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Non-Instantaneous Impulsive Boundary Value Problems Containing Caputo Fractional Derivative of a Function with Respect to Another Function and Riemann–Stieltjes Fractional Integral Boundary Conditions

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Abstract: In the present article we study existence and uniqueness results for a new class of boundary value problems consisting by non-instantaneous impulses and Caputo fractional derivative of a function with respect to another function, supplemented with Riemann–Stieltjes fractional integral boundary conditions. The existence of a unique solution is obtained via Banach's contraction mapping principle, while an existence result is established by using Leray–Schauder nonlinear alternative. Examples illustrating the main results are also constructed.

Keywords: impulsive differential equations; fractional impulsive differential equations; instantaneous impulses; non-instantaneous impulses

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1. Introduction and Preliminaries

Fractional calculus is a generalization of classical differentiation and integration to an arbitrary real order. Fractional differential equations has gained much attention in literature because of its applications for description of hereditary properties in many fields, such as physics, mechanics, engineering, game theory, stability and optimal control. With the help of fractional calculus, the natural phenomena and mathematical models can be described more accurately. Many researchers have shown their interests in fractional differential equations, and the theory and applications of the fractional differential equations have been greatly developed. For the basic theory of fractional calculus and fractional differential equations we refer to the monographs [1–8] and references therein.

The theory of impulsive differential equations arise naturally in biology, physics, engineering, and medical fields where at certain moments they change their state rapidly. There are two type of impulses. One is called instantaneous impulses in which the duration of these changes is relatively short, and the other is called non-instantaneous impulses in which an impulsive action, starting abruptly at some points and continue to be active on a finite time interval. Some examples of such processes can be found in physics, biology, population dynamics, ecology, pharmacokinetics, and others. For results with instantaneous impulses see, e.g., the monographs [9–14], the papers [15–19], and the references cited therein. Non-instantaneous impulsive differential equation was introduced by Hernández and O'Regan in [20] pointed out that the instantaneous impulses cannot characterize some processes such as evolution processes in pharmacotherapy. Some practical problems involving non-instantaneous impulses within the area of psychology have been reviewed in [21]. For some recent works, on non-instantaneous impulsive fractional differential equations we refer the reader to [22–25] and references therein.

The scope of this investigation is to establish existence results of the new class of boundary value problems consisting by non-instantaneous impulses and Caputo fractional derivative of a function with respect to another function, supplemented with Riemann–Stieltjes fractional integral boundary conditions of the form

$$\begin{cases} {}_{s_i}D^{\alpha_i}_{g_i}x(t) = f(t,x(t)), \quad t \in [s_i, t_{i+1}), \quad i = 0,1,2,\ldots,m, \\ x(t) = \varphi_i(t) + \psi_i(t)x(t_i^-), \quad t \in [t_i, s_i), \quad i = 1,2,3,\ldots,m, \\ \beta_1 x(0) + \beta_2 x(T) = \sum_{k=0}^{m} \mu_k \int_{s_k}^{t_{k+1}} \left({}_{s_k}I^{\gamma_k}_{g_k}x\right)(u)\, dH_k(u). \end{cases} \tag{1}$$

Here ${}_{s_i}D^{\alpha_i}_{g_i}$ is the Caputo fractional derivative of order $\alpha_i \in (0,1)$, with respect to a function g_i starting at the point s_i, over the interval $[s_i, t_{i+1})$, ${}_{s_i}I^{\gamma_i}_{g_i}$ is the Riemann–Liouville fractional integral with respect to the function g_i on $[s_i, t_{i+1})$ of order $\gamma_i > 0$, $\mu_i \in \mathbb{R}$, the bounded variation function H_i of the Riemann–Stieltjes on $[s_i, t_{i+1})$ and a function $f : \bigcup[s_i, t_{i+1}) \to \mathbb{R}$, for $i = 0,1,2,\ldots,m$. (For details on Riemann–Stieltjes integral we refer to [26]). In impulsive interval $[t_i, s_i)$, φ_i, ψ_i, $i = 1,2,3,\ldots,m$, are given functions. The points

$$0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_m < s_m < t_{m+1} = T,$$

are fixed in $[0,T]$ and β_1, β_2 are known constants. Note that in problem (1), we have $x(s_i^+) = x(s_i^-)$ and if $\psi_i(t) \neq 1$, $\varphi_i(t) \neq 0$ at t_i for all $i = 1,2,3,\ldots,m$, then $x(t_i^+) \neq x(t_i^-)$.

For $\gamma > 0$, the Riemann–Liouville fractional integral of an integrable function $h : [a,b] \to \mathbb{R}$ with respect to another function $g \in C^1([a,b],\mathbb{R})$ such that $g'(t) > 0$, for all $t \in [a,b]$ is defined by [2,27,28]

$${}_aI^{\gamma}_g h(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{g'(s)h(s)}{[g(t)-g(s)]^{1-\gamma}} ds, \tag{2}$$

where Γ is the gamma function. The Riemann–Liouville type of fractional derivative of a function h, with respect to another function g on $[a,b]$ is defined as

$${}^{\star}_aD^{\alpha}_g h(t) = D^n_g\, {}_aI^{n-\alpha}_g h(t) = \frac{1}{\Gamma(n-\alpha)} D^n_g \int_a^t \frac{g'(s)h(s)}{[g(t)-g(s)]^{1+\alpha-n}} ds, \tag{3}$$

while the Caputo type is defined by

$${}_aD^{\alpha}_g h(t) = {}_aI^{n-\alpha}_g D^n_g h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g'(s)D^n_g h(s)}{[g(t)-g(s)]^{1+\alpha-n}} ds, \tag{4}$$

where $D^n_g = \underbrace{D_g \cdots D_g}_{n-times}$, $n-1 < \alpha < n$, n is a positive integer and D_g is defined by

$$D_g = \frac{1}{g'(t)} \frac{d}{dt}. \tag{5}$$

There are relations of fractional integral and derivatives of the Riemann–Liouville and Caputo types which will be used in our investigation, see [2], as

$${}_aI^{\gamma}_g \left({}^{\star}_aD^{\gamma}_g h\right)(t) = h(t) - \sum_{j=1}^{n} \frac{(g(t)-g(a))^{\gamma-j}}{\Gamma(\gamma-j+1)} D^{n-j}_g \left({}_aI^{n-\gamma}_g h\right)(a), \tag{6}$$

and

$${}_aI^{\gamma}_g \left({}_aD^{\gamma}_g h\right)(t) = h(t) - \sum_{j=0}^{n-1} \frac{(g(t)-g(a))^{j}}{j!} D^{j}_g h(a). \tag{7}$$

In addition, for $\gamma, \delta > 0$, the relation

$$ {}_aI_g^{\gamma}(g(t)-g(a))^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\gamma+\delta+1)}(g(t)-g(a))^{\gamma+\delta}, \tag{8} $$

is applied in the main results ([2]). For some recent results we refer the interesting reader to the papers [29–31].

Note that (2) is reduced to the Riemann–Liouville and Hadamard fractional integrals when $g(t) = t$ and $g(t) = \log t$, respectively, where $\log(\cdot) = \log_e(\cdot)$. The Hadamard and Hadamard–Caputo types fractional derivatives can be obtained by substituting $g(t) = \log t$ in (3) and (4), respectively. Also the Riemann–Liouville and Caputo fractional derivatives are presented by replacing $g(t) = t$ in (3) and (4), respectively. Therefore, the problem (1) generates many types and also mixed types of impulsive fractional differential equations with boundary conditions. There are some papers that have studied either Hadamard or Caputo fractional derivatives containing in noninstantaneous impulsive equations, see [32–34].

The significance of this studying is to mixed different calculus within the system of non-instantaneous impulsive differential equations. For example if putting $m = 1$, $t_1 = 1$, $s_1 = 2$, $t_2 = 3$, $\alpha_0 = \alpha_1 = 1/2$, $g_0(t) = t$ and $g_1(t) = \log_e t$ in the first two equations of (1), then we obtain

$$ \begin{cases} \left(\dfrac{d}{dt}\right)^{\frac{1}{2}} x = f(t, x(t)), \quad t \in [0,1), \\ x(t) = \varphi(t) + \psi(t)x(1^-), \quad t \in [1,2), \\ \left(t\dfrac{d}{dt}\right)^{\frac{1}{2}} x = f(t, x(t)), \quad t \in [2,3), \end{cases} $$

which is a special case of mixed Riemann–Liouville and Hadamard fractional impulsive system. In addition, if $H_k(t) = g_k(t)$, for all $t \in [s_i, t_{i+1})$, $k = 0, 1, 2, \ldots, m$, then the nonlocal condition in (1), is reduced to

$$ \beta_1 x(0) + \beta_2 x(T) = \sum_{k=0}^{m} \mu_k \left({}_{s_k}I_{g_k}^{\gamma_k+1} x \right)(t_{k+1}). $$

If $\varphi_i(t) = 0$, $\psi_i(t) = 1$ and $s_i \to t_i$, $i = 1, 2, 3, \ldots, m$, then (1) is reduced to a non impulsive fractional boundary value problem.

In fact, to the best of the authors knowledge, this is the first paper investigating Riemann–Stieltjes integration acting on fractional integral boundary conditions. Existence and uniqueness results are established for the the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem (1) by using classical fixed point theorems. We make use of Banach's contraction mapping principle to obtain the uniqueness result, while the Leray–Schauder nonlinear alternative is applied to obtain the existence result. The main results are presented in Section 3. In Section 2 we prove an auxiliary result concerning a linear variant of the problem (1) which is of great importance in the proof of main results. Illustrative examples are also presented.

2. An Auxiliary Result

Let us set some constants which will be used in our proofs.

$$ \Lambda_k = \frac{1}{\Gamma(\gamma_k+1)} \int_{s_k}^{t_{k+1}} (g_k(u) - g_k(s_k))^{\gamma_k} dH_k(u), \quad k = 1, 2, 3, \ldots, m, \tag{9} $$

$$ \Lambda^*(i) = \sum_{j=1}^{i} \left(\prod_{j}^{i-1} \psi_{j+1}(s_{j+1}) \right) \varphi_j(s_j), \qquad i = 1, 2, 3, \ldots, m, \tag{10} $$

$$ \Omega = \beta_1 + \beta_2 \left(\prod_{j=1}^{m} \psi_j(s_j) \right) - \sum_{k=0}^{m} \mu_k \left(\prod_{j=1}^{k} \psi_j(s_j) \right) \Lambda_k. \tag{11} $$

Lemma 1. *Let $\Omega \neq 0$ and $h \in C([0,T],\mathbb{R})$. Then the integral equation equivalent to problem (1) can be written as*

$$\begin{aligned} x(t) &= \frac{1}{\Omega}\left(\prod_{j=1}^{i}\psi_j(s_j)\right)\Bigg\{\sum_{k=0}^{m}\mu_k\Lambda^*(k)\Lambda_k - \beta_2\Lambda^*(m) \\ &+ \sum_{k=0}^{m}\mu_k\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Lambda_k \\ &+ \sum_{k=0}^{m}\mu_k\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}f_x(u)\,dH_k(u) \\ &- \beta_2\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}\psi_j(s_j)\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Bigg\} \\ &+ \Lambda^*(i) + \sum_{j=1}^{i}\left[\left(\prod_{j}^{i}\psi_j(s_j)\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right] + {}_{s_i}I_{g_i}^{\alpha_i}f_x(t), \end{aligned} \tag{12}$$

for $t \in [s_i, t_{i+1})$, $i = 0,1,2,\ldots,m$, and

$$\begin{aligned} x(t) &= \varphi_i(t) + \psi_i(t)\Bigg[\frac{1}{\Omega}\left(\prod_{j=1}^{i-1}\psi_j(s_j)\right)\Bigg\{\sum_{k=0}^{m}\mu_k\Lambda^*(i)\Lambda_k \\ &+ \sum_{k=0}^{m}\mu_k\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Lambda_k \\ &+ \sum_{k=0}^{m}\mu_k\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}f_x(u)\,dH_k(u) - \beta_2\Lambda^*(m) \\ &- \beta_2\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}\psi_j(s_j)\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Bigg\} \\ &+ \Lambda^*(i-1) + \sum_{j=1}^{i}\left(\prod_{j}^{i-1}\psi_j(s_j)\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\Bigg], \end{aligned} \tag{13}$$

for $t \in [t_i, s_i)$, $i = 1,2,3,\ldots,m$, where $f_x(t) = f(t, x(t))$.

Proof. For $t \in (s_0, t_1]$, taking the fractional integral with respect to a function $g_0(t)$ of order $\alpha_0 > 0$, from s_0 to t in the first equation of (1) and setting $x(0) = A$, we have

$$x(t) = A + {}_{s_0}I_{g_0}^{\alpha_0}f_x(t). \tag{14}$$

In particular, we get for $t = t_1^-$, that $x(t_1^-) = A + {}_{s_0}I_{g_0}^{\alpha_0}f_x(t_1^-)$.

In the second interval $[t_1, s_1)$, we have from the second equation of (1) as

$$\begin{aligned} x(t) &= \varphi_1(t) + \psi_1(t)x(t_1^-) \\ &= \varphi_1(t) + A\psi_1(t) + \psi_1(t)_{s_0}I_{g_0}^{\alpha_0}f_x(t_1^-), \end{aligned} \tag{15}$$

and also $x(s_1) = \varphi_1(s_1) + A\psi_1(s_1) + \psi_1(s_1)_{s_0}I_{g_0}^{\alpha_0}f_x(t_1^-)$.

In the third interval $[s_1, t_2)$, again taking the Riemann–Liouville fractional integral with respect to a function $g_1(t)$ of order α_1, we obtain

$$\begin{aligned} x(t) &= x(s_1) + {}_{s_1}I_{g_1}^{\alpha_1}f_x(t) \\ &= \varphi_1(s_1) + A\psi_1(s_1) + \psi_1(s_1)_{s_0}I_{g_0}^{\alpha_0}f_x(t_1^-) + {}_{s_1}I_{g_1}^{\alpha_1}f_x(t), \end{aligned}$$

which has particular case as $x(t_2^-) = \varphi_1(s_1) + A\psi_1(s_1) + \psi_1(s_1)_{s_0}I_{g_0}^{\alpha_0}f_x(t_1^-) + {}_{s_1}I_{g_1}^{\alpha_1}f_x(t_2^-)$.

In the fourth interval $[t_2, s_2)$, it follows that

$$x(t) = \varphi_2(t) + \psi_2(t)\left[\varphi_1(s_1) + A\psi_1(s_1) + \psi_1(s_1)\,{}_{s_0}I^{\alpha_0}_{g_0}f_x(t_1^-) + {}_{s_1}I^{\alpha_1}_{g_1}f_x(t_2^-)\right].$$

By the previous procedure we can find that

$$x(t) = \begin{cases} A\left(\prod_{j=1}^{i}\psi_j(s_j)\right) + \sum_{j=1}^{i}\left(\prod_{j}^{i-1}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j) \\ + \sum_{j=1}^{i}\left[\left(\prod_{j}^{i}\psi_j(s_j)\right){}_{s_{j-1}}I^{\alpha_{j-1}}_{g_{j-1}}f_x(t_j^-)\right] + {}_{s_i}I^{\alpha_i}_{g_i}f_x(t), \\ \qquad t \in [s_i, t_{i+1}),\ i = 0,1,2,\ldots,m, \\ \varphi_i(t) + \psi_i(t)\left[A\prod_{j=1}^{i-1}\psi_j(s_j) + \sum_{j=1}^{i-1}\left(\prod_{j}^{i-2}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j)\right. \\ \left. + \sum_{j=1}^{i}\left(\prod_{j}^{i-1}\psi_j(s_j)\right){}_{s_{j-1}}I^{\alpha_{j-1}}_{g_{j-1}}f_x(t_j^-)\right], \quad t \in [t_i, s_i),\ i = 1,2,3,\ldots,m. \end{cases} \tag{16}$$

By using the mathematical induction, we will claim that the formula (16) holds. Putting $i = 0$ and $i = 1$ in the first and second parts of (16), respectively, we have results in (14) and (15). Assume that the first part of (16) is true for $i = k$, that is, for $t \in [s_k, t_{k+1})$,

$$\begin{aligned} x(t) &= A\left(\prod_{j=1}^{k}\psi_j(s_j)\right) + \sum_{j=1}^{k}\left(\prod_{j}^{k-1}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j) \\ &\quad + \sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I^{\alpha_{j-1}}_{g_{j-1}}f_x(t_j^-)\right] + {}_{s_k}I^{\alpha_k}_{g_k}f_x(t). \end{aligned}$$

Then for $t \in [t_{k+1}, s_{k+1})$, we have

$$\begin{aligned} x(t) &= \varphi_{k+1}(t) + \psi_{k+1}(t)x(t_{k+1}^-) \\ &= \varphi_{k+1}(t) + \psi_{k+1}(t)\left\{A\left(\prod_{j=1}^{k}\psi_j(s_j)\right) + \sum_{j=1}^{k}\left(\prod_{j}^{k-1}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j),\right. \\ &\quad \left. + \sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I^{\alpha_{j-1}}_{g_{j-1}}f_x(t_j^-)\right] + {}_{s_k}I^{\alpha_k}_{g_k}f_x(t_{k+1})\right\} \\ &= \varphi_{k+1}(t) + \psi_{k+1}(t)\left\{A\left(\prod_{j=1}^{k}\psi_j(s_j)\right) + \sum_{j=1}^{k}\left(\prod_{j}^{k-1}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j),\right. \\ &\quad \left. + \sum_{j=1}^{k+1}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I^{\alpha_{j-1}}_{g_{j-1}}f_x(t_j^-)\right]\right\}, \end{aligned}$$

which implies that the second part of (16) holds. Similarly suppose that the second part of (16) is satisfied for $i = k$. Then for $t \in [s_k, t_{k+1})$, we obtain

$$\begin{aligned} x(t) &= x(s_k) + {}_{s_k}I^{\alpha_k}_{g_k}f_x(t) \\ &= \varphi_k(s_k) + \psi_k(s_k)\left[A\prod_{j=1}^{k-1}\psi_j(s_j) + \sum_{j=1}^{k-1}\left(\prod_{j}^{k-2}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j)\right. \\ &\quad \left. + \sum_{j=1}^{k}\left(\prod_{j}^{k-1}\psi_j(s_j)\right){}_{s_{j-1}}I^{\alpha_{j-1}}_{g_{j-1}}f_x(t_j^-)\right] + {}_{s_k}I^{\alpha_k}_{g_k}f_x(t) \end{aligned}$$

$$
\begin{aligned}
&= A\left(\prod_{j=1}^{k}\psi_j(s_j)\right)+\sum_{j=1}^{k}\left(\prod_{j}^{k-1}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j)\\
&\quad+\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]+{}_{s_k}I_{g_k}^{\alpha_k}f_x(t).
\end{aligned}
$$

Thus the first part of (16) is fulfilled. Therefore, the relation (16) holds for all $t \in [0,T]$. Now, we put $t = T$ in (16), we have

$$
\begin{aligned}
x(T) &= A\left(\prod_{j=1}^{m}\psi_j(s_j)\right)+\sum_{j=1}^{m}\left(\prod_{j}^{m-1}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j)\\
&\quad+\sum_{j=1}^{m}\left[\left(\prod_{j}^{m}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]+{}_{s_m}I_{g_m}^{\alpha_m}f_x(T)\\
&= A\left(\prod_{j=1}^{m}\psi_j(s_j)\right)+\Lambda^*(m)+\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]. \qquad (17)
\end{aligned}
$$

By taking the Riemann–Liouville fractional integral of order $\gamma_k > 0$ to (16), with respect to a function $g_k(t)$ on $[s_k, t_{k+1})$ for $k = 0, 1, 2, \ldots, m$, we obtain

$$
\begin{aligned}
{}_{s_k}I_{g_k}^{\gamma_k}x(t) &= A\frac{\big(g_k(t)-g_k(s_k)\big)^{\gamma_k}}{\Gamma(\gamma_k+1)}\left(\prod_{j=1}^{k}\psi_j(s_j)\right)\\
&\quad+\left[\sum_{j=1}^{k}\left(\prod_{j}^{k-1}\psi_{j+1}(s_{j+1})\right)\varphi_j(s_j)\right]\times\frac{\big(g_k(t)-g_k(s_k)\big)^{\gamma_k}}{\Gamma(\gamma_k+1)}\\
&\quad+\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\frac{\big(g_k(t)-g_k(s_k)\big)^{\gamma_k}}{\Gamma(\gamma_k+1)}+{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}f_x(t),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\sum_{k=0}^{m}\mu_k\int_{s_k}^{t_{k+1}}\left({}_{s_k}I_{g_k}^{\gamma_k}x\right)(u)\,dH_k(u)\\
&= A\sum_{k=0}^{m}\mu_k\left(\prod_{j=1}^{k}\psi_j(s_j)\right)\Lambda_k+\sum_{k=0}^{m}\mu_k\Lambda^*(k)\Lambda_k\\
&\quad+\sum_{k=0}^{m}\mu_k\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Lambda_k\\
&\quad+\sum_{k=0}^{m}\mu_k\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}f_x(u)\,dH_k(u). \qquad (18)
\end{aligned}
$$

The condition in (1) with (17) and (18) implies

$$
\begin{aligned}
A &= \frac{1}{\Omega}\Bigg\{\sum_{k=0}^{m}\mu_k\Lambda^*(k)\Lambda_k+\sum_{k=0}^{m}\mu_k\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Lambda_k\\
&\quad+\sum_{k=0}^{m}\mu_k\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}f_x(u)\,dH_k(u)-\beta_2\Lambda^*(m)\\
&\quad-\beta_2\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Bigg\}. \qquad (19)
\end{aligned}
$$

By substituting the constant A, (19), into (16), the obtained integral Equations (12) and (13) are presented.

Conversely, by taking the operator ${}_{s_i}D_{g_i}^{\alpha_i}$ over $[s_i, t_{i+1})$ to (12), we get ${}_{s_i}D_{g_i}^{\alpha_i}x(t) = f(t, x(t))$. Putting $t = t_i$ and replacing i by $i-1$ in (12), then (13) implies $x(t) = \varphi_i(t) + \psi_i(t)x(t_i^-)$, $t \in [t_i, s_i)$. By direct computation as substituting $t = 0$, $t = T$ and applying the Riemann–Stieltjes fractional integral of order γ_k with respect to g_k to the unknown function $x(t)$ in (12) over $[s_k, t_{k+1})$, then the condition in (1) is satisfied. Therefore the proof is completed. □

3. Existence and Uniqueness Results

Before going to prove our main results, we have to define the space of functions and the operator which are involved to problem (1). Let $J = [0, T]$ be an interval and let $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ be the spaces of piecewise continuous function defined by $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} |\ x(t)$ is continuous everywhere except for some t_i at which $x(t_i^+)$ and $x(t_i^-)$ exist for $i = 1, 2, 3, \ldots, m\}$ and $PC^1(J, \mathbb{R}) = \{x \in PC(J, \mathbb{R}) |\ x'(t)$ is continuous everywhere except for some t_i at which $x'(t_i^+)$ and $x'(t_i^-)$ exist for $i = 1, 2, 3, \ldots, m\}$. Let $E = PC(J, \mathbb{R}) \cap PC^1(J, \mathbb{R})$. Then E is the Banach space with norm $\|x\| = \sup\{|x(t)|, t \in J\}$. Now, we define the operator on E by

$$\mathcal{Q}x(t) = \begin{cases} \frac{1}{\Omega}\left(\prod_{j=1}^{i}\psi_j(s_j)\right)\Bigg\{\sum_{k=0}^{m}\mu_k\Lambda^*(k)\Lambda_k - \beta_2\Lambda^*(m) \\ +\sum_{k=0}^{m}\mu_k\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Lambda_k \\ +\sum_{k=0}^{m}\mu_k\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}f_x(u)\,dH_k(u) \\ -\beta_2\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Bigg\} \\ +\Lambda^*(i) + \sum_{j=1}^{i}\left[\left(\prod_{j}^{i}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right] + {}_{s_i}I_{g_i}^{\alpha_i}f_x(t), \\ \qquad t \in [s_i, t_{i+1}),\ i = 0, 1, 2, \ldots, m, \\ \varphi_i(t) + \psi_i(t)\left[\frac{1}{\Omega}\left(\prod_{j=1}^{i-1}\psi_j(s_j)\right)\Bigg\{\sum_{k=0}^{m}\mu_k\Lambda^*(i)\Lambda_k\right. \\ +\sum_{k=0}^{m}\mu_k\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Lambda_k \\ +\sum_{k=0}^{m}\mu_k\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}f_x(u)\,dH_k(u) - \beta_2\Lambda^*(m) \\ -\beta_2\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right]\Bigg\} \\ \left.+\Lambda^*(i-1) + \sum_{j=1}^{i}\left(\prod_{j}^{i-1}\psi_j(s_j)\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}f_x(t_j^-)\right], \\ \qquad t \in [t_i, s_i),\ i = 1, 2, 3, \ldots, m. \end{cases}$$

Next, by applying the Banach's contraction mapping principle, and Leray–Schauder's nonlinear alternative, we derive the existence and uniqueness of solutions to problem (1). Some constants are set as follows:

$$\Phi_1 = \frac{1}{|\Omega|}\left(\prod_{j=1}^{m}|\psi_j(s_j)|\right), \quad \Phi_2 = \sum_{k=0}^{m}|\mu_k||\Lambda^*(k)||\Lambda_k|,$$

$$\begin{aligned}
\Phi_3 &= \sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}|\psi_j(s_j)|\right)\left(\frac{(g_{j-1}(t_j)-g_{j-1}(s_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}\right)\right]|\Lambda_k|,\\
\Phi_4 &= \sum_{k=0}^{m}\frac{|\mu_k|}{\Gamma(\alpha_k+\gamma_k+1)}\int_{s_k}^{t_{k+1}}(g_k(u)-g_k(s_k))^{\alpha_k+\gamma_k}\,dH_k(u),\\
\Phi_5 &= \sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right)\left(\frac{(g_{j-1}(t_j)-g_{j-1}(s_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}\right)\right],\\
\Phi_6 &= \Phi_1(\Phi_3+\Phi_4)+\Phi_5(|\beta_2|\Phi_1+1).
\end{aligned} \tag{20}$$

Theorem 1. *Suppose that the nonlinear function $f\colon J\times\mathbb{R}\to\mathbb{R}$ satisfies the condition:*
(H_1) *There exists a constant $L>0$ such that for all $t\in J$ and $x,y\in\mathbb{R}$,*

$$|f(t,x)-f(t,y)|\le L|x-y|.$$

If $L\Phi_6<1$, where Φ_6 is defined by (20), then the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem (1) has a unique solution on J.

Proof. Let B_r be the subset of E defined by $B_r=\{x\in E:\|x\|\le r\}$, where a fixed constant r satisfies

$$r\ge\frac{\Phi_1\Phi_2+|\Lambda^*(m)|(|\beta_2|\Phi_1+1)+M\Phi_6}{1-L\Phi_6}. \tag{21}$$

Now we will prove that $\mathcal{Q}B_r\subset B_r$. Setting $M=\sup\{|f(t,0)|,\ t\in J|\}$, we have, from triangle inequality and (H_1), that $|f(t,x)|\le|f(t,x)-f(t,0)|+|f(t,0)|\le Lr+M$. Then we obtain

$$\begin{aligned}
|\mathcal{Q}x(t)| \le{}& \frac{1}{|\Omega|}\left(\prod_{j=1}^{i}|\psi_j(s_j)|\right)\Bigg\{\sum_{k=0}^{m}|\mu_k||\Lambda^*(k)||\Lambda_k|+|\beta_2||\Lambda^*(m)|\\
&+\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x|(t_j^-)\right]|\Lambda_k|\\
&+\sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}|f_x|(u)\,dH_k(u)\\
&+|\beta_2|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x|(t_j^-)\right]\Bigg\}\\
&+|\Lambda^*(i)|+\sum_{j=1}^{i}\left[\left(\prod_{j}^{i}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x|(t_j^-)\right]+{}_{s_i}I_{g_i}^{\alpha_i}|f_x|(t)
\end{aligned}$$

for $t\in[s_i,t_{i+1})$, $i=0,1,2,\ldots,m$, and

$$\begin{aligned}
|\mathcal{Q}x(t)| \le{}& |\varphi_i(t)|+|\psi_i(t)|\left[\frac{1}{|\Omega|}\left(\prod_{j=1}^{i-1}|\psi_j(s_j)|\right)\Bigg\{\sum_{k=0}^{m}|\mu_k||\Lambda^*(i)||\Lambda_k|\right.\\
&+\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x|(t_j^-)\right]\Lambda_k\\
&+\sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}|f_x|(u)\,dH_k(u)+|\beta_2||\Lambda^*(m)|\\
&+|\beta_2|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x|(t_j^-)\right]\Bigg\}
\end{aligned}$$

$$+|\Lambda^*(i-1)|+\sum_{j=1}^{i}\left(\prod_{j}^{i-1}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x|(t_j^-)\Bigg],$$

for $t\in[t_i,s_i)$, $i=1,2,3,\ldots,m$. Then we have

$$\begin{aligned}
\sup_{t\in J}|\mathcal{Q}x(t)| \leq\;& \frac{1}{|\Omega|}\left(\prod_{j=1}^{m}|\psi_j(s_j)|\right)\Bigg\{\sum_{k=0}^{m}|\mu_k||\Lambda^*(k)||\Lambda_k|+|\beta_2||\Lambda^*(m)| \\
&+(Lr+M)\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right]|\Lambda_k| \\
&+(Lr+M)\sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}(1)(u)\,dH_k(u) \\
&+(Lr+M)|\beta_2|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right]\Bigg\} \\
&+|\Lambda^*(m)|+(Lr+M)\sum_{j=1}^{m}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right] \\
&+(Lr+M)_{s_m}I_{g_m}^{\alpha_m}(1)(T) \\
=\;& \Phi_1\Phi_2+|\Lambda^*(m)|(|\beta_2|\Phi_1+1)+rL\{\Phi_1(\Phi_3+\Phi_4)+\Phi_5(|\beta_2|\Phi_1+1)\} \\
&+M\{\Phi_1(\Phi_3+\Phi_4)+\Phi_5(|\beta_2|\Phi_1+1)\} \\
=\;& \Phi_1\Phi_2+|\Lambda^*(m)|(|\beta_2|\Phi_1+1)+rL\Phi_6+M\Phi_6,
\end{aligned}$$

since

$$\begin{aligned}
{}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-) &= \frac{(g_{j-1}(t_j)-g_{j-1}(s_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}, \\
\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}(1)(u)\,dH_k(u) &= \int_{s_k}^{t_{k+1}}\frac{(g_k(u)-g_k(s_k))^{\alpha_k+\gamma_k}}{\Gamma(\alpha_k+\gamma_k+1)}\,dH_k(u).
\end{aligned}$$

Thus $\|\mathcal{Q}x\|\leq r$, where r satisfies (21). Therefore, we conclude that $\mathcal{Q}B_r\subset B_r$.

Next we will prove that the operator $\mathcal{Q}$ is a contraction. For any $x,y\in B_r$ we have

$$\begin{aligned}
&|\mathcal{Q}x(t)-\mathcal{Q}y(t)| \\
\leq\;& \frac{1}{|\Omega|}\left(\prod_{j=1}^{i}|\psi_j(s_j)|\right)\Bigg\{\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x-f_y|(t_j^-)\right]|\Lambda_k| \\
&+\sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}}{}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}|f_x-f_y|(u)\,dH_k(u) \\
&+|\beta_2|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x-f_y|(t_j^-)\right]\Bigg\} \\
&+\sum_{j=1}^{i}\left[\left(\prod_{j}^{i}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x-f_y|(t_j^-)\right]+{}_{s_i}I_{g_i}^{\alpha_i}|f_x-f_y|(t)
\end{aligned}$$

for $t\in[s_i,t_{i+1})$, $i=0,1,2,\ldots,m$, and

$$\begin{aligned}
&|\mathcal{Q}x(t)-\mathcal{Q}y(t)| \\
\leq\;& |\varphi_i(t)|+|\psi_i(t)|\Bigg[\frac{1}{|\Omega|}\left(\prod_{j=1}^{i-1}|\psi_j(s_j)|\right)\Bigg\{\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\Bigg[\left(\prod_{j}^{k}|\psi_j(s_j)|\right)
\end{aligned}$$

$$\times {}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x - f_y|(t_j^-)\Bigg]\Lambda_k + \sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}} {}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}|f_x - f_y|(u)\,dH_k(u)$$
$$+|\beta_2|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x - f_y|(t_j^-)\right]\Bigg\}$$
$$+\sum_{j=1}^{i}\left(\prod_{j}^{i-1}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x - f_y|(t_j^-)\Bigg]$$

for $t \in [t_i, s_i)$, $i = 1, 2, 3, \ldots, m$. Consequently

$$\begin{aligned}
&|\mathcal{Q}x(t) - \mathcal{Q}y(t)| \\
\le\ & \frac{1}{|\Omega|}\left(\prod_{j=1}^{m}|\psi_j(s_j)|\right)\Bigg\{L\|x-y\|\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right]|\Lambda_k| \\
&+L\|x-y\|\sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}} {}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}(1)(u)\,dH_k(u) \\
&+L\|x-y\||\beta_2|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right]\Bigg\} \\
&+L\|x-y\|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right] \\
=\ & L\Phi_6\|x-y\|,
\end{aligned}$$

which yields $\|\mathcal{Q}x - \mathcal{Q}y\| \le L\Phi_6\|x-y\|$. As $L\Phi_6 < 1$, $\mathcal{Q}$ is a contraction. Therefore, we deduce by Banach's contraction mapping principle, that $\mathcal{Q}$ has a fixed point which is the solution of the boundary value problem (1). The proof is completed. □

Remark 1. *If $\beta_1 \neq 0$, $\beta_2 = 0$, then the problem (1) is reduced to the initial and integral values problem. The constants Ω^*, Φ_6^* and Φ_1^*, given by*

$$\Omega^* = \beta_1 - \sum_{k=0}^{m}\mu_k\left(\prod_{j=1}^{k}\psi_j(s_j)\right)\Lambda_k,\ \ \Phi_6^* = \Phi_1^*(\Phi_3 + \Phi_4) + \Phi_5,\ \ \Phi_1^* = \frac{1}{|\Omega^*|}\left(\prod_{j=1}^{m}|\psi_j(s_j)|\right),$$

with conditions (H_1) and $L\Phi_6^* < 1$ are used to obtain the existence of a unique solution of such a problem on J.

The following theorem of Leray–Schauder's nonlinear alternative will be applied to the next result.

Theorem 2 ([35])**.** *Given E is a Banach space, and B is a closed, convex subset of E. In addition let G be an open subset of B such that $0 \in G$. Suppose that $\mathcal{Q} : \overline{G} \to B$ is a continuous, compact (that is, $\mathcal{Q}(\overline{G})$ is a relatively compact subset of B) map. Then either*

(i) $\mathcal{Q}$ has a fixed point in $\overline{G}$,
(ii) there is a $x \in \partial G$ (the boundary of G in B) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{Q}(x)$.

Theorem 3. *Suppose that $f : J \times \mathbb{R}$ is a continuous function. In addition we assume that:*

(H_2) There exist a continuous nondecreasing function $\Psi : [0,\infty) \to (0,\infty)$ and continuous function $w : J \to \mathbb{R}^+$, such that

$$|f(t,x)| \le w(t)\Psi(|x|),$$

for each $(t,x) \in J \times \mathbb{R}$;

(H_3) *There exists a constant $N > 0$ such that*

$$\frac{N}{\Phi_1\Phi_2 + |\Lambda^*(m)|(|\beta_2|\Phi_1 + 1) + \|w\|\Psi(N)\Phi_6} > 1.$$

Then the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem (1) has at least one solution on J.

Proof. Let ρ be a radius of a ball $B_\rho = \{x \in E : \|x\| \le \rho\}$. It is obvious that B_ρ is a closed, convex subset of E. Now, we will show that the operator $\mathcal{Q}$ is fulfilled all conditions of Theorem 2. Firstly the continuity of operator $\mathcal{Q}$ is proved by defining a sequence $\{x_n\}$ which is converse to x. Then

$$\begin{aligned}
&|\mathcal{Q}x_n(t) - \mathcal{Q}x(t)| \\
\le\ & \frac{1}{|\Omega|}\left(\prod_{j=1}^{i}|\psi_j(s_j)|\right)\Bigg\{\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_{x_n} - f_y|(t_j^-)\right]|\Lambda_k| \\
&+ \sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}} {}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}|f_{x_n} - f_x|(u)\,dH_k(u) \\
&+ |\beta_2|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_{x_n} - f_x|(t_j^-)\right]\Bigg\} \\
&+ \sum_{j=1}^{i}\left[\left(\prod_{j}^{i}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_x - f_y|(t_j^-)\right] + {}_{s_i}I_{g_i}^{\alpha_i}|f_{x_n} - f_x|(t) \\
\to\ & 0, \qquad \text{as} \quad n \to \infty,
\end{aligned}$$

for $t \in [s_i, t_{i+1})$, $i = 0, 1, 2, \dots, m$, and

$$\begin{aligned}
&|\mathcal{Q}x_n(t) - \mathcal{Q}x(t)| \\
\le\ & |\varphi_i(t)| + |\psi_i(t)|\Bigg[\frac{1}{|\Omega|}\left(\prod_{j=1}^{i-1}|\psi_j(s_j)|\right)\Bigg\{\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\Bigg[\left(\prod_{j}^{k}|\psi_j(s_j)|\right) \\
&\times {}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_{x_n} - f_x|(t_j^-)\Bigg]\Lambda_k + \sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}} {}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}|f_{x_n} - f_x|(u)\,dH_k(u) \\
&+ |\beta_2|\sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_{x_n} - f_x|(t_j^-)\right]\Bigg\} \\
&+ \sum_{j=1}^{i}\left(\prod_{j}^{i-1}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}|f_{x_n} - f_x|(t_j^-)\Bigg] \to 0, \qquad \text{as} \quad n \to \infty,
\end{aligned}$$

for $t \in [t_i, s_i)$, $i = 1, 2, 3, \dots, m$. Then $\mathcal{Q}$ is continuous.

Next the compactness of the operator Q will be proved. Assume that $x \in B_\rho$, then we have

$$\begin{aligned}
|\mathcal{Q}x(t)| \le\ & \frac{1}{|\Omega|}\left(\prod_{j=1}^{m}|\psi_j(s_j)|\right)\Bigg\{\sum_{k=0}^{m}|\mu_k||\Lambda^*(k)||\Lambda_k| + |\beta_2||\Lambda^*(m)| \\
&+ \|w\|\Psi(\rho)\sum_{k=0}^{m}|\mu_k|\sum_{j=1}^{k}\left[\left(\prod_{j}^{k}|\psi_j(s_j)|\right){}_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right]|\Lambda_k| \\
&+ \|w\|\Psi(\rho)\sum_{k=0}^{m}|\mu_k|\int_{s_k}^{t_{k+1}} {}_{s_k}I_{g_k}^{\alpha_k+\gamma_k}(1)(u)\,dH_k(u)
\end{aligned}$$

$$
\begin{aligned}
& + \|w\|\Psi(\rho)|\beta_2| \sum_{j=1}^{m+1}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right]\Bigg\} \\
& + |\Lambda^*(m)| + \|w\|\Psi(\rho)\sum_{j=1}^{m}\left[\left(\prod_{j}^{m}|\psi_j(s_j)|\right)_{s_{j-1}}I_{g_{j-1}}^{\alpha_{j-1}}(1)(t_j^-)\right] \\
& + \|w\|\Psi(\rho)_{s_m}I_{g_m}^{\alpha_m}(1)(T) \\
= \;& \Phi_1\Phi_2 + |\Lambda^*(m)|(|\beta_2|\Phi_1 + 1) + \|w\|\Psi(\rho)\Phi_6 \\
:= \;& \Phi_7, \qquad (22)
\end{aligned}
$$

which yields $\|\mathcal{Q}x\| \leq \Phi_7$ and then $\mathcal{Q}B_\rho$ is a uniformly bounded set. To prove equicontinuity of $\mathcal{Q}B_\rho$, we let the points $\theta_1, \theta_2 \in [0,T]$ such that $\theta_1 < \theta_2$. Then for any $x \in B_\rho$, it follows that

$$
\begin{aligned}
& |\mathcal{Q}x(\theta_2) - \mathcal{Q}x(\theta_1)| \\
= \;& \left|{}_{s_i}I_{g_i}^{\alpha_i} f_x(\theta_2) - {}_{s_i}I_{g_i}^{\alpha_i} f_x(\theta_1)\right| \\
\leq \;& \|w\|\Psi(\rho)\left|{}_{s_i}I_{g_i}^{\alpha_i}(1)(\theta_2) - {}_{s_i}I_{g_i}^{\alpha_i}(1)(\theta_1)\right| \\
= \;& \frac{\|w\|\Psi(\rho)}{\Gamma(\alpha_i+1)}\left\{2(g(\theta_2)-g(\theta_1))^{\alpha_i} + \left|g((\theta_2)-g(s_i))^{\alpha_i} - (g(\theta_1)-g(s_i))^{\alpha_i}\right|\right\} \to 0,
\end{aligned}
$$

as $\theta_1 \to \theta_2$ for $t \in [s_i, t_{i+1})$, $i = 0,1,2,\ldots,m$, and

$$
\begin{aligned}
|\mathcal{Q}x(\theta_2) - \mathcal{Q}x(\theta_1)| &= |\varphi_i(\theta_2) - \varphi_i(\theta_1)| + |\psi_i(\theta_2) - \psi_i(\theta_1)| \times \text{const.} \\
&\to 0, \quad \text{as} \quad \theta_1 \to \theta_2,
\end{aligned}
$$

for $t \in [t_i, s_i)$, $i = 1,2,3,\ldots,m$. The above two inequalities are convergent to zero independently of x. Then $\mathcal{Q}B_\rho$ is equicontinuous set. Therefore, we deduce that $\mathcal{Q}B_\rho$ is relatively compact which implies by the Arzelá–Ascoli theorem, that the operator $\mathcal{Q}$ is completely continuous.

In the last step, we will illustrate that the condition (ii) of Theorem 2 dose not hold. Let x be a solution of problem (1). Now, we consider the operator equation $x = \lambda\mathcal{Q}x$ for any fixed constant $\lambda \in (0,1)$. Consequently, from above computation getting (22), we obtain

$$
\frac{\|x\|}{\Phi_1\Phi_2 + |\Lambda^*(m)|(|\beta_2|\Phi_1 + 1) + \|w\|\Psi(\|x\|)\Phi_6} \leq 1.
$$

The hypothesis (H_3) implies that there exists a positive constants N such that $\|x\| \neq N$. Define the open subset of B_ρ by $G = \{x \in B_\rho : \|x\| < N\}$. It is easy to see that $\mathcal{Q} : \overline{G} \to E$ is continuous and completely continuous. Thus, there is no $x \in \partial G$ such that $x = \lambda\mathcal{Q}x$ for some $\lambda \in (0,1)$. Hence the condition (ii) of Theorem 2 is not true. Therefore, by the conclusion from Theorem 2 (i), the operator $\mathcal{Q}$ has a fixed point $x \in \overline{G}$ which is a solution of the problem (1) on J. This is the end of the proof. □

A special case can be obtain by setting $p(t) \equiv 1$ and $\Psi(x) = \kappa_1 x + \kappa_2$, $\kappa_1 \geq 0$, $\kappa_2 > 0$ in Theorem 3.

Corollary 1. *If*

$$
|f(t,x)| \leq \kappa_1 x + \kappa_2,
$$

and if $\kappa_1\Phi_6 < 1$, then the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem (1) has at least one solution on J.

Remark 2. *In the same way of Remark 1, if $\beta_1 \neq 0$, $\beta_2 = 0$, and conditions (H_2)-(H_3) are fulfilled with*

$$
\frac{N}{\Phi_1^*\Phi_2 + |\Lambda^*(m)| + \|w\|\Psi(N)\Phi_6^*} > 1,
$$

then the initial and integral values problem has at least one solution on J.

Example 1. *Consider the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem*

$$\begin{cases} {}_{2i}D^{\frac{4i+5}{4i+6}}_{\frac{e^t}{(e^t+4+i-t)}} x(t) = f(t,x(t)), \quad t \in [2i,2i+1), \quad i = 0,1,2,3, \\ x(t) = \dfrac{1}{2}\log_e(i+t) + \left(\dfrac{1}{i+\tan^{-1}(t)}\right)x(t_i^-), \quad t \in [2i-1,2i), \quad i = 1,2,3, \\ \dfrac{3}{11}x(0) + \dfrac{4}{13}x(7) = \dfrac{5}{17}\displaystyle\int_0^1 \left({}_0I^{\frac{1}{4}}_{(\frac{e^u}{(e^u+4-u)}} x\right)(u)\, d(u^2+u) \\ \qquad + \dfrac{6}{19}\displaystyle\int_2^3 \left({}_2I^{\frac{1}{2}}_{\frac{e^u}{(e^u+5-u)}} x\right)(u)\, d(u^2+2u) \\ \qquad + \dfrac{7}{23}\displaystyle\int_4^5 \left({}_4I^{\frac{3}{4}}_{\frac{e^u}{(e^u+6-u)}} x\right)(u)\, d(u^2+3u) \\ \qquad + \dfrac{8}{29}\displaystyle\int_6^7 \left({}_6I^{\frac{3}{2}}_{\frac{e^u}{(e^u+7-u)}} x\right)(u)\, d(u^2+4u). \end{cases} \tag{23}$$

Here $\alpha_i = (4i+5)/(4i+6)$, $g_i(t) = e^t/(e^t+4+i-t)$, for $t \in [2i,2i+1)$, $i = 0,1,2,3$, $\varphi_i(t) = (1/2)\log_e(i+t)$, $\psi_i(t) = 1/(i+\tan^{-1} t)$, $t \in [2i-1,2i)$, $i = 1,2,3$, $\beta_1 = 3/11$, $\beta_2 = 4/13$. Since $[2i,2i+1) \cup [2j-1,2j) \cup \{7\} = [0,7]$, for $i = 0,1,2,3$, $j = 1,2,3$, we put $T = 7$. Setting $\mu_0 = 5/17$, $\mu_1 = 6/19$, $\mu_2 = 7/23$, $\mu_3 = 8/29$, $H_i(t) = t^2 + it$, $i = 1,2,3,4$, $\gamma_0 = 1/4$, $\gamma_1 = 1/2$, $\gamma_2 = 3/4$, $\gamma_3 = 3/2$. Remark that $g_i'(t) > 0$ for all $t \in [0,7]$, $i = 0,1,2,3$. Then from all information, we can compute that $|\Omega| \approx 0.5181070744$, $\Phi_1 \approx 0.06251397190$, $\Phi_2 \approx 0.8574153788$, $\Phi_3 \approx 0.1639270834$, $\Phi_4 \approx 0.1706687388$, $\Phi_5 \approx 0.1889629435$, $\Phi_6 \approx 0.2135145724$ and $\Lambda^*(3) \approx 1.376938726$.

(i) Consider a nonlinear function $f : [0,7] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,x) = \frac{4}{3}e^{-t}\left(\frac{2x^2+3|x|}{1+|x|}\right) + \frac{1}{2}t + 1. \tag{24}$$

It is easy to check that the function $f(t,x)$ satisfies the Lipchitz condition with $L = 4$, as $|f(t,x) - f(t,y)| \le 4|x-y|$, for all $t \in [0,7]$ and $x,y \in \mathbb{R}$. Since $L\Phi_6 \approx 0.8540582896 < 1$, by applying the result in Theorem 1, we have that the problem (23), with f given by (24), has a unique solution on $[0,7]$.

(ii) Let now a nonlinear function f defined by

$$f(t,x) = \frac{1}{t+2}\left(\frac{x^{16}}{1+x^{14}} + \frac{2}{3}\sin^2 x + \frac{1}{3}e^{-x^2}\right). \tag{25}$$

Note that

$$|f(t,x)| \le \frac{1}{t+2}\left(x^2+1\right),$$

which satisfies (H_2) with $p(t) = 1/(t+2)$ and $\Psi(x) = x^2+1$. Accordingly, $\|p\| = 1/2$ and there exists a constant $N \in (1.984010360, 7.383031794)$ satisfying the condition (H_3) of Theorem 3. Therefore, by applying Theorem 3, we deduce that the problem (23), with f given by (25), has at least one solution on $[0,7]$.

(iii) If the term x^{16} is replaced by $|x|^{15}$ in (25) then

$$f(t,x) = \frac{1}{t+2}\left(\frac{|x|^{15}}{1+x^{14}} + \frac{2}{3}\sin^2 x + \frac{1}{3}e^{-x^2}\right). \tag{26}$$

Hence we get $|f(t,x)| \le (1/2)|x| + (1/2)$. Putting $\kappa_1 = 1/2$ and $\kappa_2 = 1/2$, it follows that $\kappa_1\Phi_6 \approx 0.1067572862 < 1$, which implies, by Corollary 1, that the problem (23) with (26) has at least one solution on $[0,7]$.

4. Conclusions

We have presented the sufficient criteria for the existence and uniqueness of solutions for a non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem. The given boundary value problem is converted into an equivalent fixed point operator equation, which is solved by applying the standard fixed point theorems. We make use of Banach's contraction mapping principle to obtain the uniqueness result, while the Leray–Schauder nonlinear alternative is applied to obtain the existence result. We have demonstrated the application of the obtained results by constructing examples.

Our problem generates many types and also mixed types of impulsive fractional boundary value problems. For example, our results are reduced to Riemann–Liouville and Hadamard impulsive fractional boundary value problems when $g(t) = t$ and $g(t) = \log t$, respectively. Our results are new in the given configuration and contributes to the theory of fractional boundary value problems.

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References

1. Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer: New York, NY, USA, 2010.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of the Fractional Differential Equations*; Elseiver: Amsterdam, The Netherlands, 2006.
3. Lakshmikantham, V.; Leela, S.; Devi, J.V. *Theory of Fractional Dynamic Systems*; Cambridge Scientific Publishers: Cambridge, UK, 2009.
4. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Differential Equations*; John Wiley: New York, NY, USA, 1993.
5. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Academic Press: New York, NY, USA, 1998.
6. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer International Publishing: Cham, Switzerland, 2017.
7. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives Theory and Applications*; Gordon and Breach: New York, NY, USA, 1993.
8. Zhou, Y.; Wang, J.R.; Zhang, L. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.
9. Samoilenko, A.M.; Perestyuk, N.A. *Impulsive Differential Equations*; World Scientific Publishing: Singapore, 1995.
10. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific: Singapore, 1989.
11. Benchohra, M.; Henderson, J.; Ntouyas, S.K. *Impulsive Differential Equations and Inclusions*; Hindawi Publishing Corporation: New York, NY, USA, 2006.
12. Kostic, M. *Abstract Voltera Integro-Differential Equations*; ORC Press: Boca Raton, FL, USA, 2015.
13. Song, X.; Gno, H.; Shi, X. *Theory and Applications of Impulsive Differential Equations*; Science Press: Beijing, China, 2011.
14. Stamov, G.T. *Almost Periodic Solutions of Impulsive Differential Equations*; Springer: Berlin, Germany, 2012.
15. Yukunthorn, W.; Ntouyas, S.K.; Tariboon, J. Impulsive multiorders Riemann-Liouville fractional differential equations. *Discret. Dyn. Nat. Soc.* **2015**, *2015*, 603893. [CrossRef]
16. Thaiprayoon, C.; Tariboon, J.; Ntouyas, S.K. Impulsive fractional boundary-value problems with fractional integral jump conditions. *Boun. Value Probl.* **2014**, *2014*, 16. [CrossRef]
17. Yukunthorn, W.; Ahmad, B.; Ntouyas, S.K.; Tariboon, J. On Caputo- Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions. *Nonlinear Anal. Hybrid Syst.* **2016**, *19*, 77–92. [CrossRef]
18. Heidarkhani, S.; Caristi, G.; Salari, A. Nontrivial solutions for impulsive elastic beam equations of Kirchhoff-type. *J. Nonlinear Funct. Anal.* **2020**, *2020*, 4.

19. Kamenski, M.; Petrosyan, G.; Wen, C.F. An existence result for a periodic boundary value problem of fractional semilinear differential equations in Banach spaces. *J. Nonlinear Var. Anal.* **2021**, *5*, 155–177.
20. Hernández, E.; O'Regan, D. On a new class of abstract impulsive differential equation. *Proc. Am. Math. Soc.* **2013**, *141*, 1641–1649. [CrossRef]
21. Terzieva, R. Some phenomena for non-instantaneous impulsive differential equations. *Int. J. Pure Appl. Math.* **2018**, *119*, 483–490.
22. Li, P.L.; Xu, C.J. Mild solution of fractional order differential equations with not instantaneous impulses. *Open Math.* **2015**, *23*, 436–443. [CrossRef]
23. Wang, J.; Li, Z. Periodic BVP for integer/fractional order nonlinear differential equations with non-instantaneous impulses. *J. Appl. Math. Comput.* **2014**, *46*, 321–334. [CrossRef]
24. Agarwal, R.; Hristova, S.; O'Regan, D. non-instantaneous impulses in Caputo fractional differential equations and practical stability via Lyapunov functions. *J. Frankl. Inst.* **2017**, *354*, 3097–3119. [CrossRef]
25. Agarwal, R.; O'Regan, D.; Hristova, S. Monotone iterative technique for the initial value problem for differential equations with non-instantaneous impulses. *Appl. Math. Comput.* **2017**, *298*, 45–56. [CrossRef]
26. Rudin, W. *Principles of Mathematical Analysis*; McGraw-Hill: New York, NY, USA, 1964.
27. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simulat.* **2017**, *44*, 460–481. [CrossRef]
28. Almeida, R.; Malinowska, A.; Monteiro, T. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Math. Methods Appl. Sci.* **2018**, *41*, 336–352. [CrossRef]
29. Ameen, R.; Jarad, F.; Abdeljawad, T. Ulam stability for delay fractional differential equations with a generalized Caputo derivative. *Filomat* **2018**, *32*, 5265–5274. [CrossRef]
30. Samet, B.; Aydi, H. Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative. *J. Inequal. Appl.* **2018**, *2018*, 286. [CrossRef] [PubMed]
31. Katugampola, U. A new approach to generalized fractional derivatives. *Bull. Math. Anal. Appl.* **2014**, *6*, 1–15.
32. Zhang, X.; Liu, Z.; Peng, H.; Zhang, X.; Yang, S. The general solution of differential equations with Caputo-Hadamard fractional derivatives and nininstantaneous impulses. *Adv. Math. Phy.* ***2017***, 2017, *3094173*.
33. Agarwal, R.; Hristova, S.; O'Regan, D. Lipschitz stability for non- instantaneous impulsive Caputo fractional differential equations with state dependent delays. *Axioms* **2019**, *8*, 4, doi:10.3390/axioms8010004. [CrossRef]
34. Hristova, S.; Ivanova, K. Caputo fractional differential equations with non-instantaneous random Erlang distributed impulses. *Fractal Fract.* **2019**, *3*, 28. [CrossRef]
35. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.