}_{1a}, \mathsf{KDF})$. (**b**) Invalid diagram for $(\mathcal{F}_{2a}, \mathsf{KD})$.

Figure 10. Invalid (!) diagrams that approximate the PCDs for $(\mathcal{F}_{1a}, \mathsf{KDF})$ and $(\mathcal{F}_{2a}, \mathsf{KD})$.

5.3. Examples from Non-Normal Modal Logic

Consider the following fragment:

$$\mathcal{F}_{2b} := \{\Box p, \Box\top, \Diamond\bot, \Diamond\neg p\}.$$

For any (non-absurd) normal modal logic $\mathsf{K}\alpha$, we find that $\Box p$ and $\Diamond\neg p$ are $\mathsf{K}\alpha$-contingent and $\mathsf{K}\alpha$-contradictory to each other, while $\Box\top$ and $\Diamond\bot$ are not $\mathsf{K}\alpha$-contingent. The Aristotelian diagram for $(\mathcal{F}_{2b}, \mathsf{K}\alpha)$ is thus a PCD, and the induced partition is

$\Pi_{\mathsf{K}\alpha}(\mathcal{F}_{2b}) = \{\Box p, \Diamond\neg p\}$. Within the large class of non-absurd *normal* modal logics, the fragment $\mathcal{F}_{2b}$ is thus not at all logic-sensitive. However, this changes drastically as soon as we move to *non-normal* modal logics.

With respect to EM, we find that all formulas of $\mathcal{F}_{2b}$ are EM-contingent, and constitute a classical square of opposition. For example, there is an EM-subalternation from $\Box p$ to $\Box\top$, since $\models_{\mathsf{EM}} \Box p \to \Box\top$ and $\not\models_{\mathsf{EM}} \Box\top \to \Box p$. Furthermore, the partition that is induced by $\mathcal{F}_{2b}$ in EM looks as follows:

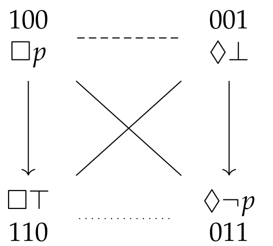
$$\Pi_{\mathsf{EM}}(\mathcal{F}_{2b}) = \{\Box p, \Box\top \wedge \Diamond\neg p, \Diamond\bot\}.$$

This partition yields bitstrings of length 3. The classical square for $(\mathcal{F}_{2b}, \mathsf{EM})$ and the corresponding $\Pi_{\mathsf{EM}}(\mathcal{F}_{2b})$-bitstrings are shown in Figure 11a.

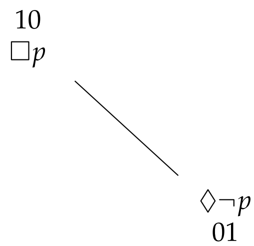
We now turn to EMN. The formulas $\Box p$ and $\Diamond\neg p$ are still EMN-contingent. By contrast, $\Box\top$ and $\Diamond\bot$ go from being EM-contingent to not being EMN-contingent, since $\models_{\mathsf{EMN}} \Box\top$ and $\models_{\mathsf{EMN}} \neg\Diamond\bot$. Consequently, these last two formulas are not included in the Aristotelian diagram for $(\mathcal{F}_{2b}, \mathsf{EMN})$, which is thus a PCD, as shown in Figure 11b. It is again easy to see that there does not exist an Aristotelian isomorphism $f\colon (\mathcal{F}_{2b}, \mathsf{EM}) \to (\mathcal{F}_{2b}, \mathsf{EMN})$. After all, if such an isomorphism f would exist, then the fact that the EM-contingent formulas $\Box p$ and $\Diamond\bot$ are EM-contrary would entail that $f(\Box p)$ and $f(\Diamond\bot)$ are EMN-contrary as well, but $(\mathcal{F}_{2b}, \mathsf{EMN})$ does not contain any pairs of contraries. The PCD for $(\mathcal{F}_{2b}, \mathsf{EMN})$ is thus not Aristotelian isomorphic to the classical square for $(\mathcal{F}_{2b}, \mathsf{EM})$. Another easy calculation yields the following partition:

$$\Pi_{\mathsf{EMN}}(\mathcal{F}_{2b}) = \{\Box p, \Diamond\neg p\}.$$

In comparison with $\Pi_{\mathsf{EM}}(\mathcal{F}_{2b})$, we find that the second anchor formula has been simplified and the third one, that is, $\Diamond\bot$, has been dropped (since it is EM-consistent but EMN-inconsistent). The partition $\Pi_{\mathsf{EMN}}(\mathcal{F}_{2b})$ thus yields bitstrings of length 2, which are obtained by deleting the third bit from the $\Pi_{\mathsf{EM}}(\mathcal{F}_{2b})$-bitstrings; cf. Figure 11b.



(**a**) Classical square for $\mathcal{F}_{2b}$ in EM.



(**b**) PCD for $\mathcal{F}_{2b}$ in EMN.

Figure 11. Aristotelian diagrams for $\mathcal{F}_{2b} = \{\Box p, \Box\top, \Diamond\bot, \Diamond\neg p\}$ in two non-normal modal logics.

5.4. Theory and Further Examples

Just as in Section 4.4, we now take a more theoretical perspective, and show that logic-sensitivity with respect to contingency (as studied in this section) can also be systematically linked to logic-sensitivity with respect to Aristotelian families (as studied in Section 3). In particular, Theorem 2 shows how the logical contingency or non-contingency of certain formulas is related to the Aristotelian families of classical and degenerate squares of oppositions.

Theorem 2. *Consider the function $f_2\colon \wp(\mathcal{L}) \to \wp(\mathcal{L})$, which maps $\mathcal{F} = \{\alpha, \beta, \neg\alpha, \neg\beta\}$ onto*

$$f_2(\mathcal{F}) := \{\alpha \wedge \neg\beta, \alpha, \neg\alpha, \neg\alpha \vee \beta\}.$$

For any logical system S *as in Definition 7, the following hold:*

1. *If the Aristotelian diagram for* $(\mathcal{F}, \mathsf{S})$ *is a degenerate square, then the Aristotelian diagram for* $(f_2(\mathcal{F}), \mathsf{S})$ *is a classical square (with an* S*-subalternation from* $\alpha \wedge \neg\beta$ *to* α*);*
2. *If the Aristotelian diagram for* $(\mathcal{F}, \mathsf{S})$ *is a classical square (with an* S*-subalternation from* α *to* β*), then* $\alpha \wedge \neg\beta$ *and* $\neg\alpha \vee \beta$ *are not* S*-contingent and the Aristotelian diagram for* $(f_2(\mathcal{F}), \mathsf{S})$ *is a PCD.*

Furthermore, in both cases, $\Pi_{\mathsf{S}}(f_2(\mathcal{F}))$ *is a coarsening of* $\Pi_{\mathsf{S}}(\mathcal{F})$*.*

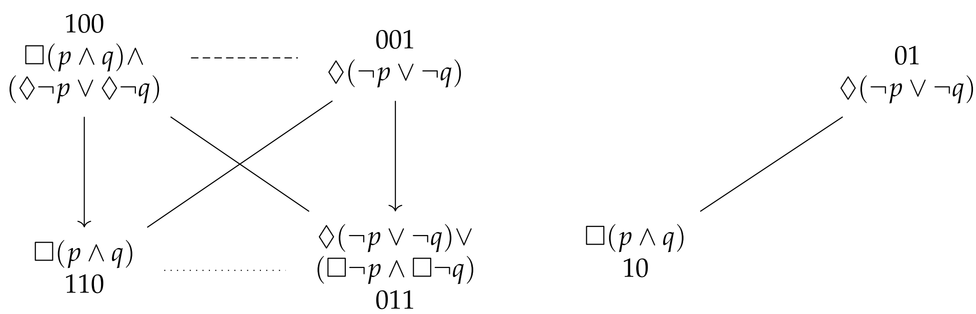
Proof. Completely analogous to the proof of Theorem 1. □

This theorem immediately yields further examples of logic-sensitivity with respect to contingency, in normal and non-normal modal logics alike. For an example from normal modal logic, we apply f_2 to the fragment $\mathcal{F}_{1a}$ to obtain $f_2(\mathcal{F}_{1a}) = \{\Box p \wedge \Box\neg p, \Box p, \Diamond\neg p, \Diamond p \vee \Diamond\neg p\}$; note that the first and last formula can be simplified (in any normal modal logic) to resp. $\Box\bot$ and $\Diamond\top$. Recall from Section 3.2 that $(\mathcal{F}_{1a}, \mathsf{K})$ yields a degenerate square, while $(\mathcal{F}_{1a}, \mathsf{KD})$ yields a classical square. By Theorem 2, it follows immediately that $(f_2(\mathcal{F}_{1a}), \mathsf{K})$ yields a classical square, which is shown in Figure 12a, while $(f_2(\mathcal{F}_{1a}), \mathsf{KD})$ yields a PCD, which is shown in Figure 12b. Furthermore, we have $\Pi_{\mathsf{K}}(f_2(\mathcal{F}_{1a})) = \{\Box\bot, \Box p \wedge \Diamond\top, \Diamond\neg p\}$ and $\Pi_{\mathsf{KD}}(f_2(\mathcal{F}_{1a})) = \{\Box p, \Diamond\neg p\}$; the corresponding bitstrings of length 3 and 2 are shown in Figure 12a,b, respectively.

(**a**) Classical square for $f_2(\mathcal{F}_{1a})$ in K. (**b**) PCD for $f_2(\mathcal{F}_{1a})$ in KD.

Figure 12. Aristotelian diagrams for $f_2(\mathcal{F}_{1a}) = \{\Box\bot, \Box p, \Diamond\neg p, \Diamond\top\}$ in two normal modal logics.

For an example from non-normal modal logic, note that applying f_2 to $\mathcal{F}_{1b}$ yields $f_2(\mathcal{F}_{1b}) = \{\Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q), \Box(p \wedge q), \Diamond(\neg p \vee \neg q), \Diamond(\neg p \vee \neg q) \vee (\Box\neg p \wedge \Box\neg q)\}$; we already saw in Section 3.3 that $(\mathcal{F}_{1b}, \mathsf{E})$ yields a degenerate square, while $(\mathcal{F}_{1b}, \mathsf{EM})$ yields a classical square. Theorem 2 now entails that $(f_2(\mathcal{F}_{1b}), \mathsf{E})$ yields a classical square, which is shown in Figure 13a, while $(f_2(\mathcal{F}_{1b}), \mathsf{EM})$ yields a PCD, which is shown in Figure 13b. Finally, we have $\Pi_{\mathsf{E}}(f_2(\mathcal{F}_{1b})) = \{\Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q), \Box(p \wedge q) \wedge \Box\neg p \wedge \Box\neg q, \Diamond(\neg p \vee \neg q)\}$ and $\Pi_{\mathsf{EM}}(f_2(\mathcal{F}_{1b})) = \{\Box(p \wedge q), \Diamond(\neg p \vee \neg q)\}$; the corresponding bitstrings of length 3 and 2 are again shown in Figure 13a,b, respectively.



(**a**) Classical square for $f_2(\mathcal{F}_{1b})$ in E. (**b**) PCD for $f_2(\mathcal{F}_{1b})$ in EM.

Figure 13. Aristotelian diagrams for $f_2(\mathcal{F}_{1b}) = \{\Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q), \Box(p \wedge q), \Diamond(\neg p \vee \neg q), \Diamond(\neg p \vee \neg q) \vee (\Box\neg p \wedge \Box\neg q)\}$ in two non-normal modal logics.

6. Logic-Sensitivity and Boolean Subfamilies

6.1. Introduction

In this section, we discuss one final type of logic-sensitivity, viz., with respect to Boolean subfamilies. This is more subtle than the previous three types; as far as we know, no concrete examples of this type of logic-sensitivity have previously been discussed in the literature. As explained in Sections 4.1 and 5.1, the equivalence-based and contingency-based types of logic-sensitivity can be viewed as subtypes of logic-sensitivity with respect to Aristotelian families, and in that sense, they are not genuinely new kinds of behavior. By contrast, the type of logic-sensitivity we will discuss in this section is not just another subtype of logic-sensitivity with respect to Aristotelian families, and thus does represent a fundamentally new phenomenon.

Roughly speaking, the idea is that Aristotelian diagrams for one and the same fragment with respect to two different logical systems might be completely identical, *except* for their Boolean properties. Put more precisely: for some fragments $\mathcal{F}$ and logical systems S_1 and S_2, (i) there exists an Aristotelian isomorphism between $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$, and this Aristotelian isomorphism can even be taken to be $id_{\mathcal{F}}$, (ii) for all $\varphi, \psi \in \mathcal{F}$ we have $\varphi \equiv_{\mathsf{S}_1} \psi$ iff $\varphi \equiv_{\mathsf{S}_2} \psi$, (iii) for all $\varphi \in \mathcal{F}$ we have that φ is S_1-contingent iff φ is S_2-contingent, but nevertheless, (iv) there does not exist a Boolean isomorphism between $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$. Condition (i) entails that we are not dealing with logic-sensitivity with respect to Aristotelian families (as discussed in Section 3); condition (ii) entails that we are not dealing with logic-sensitivity with respect to logical equivalence (as discussed in Section 4); finally, condition (iii) entails that we are not dealing with logic-sensitivity with respect to logical equivalence (as discussed in Section 5). Nevertheless, condition (iv) shows that there are logically relevant differences between the Aristotelian diagrams for $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$, viz., in terms of their Boolean properties. Using more classification-oriented terminology, the diagrams for $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$ belong to one and the same Aristotelian family, but they belong to different Boolean subfamilies of this family.

6.2. Theory and Examples

Rather than first discussing some independent examples, we immediately take a more theoretical perspective and show how logic-sensitivity with respect to Boolean subfamilies can once again be systematically linked to logic-sensitivity with respect to Aristotelian families (as studied in Section 3). In particular, Theorem 3 shows how the two Boolean subfamilies (strong/weak) of JSB hexagons are related to the two Aristotelian families of squares (classical/degenerate). Although the proof is analogous to, and thus not harder than, those of Theorems 1 and 2, it is spelt out in detail precisely to emphasize the high degree of similarity: logic-sensitivity with respect to Boolean subfamilies is conceptually speaking quite subtle, but on a mathematical level it is on par with the other types of logic-sensitivity.

Theorem 3. *Consider the function* $f_3 \colon \wp(\mathcal{L}) \to \wp(\mathcal{L})$*, which maps* $\mathcal{F} = \{\alpha, \beta, \neg\alpha, \neg\beta\}$ *onto*

$$f_3(\mathcal{F}) := \{\alpha \wedge \beta, \neg\alpha \wedge \beta, \neg\alpha \wedge \neg\beta, \alpha \vee \beta, \alpha \vee \neg\beta, \neg\alpha \vee \neg\beta\}.$$

For any logical system S *as in Definition 7, the following hold:*

1. *If the Aristotelian diagram for* $(\mathcal{F}, \mathsf{S})$ *is a degenerate square, then the Aristotelian diagram for* $(f_3(\mathcal{F}), \mathsf{S})$ *is a weak JSB hexagon (with pairwise* S*-contrarieties between* $\alpha \wedge \beta$*,*$\neg\alpha \wedge \beta$ *and* $\neg\alpha \wedge \neg\beta$*);*
2. *If the Aristotelian diagram for* $(\mathcal{F}, \mathsf{S})$ *is a classical square (with an* S*-subalternation from* α *to* β*), then the Aristotelian diagram for* $(f_3(\mathcal{F}), \mathsf{S})$ *is a strong JSB hexagon (with the same pairwise* S*-contrarieties).*

Furthermore, in both cases, $\Pi_{\mathsf{S}}(f_3(\mathcal{F}))$ *is identical to* $\Pi_{\mathsf{S}}(\mathcal{F})$*.*

Proof. 1. If the Aristotelian diagram for $(\mathcal{F}, \mathsf{S})$ is a degenerate square, then $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha \wedge \beta, \alpha \wedge \neg\beta, \neg\alpha \wedge \beta, \neg\alpha \wedge \neg\beta\}$. The bitstring semantics $\beta_{\mathsf{S}}^{\mathcal{F}}$ corresponding to this partition maps $\alpha \wedge \beta$ to 1000, $\neg\alpha \wedge \beta$ to 0010, $\neg\alpha \wedge \neg\beta$ to 0001, $\alpha \vee \beta$ to 1110, $\alpha \vee \neg\beta$ to 1101 and $\neg\alpha \vee \neg\beta$ to 0111. These six bitstrings constitute a JSB hexagon (with pairwise contrarieties between 1000, 0010 and 0001). Furthermore, this JSB hexagon is a weak one, since $1000 \vee 0010 \vee 0001 \neq 1111$. Since $\beta_{\mathsf{S}}^{\mathcal{F}}$ is a Boolean (and thus also an Aristotelian) isomorphism, it follows that the six $f_3(\mathcal{F})$-formulas also constitute a weak JSB hexagon (with the required contrarieties) in S. Finally, an easy computation yields $\Pi_{\mathsf{S}}(f_3(\mathcal{F})) = \{\alpha \wedge \beta, \alpha \wedge \neg\beta, \neg\alpha \wedge \beta, \neg\alpha \wedge \neg\beta\} = \Pi_{\mathsf{S}}(\mathcal{F})$.

2. If the Aristotelian diagram for $(\mathcal{F}, \mathsf{S})$ is a classical square (with an S-subalternation from α to β), then $\Pi_{\mathsf{S}}(\mathcal{F}) = \{\alpha, \neg\alpha \wedge \beta, \neg\beta\}$. The bitstring semantics $\beta_{\mathsf{S}}^{\mathcal{F}}$ corresponding to this partition maps $\alpha \wedge \beta$ to 100, $\neg\alpha \wedge \beta$ to 010, $\neg\alpha \wedge \neg\beta$ to 001, $\alpha \vee \beta$ to 110, $\alpha \vee \neg\beta$ to 101 and $\neg\alpha \vee \neg\beta$ to 011. These six bitstrings constitute a JSB hexagon (with pairwise contrarieties between 100, 010 and 001). Furthermore, this JSB hexagon is a strong one, since $100 \vee 010 \vee 001 = 111$. Since $\beta_{\mathsf{S}}^{\mathcal{F}}$ is a Boolean (and thus also an Aristotelian) isomorphism, it follows that the six $f_3(\mathcal{F})$-formulas also constitute a strong JSB hexagon (with the required contrarieties) in S. Finally, an easy computation yields $\Pi_{\mathsf{S}}(f_3(\mathcal{F})) = \{\alpha, \neg\alpha \wedge \beta, \neg\beta\} = \Pi_{\mathsf{S}}(\mathcal{F})$. □

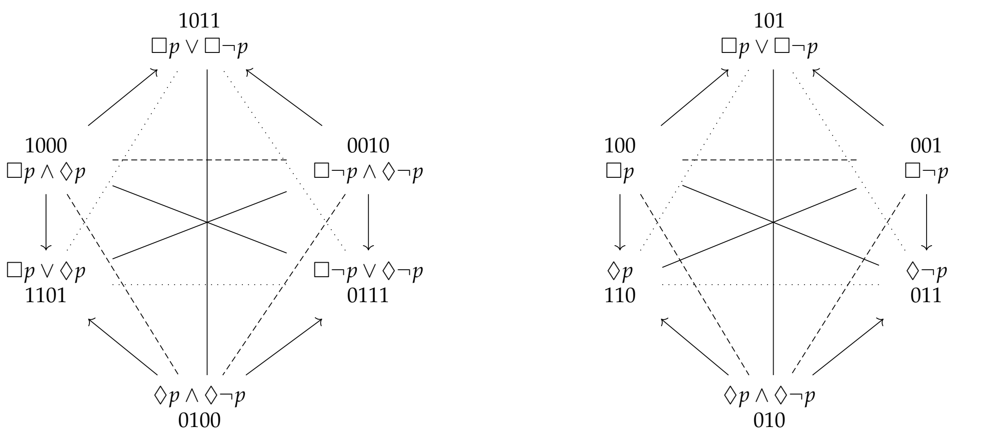
This theorem easily generates concrete examples of logic-sensitivity with respect to Boolean subfamilies. For an example from normal modal logic, we turn one last time to the fragment $\mathcal{F}_{1a} = \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$. Applying f_3 to this fragment yields

$$f_3(\mathcal{F}_{1a}) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \Box p \vee \Diamond p, \Box p \vee \Box\neg p, \Box\neg p \vee \Diamond\neg p\}.$$

Recall from Section 3.2 that $(\mathcal{F}_{1a}, \mathsf{K})$ yields a degenerate square, while $(\mathcal{F}_{1a}, \mathsf{KD})$ yields a classical square. By Theorem 3 it follows immediately that $(f_3(\mathcal{F}_{1a}), \mathsf{K})$ yields a weak JSB hexagon, which is shown in Figure 14a, while $(f_3(\mathcal{F}_{1a}), \mathsf{KD})$ yields a strong JSB hexagon, which is shown in Figure 14b. Note that in KD, four of the six formulas in $f_3(\mathcal{F}_{1a})$ can be simplified (e.g., $\Box p \wedge \Diamond p \equiv_{\mathsf{KD}} \Box p$); the JSB hexagon in Figure 14b is shown using these simplified formulas. Finally, we have

$$\Pi_{\mathsf{K}}(f_3(\mathcal{F}_{1a})) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \Box p \wedge \Box\neg p\} = \Pi_{\mathsf{K}}(\mathcal{F}_{1a}),$$
$$\Pi_{\mathsf{KD}}(f_3(\mathcal{F}_{1a})) = \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\} = \Pi_{\mathsf{KD}}(\mathcal{F}_{1a}).$$

The corresponding bitstrings of length 4 and 3 are shown in Figure 14a,b, respectively.



(**a**) Weak JSB hexagon for $f_3(\mathcal{F}_{1a})$ in K. (**b**) Strong JSB hexagon for $f_3(\mathcal{F}_{1a})$ in KD.

Figure 14. Aristotelian diagrams for $f_3(\mathcal{F}_{1a}) = \{\Box p \wedge \Diamond p, \Diamond p \wedge \Diamond\neg p, \Box\neg p \wedge \Diamond\neg p, \Box p \vee \Diamond p, \Box p \vee \Box\neg p, \Box\neg p \vee \Diamond\neg p\}$ in two normal modal logics.

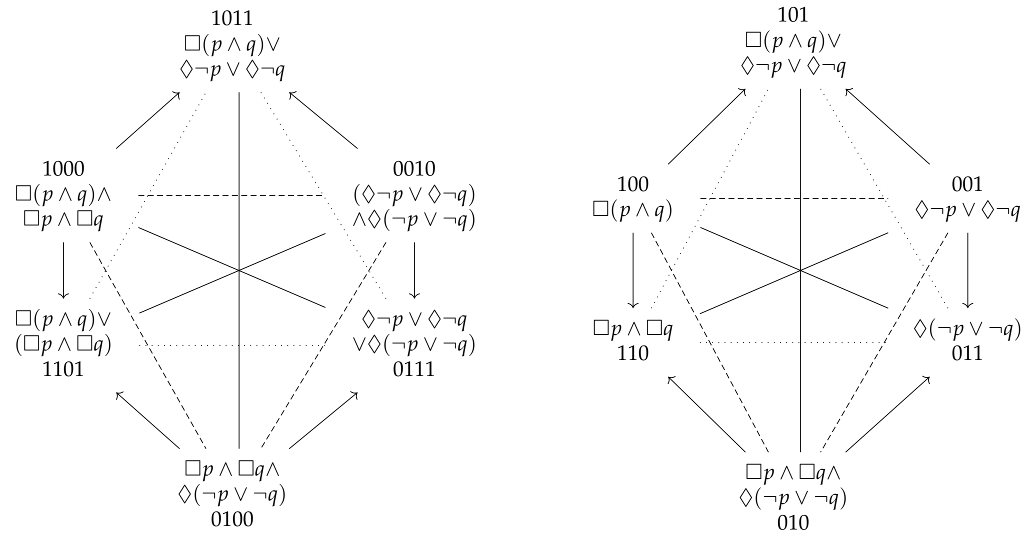
For an example from non-normal modal logic, we turn one last time to the fragment $\mathcal{F}_{1b} = \{\Box(p \wedge q), \Box p \wedge \Box q, \Diamond\neg p \vee \Diamond\neg q, \Diamond(\neg p \vee \neg q)\}$. Applying f_3 yields

$$f_3(\mathcal{F}_{1b}) = \{\Box(p \wedge q) \wedge \Box p \wedge \Box q,\ \Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q),\ (\Diamond\neg p \vee \Diamond\neg q) \wedge \Diamond(\neg p \vee \neg q),\\ \Box(p \wedge q) \vee (\Box p \wedge \Box q),\ \Box(p \wedge q) \vee \Diamond\neg p \vee \Diamond\neg q,\ \Diamond\neg p \vee \Diamond\neg q \vee \Diamond(\neg p \vee \neg q)\}.$$

Recall from Section 3.3 that $(\mathcal{F}_{1b}, \mathsf{E})$ yields a degenerate square, while $(\mathcal{F}_{1b}, \mathsf{EM})$ yields a classical square. By Theorem 3 it follows immediately that $(f_3(\mathcal{F}_{1b}), \mathsf{E})$ yields a weak JSB hexagon, which is shown in Figure 15a, while $(f_3(\mathcal{F}_{1b}), \mathsf{EM})$ yields a strong JSB hexagon, which is shown in Figure 15b. Note that in EM, four of the six formulas in $f_3(\mathcal{F}_{1b})$ can be simplified; the JSB hexagon in Figure 15b is shown using these simplified formulas. Finally, we have

$$\Pi_{\mathsf{E}}(f_3(\mathcal{F}_{1b})) = \{\Box(p \wedge q) \wedge \Box p \wedge \Box q,\ \Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q),\\ (\Diamond\neg p \vee \Diamond\neg q) \wedge \Diamond(\neg p \vee \neg q),\ \Box(p \wedge q) \wedge (\Diamond\neg p \vee \Diamond\neg q)\} = \Pi_{\mathsf{E}}(\mathcal{F}_{1b}),$$
$$\Pi_{\mathsf{EM}}(f_3(\mathcal{F}_{1b})) = \{\Box(p \wedge q), \Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q), \Diamond\neg p \vee \Diamond\neg q\} = \Pi_{\mathsf{EM}}(\mathcal{F}_{1b}).$$

The corresponding bitstrings of length 4 and 3 are shown in Figure 15a,b, respectively.



(**a**) Weak JSB hexagon for $f_3(\mathcal{F}_{1b})$ in E.

(**b**) Strong JSB hexagon for $f_3(\mathcal{F}_{1b})$ in EM.

Figure 15. Aristotelian diagrams for $f_3(\mathcal{F}_{1b}) = \{\Box(p \wedge q) \wedge \Box p \wedge \Box q,\ \Box p \wedge \Box q \wedge \Diamond(\neg p \vee \neg q),\ (\Diamond\neg p \vee \Diamond\neg q) \wedge \Diamond(\neg p \vee \neg q),\ \Box(p \wedge q) \vee (\Box p \wedge \Box q),\ \Box(p \wedge q) \vee \Diamond\neg p \vee \Diamond\neg q,\ \Diamond\neg p \vee \Diamond\neg q \vee \Diamond(\neg p \vee \neg q)\}$ in two non-normal modal logics.

7. Conclusions

In this paper, we have investigated the logic-sensitivity of Aristotelian diagrams in normal, as well as non-normal modal logics. We have distinguished between four different types of logic-sensitivity, viz. with respect to (i) Aristotelian families, (ii) logical equivalence of formulas, (iii) contingency of formulas, and (iv) Boolean subfamilies of a given Aristotelian family, and explained the differences and relations between them (cf. Sections 3.1, 4.1, 5.1 and 6.1). For the first three types, we have given examples that naturally arise in the context of normal modal logic, and shown that these examples have direct and equally natural counterparts in the context of non-normal modal logic (cf. Sections 3.2, 3.3, 4.2, 4.3, 5.2 and 5.3). Furthermore, we have also developed a more theoretical perspective that systematically links logic-sensitivity with respect to Aristotelian families to the three other types of logic-sensitivity, and we have shown that it applies to Aristotelian diagrams for normal and non-normal modal logics alike (cf. Sections 4.4, 5.4 and 6.2).

It will be interesting to investigate to what extent this theoretical perspective can be extended and generalized. Theorems 1–3 in the present paper were all based on the *binary* distinction between classical and degenerate squares. However, there also exist logic-sensitivity phenomena that are not binary in nature. For example, the Aristotelian family of Buridan octagons has *three* distinct Boolean subfamilies, which are sometimes called weak, intermediate and strong Buridan octagons; these three subfamilies can be represented by bitstrings of length 6, 5 and 4, respectively [57,65]. Can this threefold distinction still be captured in terms of the binary distinction between classical and degenerate squares? One promising idea is to search for a function $f_4 \colon \wp(\mathcal{L}) \times \wp(\mathcal{L}) \to \wp(\mathcal{L})$ such that for any logical system S as in Definition 7, we have:

- If $(\mathcal{F}_1, \mathsf{S})$ and $(\mathcal{F}_2, \mathsf{S})$ are both degenerate squares, then $(f_4(\mathcal{F}_1, \mathcal{F}_2), \mathsf{S})$ is a *weak* Buridan octagon;
- If exactly one of $(\mathcal{F}_1, \mathsf{S})$ and $(\mathcal{F}_2, \mathsf{S})$ is a degenerate square and the other is a classical square, then $(f_4(\mathcal{F}_1, \mathcal{F}_2), \mathsf{S})$ is an *intermediate* Buridan octagon;
- If $(\mathcal{F}_1, \mathsf{S})$ and $(\mathcal{F}_2, \mathsf{S})$ are both classical squares, then $(f_4(\mathcal{F}_1, \mathcal{F}_2), \mathsf{S})$ is a *strong* Buridan octagon.

The precise details have to be left for future research. However, the present paper has clearly demonstrated that however the details of f_4 may turn out, it will be applicable to normal and non-normal modal logics alike.

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