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On the Natural Density of Sets Related to Generalized Fibonacci Numbers of Order r

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Abstract: For $r \geq 2$ and $a \geq 1$ integers, let $(t_n^{(r,a)})_{n\geq 1}$ be the sequence of the (r,a)-generalized Fibonacci numbers which is defined by the recurrence $t_n^{(r,a)} = t_{n-1}^{(r,a)} + \cdots + t_{n-r}^{(r,a)}$ for $n > r$, with initial values $t_i^{(r,a)} = 1$, for all $i \in [1, r-1]$ and $t_r^{(r,a)} = a$. In this paper, we shall prove (in particular) that, for any given $r \geq 2$, there exists a positive proportion of positive integers which can not be written as $t_n^{(r,a)}$ for any $(n,a) \in \mathbb{Z}_{\geq r+2} \times \mathbb{Z}_{\geq 1}$.

Keywords: natural density; fibonacci r-numbers; recurrence sequences

1. Introduction

We start by recalling that if $\mathcal{A}$ is a set of positive integers, the *natural density* of $\mathcal{A}$, denoted by $\delta(\mathcal{A})$, is the following limit (if it exists)

$$\delta(\mathcal{A}) := \lim_{x\to\infty} \frac{\#\mathcal{A}(x)}{x},$$

where, $\mathcal{A}(x) := \mathcal{A} \cap [1, x]$, for $x > 0$ (also $\liminf_{x\to\infty} \#\mathcal{A}(x)/x$ and $\limsup_{x\to\infty} \#\mathcal{A}(x)/x$ are called *lower density* and *upper density*, respectively). For example, if $\mathcal{F} := (F_n)_{n\geq 0}$ is the *Fibonacci sequence*, one has that $\mathcal{F}(x) \leq (\log x)/(\log \phi)$, where $\phi = (1+\sqrt{5})/2$. In particular, the natural density of Fibonacci numbers is zero and, so, almost all positive integers are non-Fibonacci numbers (i.e., $\delta(\mathbb{Z}_{>0} \backslash \mathcal{F}) = 1$). Clearly, this is not a surprising fact, given the exponential nature of Fibonacci sequence. We refer the reader to the classical work of Niven [1] (and references therein) for more details about natural (and asymptotic) density (see also books [2,3] and papers [4–8] for more recent results).

It is especially interesting that the some kind of "combinations" of zero density sets may have positive density. For instance, the set of powers of two and the set of prime numbers have zero density but, in 1934, a classical result of Romanov [9] implied that the set of positive integers which are not of the form $p + 2^k$, for some p prime and $k \geq 0$, has upper density smaller than one.

It is also possible to study the density of a set in some prescribed subset of $\mathbb{Z}_{\geq 1}$. More precisely, let $\mathcal{A}$ and $\mathcal{B}$ be elements of $\mathcal{P}(\mathbb{Z}_{\geq 1})$ (the *power set* of $\mathbb{Z}_{\geq 1}$, i.e., the set of all subsets of positive integers), we name $\delta_{\mathcal{B}}(\mathcal{A})$ the *$\mathcal{B}$-density of $\mathcal{A}$*, as the limit (if it exists)

$$\delta_{\mathcal{B}}(\mathcal{A}) := \lim_{x\to\infty} \frac{\#\mathcal{A}_{\mathcal{B}}(x)}{x},$$

where, $\mathcal{A}_{\mathcal{B}}(x) := \{m \leq x : b_m \in \mathcal{A}\}$ (for $x > 0$) and $\mathcal{B} := \{b_1, b_2, \ldots\}$ (with $b_i < b_{i+1}$). By convention $\delta_{\emptyset}(\mathcal{A}) = 0$ and note that $\delta_{\mathbb{Z}_{\geq 1}} : \mathcal{P}(\mathbb{Z}_{\geq 1}) \to [0,1]$ is the standard natural density.

As any very well-studied object in mathematics, the Fibonacci sequence possesses many kinds of generalizations (see, e.g., [10–14]). One of the most well-known generalization is probably the sequence of *generalized Fibonacci numbers of order r*, denoted by $(t_n^{(r)})_{n\geq 0}$, which is defined by the rth order recurrence

$$t_n^{(r)} = t_{n-1}^{(r)} + \cdots + t_{n-r}^{(r)}$$

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with initial conditions $t_0^{(r)} = 0$ and $t_i^{(r)} = 1$, for $i \in [1, r-1]$. For $r = 2$, we have the sequence of Fibonacci numbers, for $r = 3$, we have the *Tribonacci numbers* and so on. For recent results on this sequence, we cite [15] and its annotated bibliography.

Here we are interested in a related generalization. More precisely, let $r \geq 2$ and $a \geq 0$ be integers. The (r, a)-*generalized Fibonacci sequence* $(t_n^{(r,a)})_{n\geq 1}$ is defined by

$$t_n^{(r,a)} = \begin{cases} 1, & \text{if } 1 \leq n \leq r-1; \\ a, & \text{if } n = r; \\ \sum_{i=1}^{r} t_{n-i}^{(r,a)}, & \text{if } n \geq r+1. \end{cases}$$

Note that $(t_n^{(r)})_{n\geq 1} = (t_n^{(r,r-1)})_{n\geq 1}$. For our purpose, we consider the previous sequence only after its $(r+1)$th term, i.e., we denote $\mathcal{T}_{r,a} := (t_n^{(r,a)})_{n>r+1}$. As before, by its exponential nature, it holds that $\delta(\mathcal{T}_{r,a}) = 0$. Now, we turn or attention to the a specific "combination" of these sets, namely, their union. Thus, the following question arises: Are there infinitely many positive integers which do not belong to $\mathcal{T}_r = \cup_{a\geq 1}\mathcal{T}_{r,a}$? If so, does this "exception set" represent a positive proportion (i.e., with positive natural density) of the positive integers?

In this paper, we answer (positively) this question by proving a more general result. More precisely,

Theorem 1. *Let $r \geq 2$ be an integer. Then there exists $\mathcal{B} \in \mathcal{P}(\mathbb{Z}_{\geq 1})$ (depending only on r) with*

$$\delta(\mathcal{B}) = \frac{1}{2^{r-1}(2^r - 1)}$$

such that

$$\delta_{\mathcal{B}}(\mathcal{T}_r) < \frac{1}{\alpha_r^r(\alpha_r - 1)}, \tag{1}$$

where α_r is the only positive real root of $x^r - x^{r-1} - \cdots - x - 1$. In particular, for any $r \geq 2$, there exists a positive proportion of positive integers which do not belong to $\mathcal{T}_{r,a}$, for all $a \in \mathbb{Z}_{\geq 1}$.

Remark 1. *Table 1 shows that the upper bound for $\delta_{\mathcal{B}}(\mathcal{T}_r)$ provided in the Theorem 1, say u_r, decreases significantly as r grows.*

Table 1. The upper bound for $\delta_{\mathcal{B}}(\mathcal{T}_r)$.

r	2	3	4	5	6	7	8	9	10
u_r	0.6180	0.1914	0.0781	0.0352	0.0166	0.0081	0.0039	0.0019	0.0009

We organize this paper as follows. In Section 2, we will present some helpful properties of the sequence $(t_n^{(r,a)})_n$. The third section is devoted to the proof of Theorem 1.

2. Auxiliary Results

Before proceeding further, we shall present some useful tools related to the previous sequences.

The characteristic polynomial of the sequence $(t_n^{(r,a)})_n$ is $\psi_r(x) = x^r - x^{r-1} - \cdots - x - 1$ which has only one root outside the unit circle, say α_r, which is located in the interval $(2(1 - 2^{-r}), 2)$ (see [16]). Throughout this work, in order to simplify the notations, we shall write α for α_r and for integers $a < b$, we write $[a, b]$ for $\{a, a+1, \ldots, b\}$.

In 2018, Young [17] (p. 3) found a closed formula for $t_n^{(r)}$ in the range $n \in [r+1, 2r]$, namely,

$$t_{r+i}^{(r)} = 2^{i-1}(2r - 3) + 1,$$

for all $i \in [1, r]$. The next result generalizes this fact for $(t_n^{(r,a)})_n$:

Lemma 1. *The identity*

$$t_{r+i}^{(r,a)} = 2^{i-1}(r + a - 2) + 1$$

holds for all $i \in [1, r]$.

Proof. To prove this identity, we shall use (finite) induction on $i \in [1, r]$. For the basis case $i = 1$, since $t_r^{(r,a)} = a$, one has that

$$t_{r+1}^{(r,a)} = a + \underbrace{1 + \cdots + 1}_{r-1} = a + r - 1 = 2^{1-1}(r + a - 2) + 1.$$

Suppose (by the induction hypothesis) that $t_{r+i}^{(r,a)} = 2^{i-1}(r + a - 2) + 1$, for some $i \in [1, r - 1]$. Then

$$\begin{aligned} t_{r+i+1}^{(r,a)} &= t_{r+i}^{(r,a)} + t_{r+i-1}^{(r,a)} + \cdots + t_{i+1}^{(r,a)} = 2t_{r+i}^{(r,a)} - t_i^{(r,a)} \\ &= 2 \cdot (2^{i-1}(r + a - 2)) - 1 \\ &= 2^i(r + a - 2) + 1 \end{aligned}$$

which completes the induction process. □

Before stating the next lemma, we recall that the sequence of *k-bonacci numbers* (or *k-generalized Fibonacci numbers*) $(F_n^{(r)})_{n \geq -(r-2)}$, is the kth order linear recurrence which satisfies the same recurrence as $(t_n^{(r,a)})_{n \geq 1}$, namely,

$$F_n^{(r)} = F_{n-1}^{(r)} + \cdots + F_{n-r}^{(r)},$$

but with r initial values $0, \ldots, 0, 1 = F_1^{(r)}$ (see, e.g., [18,19]).

The third lemma relates the sequences $(t_n^{(r,a)})_n$ and $(F_n^{(r)})_n$. Specifically, we have

Lemma 2. *If* $n \geq r$, *then*

$$t_n^{(r,a)} = aF_{n-r+1}^{(r)} + t_n^{(r,0)}.$$

Proof. Define the sequence $(X_n)_{n \geq r}$, by

$$X_n := aF_{n-r+1}^{(r)} + t_n^{(r,0)}.$$

Therefore, we want to prove that $X_n = t_n^{(r,a)}$, for all $n \geq r$. For that, first observe that

$$\begin{aligned} X_n &= aF_{n-r+1}^{(r)} + t_n^{(r,0)} \\ X_{n+1} &= aF_{n-r+2}^{(r)} + t_{n+1}^{(r,0)} \\ X_{n+2} &= aF_{n-r+3}^{(r)} + t_{n+2}^{(r,0)} \\ &\vdots \\ X_{n+r-1} &= aF_n^{(r)} + t_{n+r-1}^{(r,0)}. \end{aligned}$$

By summing up all previous equalities, we obtain that

$$
\begin{aligned}
X_{n+r-1}+X_{n+r-2}+\cdots+X_n &= a\sum_{j=1}^{r} F_{n-r+j}^{(r)}+\sum_{j=0}^{r-1} t_{n+j}^{(r,0)} \\
&= aF_{n+1}^{(r)}+t_{n+r}^{(r,0)} \\
&= X_{n+r}.
\end{aligned}
$$

Therefore, $(X_n)_n$ and $(t_n^{(r,a)})_n$ satisfy the same r-order recurrence relation. Now, it suffices to prove that $t_{r+i}^{(r,a)} = X_{r+i}$, for all $i \in [0, r-1]$. For $i = 0$, we have $X_r = aF_1^{(r)} + t_0^{(r,0)} = a = t_r^{(r,a)}$. In the case $i \in [1, r-1]$, one has

$$X_{r+i} = aF_{i+1}^{(r)} + t_{r+i}^{(r,0)} = a \cdot 2^{i-1} + 2^{i-1}(r-2) + 1 = 2^{i-1}(a+r-2) + 1,$$

where we used Lemma 1 together with the well-known fact that $F_n^{(r)} = 2^{n-2}$, if $n \in [2, r+1]$. □

The following lower bound for $F_n^{(r)}$, which is due to Bravo and Luca [20], is the last useful ingredient.

Lemma 3. *We have that*

$$F_n^{(r)} \geq \alpha^{n-2}$$

holds for all $n \geq 1$.

Now we are ready to deal with the proof.

3. The Proof of the Theorem 1

First, let us denote $\mathcal{S}_n^{(r)}$ by

$$\mathcal{S}_n^{(r)} := \{aF_{n-r+1}^{(r)} + t_n^{(r,0)} : a \in \mathbb{Z}_{\geq 1}\}.$$

From Lemma 2, we have that

$$\mathcal{T}_r = \bigcup_{n \geq r+2} \mathcal{S}_n^{(r)}.$$

Consider $\mathcal{B} := A_r\mathbb{Z}_{\geq 1}$, where $A_r := 2^{r-1}(2^r - 1)$. Note that the natural density of $\mathcal{B}$ is $1/A_r$, i.e.,

$$\delta(\mathcal{B}) = \frac{1}{2^{r-1}(2^r-1)}.$$

We also define $\mathcal{T}_{r,n}(x) := \{m \leq x : A_r m \in \mathcal{S}_n^{(r)}\}$ and then

$$(\mathcal{T}_r)_{\mathcal{B}}(x) := \{m \leq x : A_r m \in \mathcal{T}_r\} = \bigcup_{n \geq r+2} \mathcal{T}_{r,n}(x).$$

Thus

$$\#(\mathcal{T}_r)_{\mathcal{B}}(x) \leq \sum_{n \geq r+2} \#(\mathcal{T}_{r,n}(x)). \tag{2}$$

Now, we claim that

$$\mathcal{T}_{r,r+2}(x) = \mathcal{T}_{r,r+3}(x) = \cdots = \mathcal{T}_{r,2r}(x) = \mathcal{T}_{r,2r+1}(x) = \emptyset.$$

In fact, aiming for a contradiction, suppose the contrary, then there exists $m \le x$ such that $A_r m \in \mathcal{S}_{r+i}^{(r)}$ for at least one $i \in [2, r+1]$ and so at least one of the following relations is true

$$\begin{aligned} A_r m &= aF_3^{(r)} + t_{r+2}^{(r,0)} \\ A_r m &= aF_4^{(r)} + t_{r+3}^{(r,0)} \\ A_r m &= aF_5^{(r)} + t_{r+4}^{(r,0)} \\ &\vdots \\ A_r m &= aF_{r+1}^{(r)} + t_{2r}^{(r,0)} \\ A_r m &= aF_{r+2}^{(r)} + t_{2r+1}^{(r,0)}. \end{aligned} \tag{3}$$

However, if some of the previous equalities (except equality (3)) holds (since $F_i^{(r)} \mid A_r$, for all $i \in [3, r+1]$), we would arrive at the absurdity that $F_i^{(r)} = 2^{i-2} > 1$ divides $t_{r+i-1}^{(r,0)} = 2^{i-2}(r-2) + 1$ (see Lemma 1), for some $i \in [3, r+1]$.

It is well-known that $F_{r+2}^{(r)} = 2^r - 1$ (which also divides A_r), thus, for the case in which (3) holds, one has that $F_{r+2}^{(r)}$ should divide $t_{2r+1}^{(r,0)} = 2t_{2r}^{(r,0)} = 2^r(r-2) + 2$. Since $2^r(r-2) + 2 = (2^r - 1)(r-2) + r$, then $2^r - 1 \mid r$ which is an absurd (since $2^2 - 1 \nmid 2$ and $2^r - 1 > r$, for all $r \ge 3$). In conclusion, $\mathcal{T}_{r,r+i}(x)$ is the empty set, for all $i \in [2, r+1]$.

Going back to (2), one infers that

$$\#(\mathcal{T}_r)_{\mathcal{B}}(x) \le \sum_{n \ge 2r+2} \#(\mathcal{T}_{r,n}(x))$$

and so

$$\frac{\#(\mathcal{T}_r)_{\mathcal{B}}(x)}{x} \le \sum_{n \ge 2r+2} \frac{\#(\mathcal{T}_{r,n}(x))}{x}.$$

However,

$$\frac{\#(\mathcal{T}_{r,n}(x))}{x} \le \frac{\#\{a \ge 1 : aF_{n-r+1}^{(r)} + t_n^{(r,0)} \le x\}}{x} \le \frac{(x - t_n^{(r,0)})/F_{n-r+1}^{(r)}}{x} < \frac{1}{F_{n-r+1}^{(r)}}$$

and hence

$$\begin{aligned} \frac{\#(\mathcal{T}_r)_{\mathcal{B}}(x)}{x} &< \sum_{n \ge 2r+2} \frac{1}{F_{n-r+1}^{(r)}} \le \sum_{n \ge 2r+2} \frac{1}{\alpha^{n-r-1}} \\ &= \frac{1}{\alpha^{r+1}} \left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots \right) = \frac{1}{\alpha^r(\alpha - 1)}, \end{aligned}$$

where we used Lemma 3. Since

$$\delta_{\mathcal{B}}(\mathcal{T}_r) = \lim_{x \to \infty} \frac{\#(\mathcal{T}_r)_{\mathcal{B}}(x)}{x},$$

we then have

$$\delta_{\mathcal{B}}(\mathcal{T}_r) \le \frac{1}{\alpha^r(\alpha - 1)}$$

as desired. The proof is complete. □

4. Conclusions

In this paper, we study the natural density of some sets related to recurrence sequences. More precisely, for $r \geq 2$ and $a \geq 1$ integers, let $(t_n^{(r,a)})_{n\geq 0}$ be the sequence of the (r,a)-generalized Fibonacci numbers which is defined by the recurrence $t_n^{(r,a)} = t_{n-1}^{(r,a)} + \cdots + t_{n-r}^{(r,a)}$ for $n \geq r+1$, with initial values $t_i^{(r,a)} = 1$, for all $i \in [1, r-1]$ and $t_r^{(r,a)} = a$. This family contains many well-known sequences such as the Fibonacci, k-Fibonacci, r-Fibonacci, Tribonacci etc. The main result here is that for any $r \in \mathbb{Z}_{\geq 2}$, it is possible to find a set $\mathcal{B}$ (depending only on r) with positive density and such that the portion of terms of $\left(t_n^{(r,a)}\right)_{n\geq r+2}$ belonging to $\mathcal{B}$ is smaller than one (the novelty here is that this particular union of zero density sets will have positive density). In particular, there exist infinitely many positive integers which are not of the form $t_n^{(r,a)}$, for all $n \geq r+2$ and $a \geq 1$.

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References

1. Niven, I. The asymptotic density of sequences. *Bull. Amer. Math. Soc.* **1951**, *57*, 420–434. [CrossRef]
2. Tenenbaum, G. *Introduction to Analytic and Probabilistic Number Theory*; Cambridge Studies in Adv. Math.: Cambridge, UK, 1995.
3. Kowalski, E. *Introduction to Probabilistic Number Theory*; Cambridge Studies in Adv. Math.: Cambridge, UK, 2021.
4. Leonetti, P.; Tringali, S. On the notations of upper and lower density. *Proc. Edinb. Math. Soc.* **2020**, *63*, 139–167. [CrossRef]
5. Maschio, S. Natural density and probability, constructively. *Rep. Math. Logic* **2020**, *55*, 41–59. [CrossRef]
6. Marques, D. Sharper upper bounds for the order of appearance in the Fibonacci sequence. *Fibonacci Quart.* **2013**, *51*, 233–238.
7. Patel, V.; Siksek, S. On powers that are sums of consecutive like powers. *Res. Number Theory* **2017**, *3*, 2. [CrossRef] [PubMed]
8. Trojovský, P. Some problems related to the growth of $z(n)$. *Adv. Differ. Equ.* **2020**, *2020*, 270. [CrossRef]
9. Romanov, N.P. Über einige Sätze der aditiven Zahlentheorie. *Math. Ann.* **1934**, *101*, 668–678. [CrossRef]
10. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; Wiley: New York, NY, USA, 2001.
11. Posamentier, A.S.; Lehmann, I. *The (Fabulous) Fibonacci Numbers*; Prometheus Books: Amherst, MA, USA, 2007.
12. Vorobiev, N.N. *Fibonacci Numbers*; Birkhäuser Verlag: Basel, Switzerland, 2002.
13. Hoggat, V.E. *Fibonacci and Lucas Numbers*; Houghton-Mifflin: PaloAlto, CA, USA, 1969.
14. Ribenboim, P. *My Numbers, My Friends: Popular Lectures on Number Theory*; Springer: New York, NY, USA, 2000.
15. Trojovská, E.; Trojovský, P. On Fibonacci numbers of order r which are expressible as sum of consecutive factorial numbers. *Mathematics* **2021** , *9*, 962. [CrossRef]
16. Wolfram, A. Solving generalized Fibonacci recurrences. *Fibonacci Quart.* **1998**, *36*, 129–145.
17. Young, P.T. 2-adic valuations of generalized Fibonacci numbers of odd order. *Integers* **2018**, *18*, #A1.
18. Miles, E.P., Jr. Generalized Fibonacci Numbers and Associated Matrices. *Amer. Math. Month.* **1960**, *67*, 745–752. [CrossRef]
19. Gabai, H. Generalized Fibonacci k-sequences. *Fibonacci Quart.* **1970**, *8*, 31–38.
20. Bravo, J.J.; Luca, F. Powers of Two in Generalized Fibonacci Sequences. *Rev. Colomb. Mat.***2012**, *46*, 67–79.