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Robust Stability of Time-Varying Markov Jump Linear Systems with Respect to a Class of Structured, Stochastic, Nonlinear Parametric Uncertainties

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Abstract: This note is devoted to a robust stability analysis, as well as to the problem of the robust stabilization of a class of continuous-time Markovian jump linear systems subject to block-diagonal stochastic parameter perturbations. The considered parametric uncertainties are of multiplicative white noise type with unknown intensity. In order to effectively address the multi-perturbations case, we use scaling techniques. These techniques allow us to obtain an estimation of the lower bound of the stability radius. A first characterization of a lower bound of the stability radius is obtained in terms of the unique bounded and positive semidefinite solutions of adequately defined parameterized backward Lyapunov differential equations. A second characterization is given in terms of the existence of positive solutions of adequately defined parameterized backward Lyapunov differential inequalities. This second result is then exploited in order to solve a robust control synthesis problem.

Keywords: Markov jump systems; robust stability; structured parametric uncertainties; stability radius; scaling; linear matrix inequalities

MSC: 93E15; 93D09

1. Introduction

The class of Markovian jump linear systems is very appropriate for modeling a plant the structure of which is subject to random abrupt changes. Problems, such as stability and optimal control, as well as important applications of such systems, can be found in several references in the current literature, for instance in [1–4] and the references therein. On the other hand, robustness, with respect to *stochastic* parametric uncertainties for this class of system, has attracted a lot of interest from the research community. This is partly due to the engineering applications potential of such a modeling paradigm. We will restrict ourselves in this paper to those works that relied on the concept of *stability radii* in the treatment of the parametric uncertainties robustness problem. Without being exhaustive, we cite here [5–8]. For the interested reader, a comprehensive historical perspective of the different aspects regarding the stability radius, both in deterministic and stochastic frameworks, may be found in [9].

In the current paper, we study the robust stability and robust stabilization problems, under multi-perturbations, for a class of continuous-time Markovian jump linear systems affected by parametric uncertainties of multiplicative white noise type with unknown intensity. The only available information is that the intensities of the white noises are output-dependent, unknown, nonlinear functions. In order to effectively address the

multi-perturbations case, we will use scaling techniques (see [10,11]) in order to obtain an estimation of the lower bound of the stability radius. We first provide a lower bound of the stability radius in terms of the unique bounded and positive semidefinite solutions of adequately defined parameterized backward Lyapunov differential equations. A second characterization of a lower bound of the stability radius is given in terms of the existence of positive solutions of adequately defined parameterized backward Lyapunov differential inequalities. This second result is very useful for the robust control synthesis problem as it allows the formulation of the feedback gains computation as a convex optimization problem under the Linear Matrix Inequalities (LMIs) paradigm. The original contributions of the paper could be summarized as follows:

(i) An estimation of the lower bound of the stability radius is obtained for a class of continuous-time Markovian jump linear systems subject to block-diagonal stochastic parameter perturbations. The considered parametric uncertainties are of multiplicative white noise type with unknown intensity;

(ii) Scaling techniques have been used in order to effectively address the multi-perturbations case. This allows us to provide a lower bound of the stability radius in terms of the unique bounded and positive semidefinite solutions of adequately defined parameterized backward Lyapunov differential equations;

(iii) A second characterization of a lower bound of the stability radius is given in terms of the existence of positive solutions of adequately defined parameterized backward Lyapunov differential inequalities. This second formulation allows us to state and solve a robust stabilization problem as a convex optimization problem.

Section 2 provides some preliminary definitions and introduces the problem formulation. Several preliminary issues are given in Section 3, which are related to some Lyapunov-type operators as well as to the scaling technique. The main results are established in Section 4. In Section 5, a numerical example is provided to illustrate the theoretical results.

2. Problem Formulation

2.1. Model Description

Let us consider the system $\mathcal{G}_{\mathbf{\Delta}}$, which has the state space representation described by:

$$dx(t) = A(t, \eta_t)x(t) + \sum_{k=1}^{r} D_k(t, \eta_t)\Delta_k(t, z_k(t), \eta_t)dw_k(t) \tag{1a}$$

$$z_k(t) = E_k(t, \eta_t)x(t). \tag{1b}$$

$t \in \mathbb{R}_+ = [0, \infty)$, where $x(t) \in \mathbb{R}^n$ are state parameters at the instance time t. Here, $\{w(t)\}_{t \geqslant 0}$, $\left(w(t) = (w_1(t)\ w_2(t)\ \ldots\ w_r(t))^T\right)$ is an r-dimensional standard Wiener process defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $\{\eta_t\}_{t \geqslant 0}$ is a standard Markov process with right-continuous paths and left limits defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$, taking values in the finite set $\mathcal{N} = \{1, 2, \ldots, N\}$ and having the transition semigroup $P(t) = e^{Qt}, t \geqslant 0$, where $Q \in \mathbb{R}^{N \times N}$ is a matrix whose elements have the properties:

$$q_{ij} \geqslant 0, \text{ if } i \neq j \tag{2a}$$

$$\sum_{\ell=1}^{N} q_{i\ell} = 0 \tag{2b}$$

for all $i, j \in \mathcal{N}$. For more details regarding the properties of a standard Wiener process and of the solutions of the stochastic differential equation affected by multiplicative white noise perturbations, we refer to the monographs [12–14]. For the properties of the Markov processes, we mention [15,16].

Throughout the paper, we shall write $A(t, i)$, $D_k(t, i)$, $E_k(t, i)$, $\Delta_k(t, z_k, i)$ whenever $\eta_t = i \in \mathcal{N}$.

Regarding the stochastic processes and the functions that describe the coefficients of the system (1), we make the assumptions:

Hypothesis 1a. $\{\eta_t\}_{t \geqslant 0}$ *and* $\{w(t)\}_{t \geqslant 0}$ *are* $\mathcal{P}$*-independent stochastic processes;*

Hypothesis 1b. *the initial probability distribution of the Markov process* $\pi_0 = (\pi_0(1), \pi_0(2), \dots, \pi_0(N))$, *where* $\pi_0(i) \triangleq \mathcal{P}[\eta_0 = i] > 0, \forall i \leqslant N$.

Hypothesis 1c. $t \to A(t, i) : \mathbb{R}_+ \to \mathbb{R}^{n \times n}, t \to D_k(t, i) : \mathbb{R}_+ \to \mathbb{R}^{n \times m_k}, t \to E_k(t, i) : \mathbb{R}_+ \to \mathbb{R}^{\nu_k \times n}, k = 1, 2, \cdots, r, i \in \mathcal{N}$, *are bounded and continuous matrix valued functions.*

The functions $\Delta_k(\cdot, \cdot, t)$, which model the parametric uncertainties occurring in (1a), satisfy the assumptions:

Hypothesis 2. *The functions* $(t, z_k) \to \Delta_k(t, z_k, i) : \mathbb{R}_+ \times \mathbb{R}^{\nu_k} \to \mathbb{R}^{m_k}$ *are arbitrary nonlinear functions, which are Borel measurable with the property that* $\Delta_k(t, 0, i) = 0, \forall (t, i) \in \mathbb{R}_+ \times \mathcal{N}$, *and which satisfy a Lipschitz condition of the form:*

$$|\Delta_k(t, z_k, i) - \Delta_k(t, \hat{z}_k, i)| \leqslant \mu_k |z_k - \hat{z}_k| \tag{3}$$

$\forall z_k, \hat{z}_k \in \mathbb{R}^{\nu_k}, (t, i) \in \mathbb{R}_+ \times \mathcal{N}$. *Throughout the paper,* $|\cdot|$ *denotes the Euclidean norm of a vector.*

The system (1) can be viewed as a disruption of the following system:

$$\dot{x}(t) = A(t, \eta_t) x(t), \quad t \in \mathbb{R}_+, \tag{4}$$

which will be named the nominal system. The disruption is produced by a disturbance modeled by the stochastic process:

$$\sum_{k=1}^{r} D_k(t, \eta_t) \Delta_k(t, z_k(t), \eta_t) dw_k(t), \tag{5}$$

whose magnitude depends nonlinearly on the output:

$$z_k(t) = E_k(t, \eta_t) x(t), \ 1 \leqslant k \leqslant r.$$

Often we shall say that the nominal system is defined by the pair $(\mathbf{A}(\cdot), Q)$, where $\mathbf{A}(\cdot) = (A(\cdot, 1), A(\cdot, 2), \cdots, A(\cdot, N))$ are the matrix valued functions satisfying the assumption of Hypothesis 1c and Q is the generator matrix of the Markov process the elements of which satisfy (2).

For each $t \in \mathbb{R}_+$, $\mathcal{H}_t \in \mathcal{F}$ stands for the σ-algebra generated by the random variables $w(s)$ and η_s with $0 \leqslant s \leqslant t$. Further, we denote $\mathcal{X}_{t_0}$ the set of random vectors $x_0 : \Omega \to \mathbb{R}^n$, which are $\mathcal{H}_{t_0}$-measurable and satisfy $\mathbb{E}[|x_0|^2] < \infty$. Here, and in the sequel, $\mathbb{E}[\cdot]$ denotes the mathematical expectation.

Reasoning as in the proof of Theorem 1.1 in Chapter 5 from [12], we obtain the following result regarding the existence of the solution of a stochastic differential equation of type (1) for an arbitrary function $\Delta_k(\cdot, \cdot, i)$, satisfying the assumption of Hypothesis 2.

Proposition 1. *Assume that the matrix valued functions arising in (1) satisfy the assumption Hypotheses 1a–1c. Let* $t_0 \in \mathbb{R}_+$ *and* $x_0 \in \mathcal{X}_{t_0}$ *be arbitrary. Then, for any nonlinear function* $\Delta_k(\cdot, \cdot, i)$ *satisfying the assumption Hypothesis 2, the system* $\mathcal{G}_{\boldsymbol{\Delta}}$ *described by (1) has a unique solution* $x_{\Delta}(t; t_0, x_0), t \geqslant t_0$, *which has the properties:*

(a) $x_{\Delta}(\cdot; t_0, x_0)$ *is almost surely continuous in any* $t \geqslant t_0$;
(b) *for any* $t \geqslant t_0$, $x_{\Delta}(t; t_0, x_0) \in \mathcal{X}_t$;
(c) $x_{\Delta}(t_0; t_0, x_0) = x_0$.

Remark 1. *Since the functions $\Delta_k(t, z_k, i) \equiv 0$ satisfy the assumption of Hypothesis 2, we deduce that the result in Proposition 1 is also applicable in the case of the nominal system (4).*

2.2. Robust Stability: Stability Radius

In order to define the concept of robustness of the stability of the zero solution of the nominal system (4) with respect to the structured perturbation (5), we shall introduce a norm in the set of admissible uncertainties $\mathcal{D}$ of all nonlinear functions of type:

$$\boldsymbol{\Delta}(t, \mathbf{z}) = ((\Delta_1(t, z_1, 1), \Delta_1(t, z_1, 2), \cdots, \Delta_1(t, z_1, N)), \cdots,$$
$$(\Delta_r(t, z_r, 1), \Delta_r(t, z_r, 2), \cdots, \Delta_r(t, z_r, N))) \tag{6a}$$

$$\mathbf{z} = \left(z_1^T, z_2^T, \cdots, z_r^T \right)^T \tag{6b}$$

and $(t, z_k) \to \Delta_k(t, z_k, i) : \mathbb{R}_+ \times \mathbb{R}^{\nu_k}, 1 \leqslant k \leqslant r$ and $i \in \mathcal{N}$ are arbitrary functions satisfying the assumption of Hypothesis 2. To ease the expression of (6), we rewrite it in the form:

$$\boldsymbol{\Delta}(t, \mathbf{z}) = (\underline{\Delta}_1(t, z_1), \underline{\Delta}_2(t, z_2), \cdots, \underline{\Delta}_r(t, z_r)), \tag{7}$$

where $\underline{\Delta}_k(t, z_k) = (\Delta_k(t, z_k, 1), \cdots, \Delta_k(t, z_k, N))$.

We set:

$$|||\underline{\Delta}_k(\cdot, \cdot)||| \triangleq \inf\left\{ \mu_k > 0 \middle| |\Delta_k(t, z_k, i) - \Delta_k(t, \hat{z}_k, i)| \leqslant \mu_k |z_k - \hat{z}_k|, \forall z_k, \hat{z}_k \in \mathbb{R}^{\nu_k}, t \in \mathbb{R}_+, i \in \mathcal{N} \right\}. \tag{8}$$

From (3) and (8), together with $\Delta_k(t, 0, i) = 0$, we obtain that

$$|\Delta_k(t, z_k, i)| \leqslant |||\underline{\Delta}_k(\cdot, \cdot)||| \cdot |z_k| \tag{9}$$

for all $(t, z_k, i) \in \mathbb{R}_+ \times \mathbb{R}^{\nu_k} \times \mathcal{N}$.

Now we define

$$\|\boldsymbol{\Delta}(\cdot, \cdot)\| = \max_{k \in \underline{r}} |||\underline{\Delta}_k(\cdot, \cdot)|||, \tag{10}$$

where we have denoted

$$\underline{r} = \{1, 2, \cdots, r\}. \tag{11}$$

Remark 2. *Among the subsets of the set of admissible uncertainties $\mathcal{D}$, we distinguish the subset $\mathcal{D}_{lin}$ consisting of all uncertainties of type (6), where*

$$\Delta_k(t, z_k, i) = \nabla_k(t, i) z_k \tag{12}$$

for each $k \in \underline{r}$, for all $(t, z_k, i) \in \mathbb{R}_+ \times \mathbb{R}^{\nu_k} \times \mathcal{N}$, where $t \to \nabla_k(t, i) : \mathbb{R}_+ \to \mathbb{R}^{m_k \times \nu_k}$ are arbitrary continuous and bounded matrix valued functions. In this case, the smallest $\mu_k > 0$ for which the inequality (3) is satisfied is given by:

$$\mu_k = \sup_{t \in \mathbb{R}_+} \max_{i \in \mathcal{N}} \left(\lambda_{\max} \left[\nabla_k^T(t, i) \cdot \nabla_k(t, i) \right] \right)^{\frac{1}{2}}. \tag{13}$$

So, (8), (10) and (13) allow us to retrieve the norm of the parametric uncertainties considered in [17].

Let us recall the concept of stability of the zero equilibrium of a system of type (1) and of the nominal system (4) that will be used in the rest of the paper.

Definition 1. *We say that the zero solution of a system $\mathcal{G}_{\boldsymbol{\Delta}}, \boldsymbol{\Delta} \in \mathcal{D}$, is:*

(a)   *globally exponentially mean square stable with conditioning (GESMS–C) if there exist $\beta \geqslant 1$,*
      $\alpha > 0$ with the property:

$$\mathbb{E}\Big[|x_{\boldsymbol{\Delta}}(t, t_0, x_0)|^2 \big| \eta_{t_0} = i\Big] \leqslant \beta e^{-\alpha(t-t_0)} \mathbb{E}\Big[|x_0|^2 \big| \eta_{t_0} = i\Big], \tag{14}$$

   $\forall t \geqslant t_0 \geqslant 0$, $x_0 \in \mathcal{X}_{t_0}$, $i \in \mathcal{N}$ and any initial probability distribution π_0 of the Markov
   process;

(b)   *globally stochastically stable with conditioning (GSS–C) if there exists $c > 0$ with the property:*

$$\int_{t_0}^{\infty} \mathbb{E}\Big[|x_{\boldsymbol{\Delta}}(t, t_0, x_0)|^2 \big| \eta_{t_0} = i\Big] dt \leqslant c \mathbb{E}\Big[|x_0|^2 \big| \eta_{t_0} = i\Big], \tag{15}$$

   $\forall t_0 \geqslant 0$, $x_0 \in \mathcal{X}_{t_0}$, $i \in \mathcal{N}$ and any initial probability distribution π_0 of the Markov process.

Remark 3.

(a)   *Even if the positive constants, α and β, which appear in (14) and c from (15), do not depend*
      upon t, t_0, x_0, i, they can depend upon the admissible uncertainties $\boldsymbol{\Delta}(\cdot, \cdot) \in \mathcal{D}$.
(b)   *Since the nominal system (4) is a special case of a system of type (1) (with $\boldsymbol{\Delta}(\cdot, \cdot) = 0$), it*
      follows that the previous definition is also applicable in the case of the nominal system. It is
      worth noting that the system (4) is a linear system and this is why the "global" epithet of the
      stability is redundant. At the same time, in the linear case, the stability property is not related
      to a solution, it is a property of the whole system. Therefore, we shall say that the nominal
      system is exponentially stable in mean square with conditioning (ESMS–C) if its solutions
      have a behavior like that described by (14).
(c)   *Applying Theorem 8.3.7 from [4], we deduce that the zero solution of a system $\mathcal{G}_{\boldsymbol{\Delta}}$ is GESMS–C*
      if and only if it is GSS–C.

Roughly speaking, the mean square exponential stability with the conditioning of the nominal system (4) is robust with respect to the structured uncertainties of type (5) if the zero solution of all perturbed systems $\mathcal{G}_{\boldsymbol{\Delta}}$ is ESMS–C for all admissible uncertainties $\boldsymbol{\Delta}(\cdot, \cdot)$ from a subset of $\mathcal{D}$.

To measure the level of the robustness of the stability of a linear system of type (4) with respect to the structured uncertainties of type (5) we introduce the following concept of *stability radius*. This definition is an extension of Definition 3.2 from [18].

Definition 2. *The stability radius of the nominal system (4) or, equivalently, of the pair $(\mathbf{A}(\cdot), Q)$ with respect to the structured disturbances (5), the structure of which is determined by the pairs of matrix valued functions $(D_k(\cdot, i), E_k(\cdot, i))$, $(k, i) \in \underline{r} \times \mathcal{N}$, is the number:*

$$\rho_{\mathcal{D}}\big(\mathbf{A}(\cdot), Q\big| (D_k(\cdot, i), E_k(\cdot, i)), (k, i) \in \underline{r} \times \mathcal{N}\big) = \inf\big\{\rho > 0 \big| \exists \boldsymbol{\Delta}(\cdot, \cdot) \in \mathcal{D}, \|\boldsymbol{\Delta}(\cdot, \cdot)\| \leqslant \rho$$
$$\text{for which the zero solution of the corresponding system } \mathcal{G}_{\boldsymbol{\Delta}} \text{ is not GESMS.}\big\} \tag{16}$$

Our aim is to obtain a lower bound for the stability radius of the nominal system (4) with respect to disturbances of type (5). To this end, we shall adapt to this framework of systems with Markov jumping parameters—the scaling technique introduced in [18].

3. Several Preliminary Issues

3.1. The Lyapunov Type Operators and Lyapunov Differential Equations

Let $\mathcal{S}_n$ be the linear space of symmetric matrices of size $n \times n$. We set $\mathcal{S}_n^N \triangleq \underbrace{\mathcal{S}_n \times \mathcal{S}_n \times \cdots \times \mathcal{S}_n}_{\text{N } times}$. The elements of $\mathcal{S}_n^N$ are of the form $\mathbf{X} = (X(1), \cdots, X(N))$, where $X(i) \in \mathcal{S}_n$, $i \in \mathcal{N}$. Equipped with the inner product:

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^{N} \mathrm{Tr}[X(i)Y(i)], \tag{17}$$

$\mathcal{S}_n^N$ is a finite dimensional ordered Hilbert space. The order relation on $\mathcal{S}_n^N$ denoted by "$\geq$" is induced by the convex cone $\mathcal{S}_n^{N+} = \{ \mathbf{X} = (X(1), \cdots, X(N)) \in \mathcal{S}_n^N \, | \, X(i) \geq 0, 1 \leq i \leq N \}$. Here, $X(i) \geq 0$ means that $X(i)$ is a positive semidefinite matrix.

Based on the coefficients of the nominal system (4), we consider the operator valued function $\mathcal{L}(\cdot) : \mathbb{R}_+ \to \mathbf{B}(\mathcal{S}_n^N)$ defined by:

$$\mathcal{L}(t)[\mathbf{X}] = (\mathcal{L}(t)[\mathbf{X}](1), \mathcal{L}(t)[\mathbf{X}](2), \cdots, \mathcal{L}(t)[\mathbf{X}](N)) \tag{18a}$$

$$\mathcal{L}(t)[\mathbf{X}](i) = A(t,i)X(i) + X(i)A^T(t,i) + \sum_{j=1}^{N} q_{ji}X(j). \tag{18b}$$

$\forall \mathbf{X} \in \mathcal{S}_n^N, i \in \mathcal{N}$.

Here, $\mathbf{B}(\mathcal{S}_n^N)$ denotes the space of the linear operators defined in $\mathcal{S}_n^N$. In (18b), q_{ji} are the real numbers, which satisfy conditions (2).

The adjoint $\mathcal{L}^*(t)$ of the linear operator $\mathcal{L}(t)$, with respect to the inner product (17), is described by

$$\mathcal{L}^*(t)[\mathbf{X}] = (\mathcal{L}^*(t)[\mathbf{X}](1), \mathcal{L}^*(t)[\mathbf{X}](2), \cdots, \mathcal{L}^*(t)[\mathbf{X}](N)) \tag{19a}$$

$$\mathcal{L}^*(t)[\mathbf{X}](i) = A^T(t,i)X(i) + X(i)A(t,i) + \sum_{j=1}^{N} q_{ij}X(j). \tag{19b}$$

$i \in \mathcal{N}$.

Let $\mathbf{T}(t, t_0) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ be the linear evolution operator defined by the linear differential equation

$$\dot{\mathbf{X}}(t) = \mathcal{L}(t)[\mathbf{X}(t)], \tag{20}$$

that is, $\mathbf{T}(t, t_0)[\mathbf{H}] = \mathbf{X}(t; t_0, \mathbf{H})$, where $\mathbf{X}(\cdot; t_0, \mathbf{H})$ is the solution of the Lyapunov type differential Equation (20), satisfying $\mathbf{X}(t_0; t_0, \mathbf{H}) = \mathbf{H}$.

Applying Theorem 2.7.5 and Theorem 3.2.2 (a) from [4], we obtain:

Proposition 2. *If the nominal system (4) is ESMS–C, then, for each bounded and continuous function* $\mathbf{H} : \mathbb{R}_+ \to \mathcal{S}_n^N$, *the backward Lyapunov differential equation (BLDE),*

$$\dot{\mathbf{X}}(t) + \mathcal{L}^*(t)[\mathbf{X}(t)] + \mathbf{H}(t) = 0,$$

has a unique solution $\tilde{\mathbf{X}}(\cdot) = (\tilde{X}(\cdot, 1), \cdots, \tilde{X}(\cdot, N))$, *which is bounded on* $\mathbb{R}_+$. *This solution has the representation*

$$\tilde{\mathbf{X}}(t) = \int_t^{\infty} \mathbf{T}^*(s, t)[\mathbf{H}(s)]ds, \ t \in \mathbb{R}_+.$$

Furthermore, $\tilde{\mathbf{X}}(t) \geq 0$ *if* $\mathbf{H}(t) \geq 0, t \in \mathbb{R}_+$.

3.2. The Scaling of the Uncertainties

The system (1) may be rewritten into an equivalent form:

$$dx(t) = A(t, \eta_t)x(t)dt + \sum_{k=1}^{r} \frac{1}{\alpha_k(\eta_t)} D_k(t, \eta_t) \Delta_k^{\alpha}(t, z_k^{\alpha}(t), \eta_t)dw_k(t) \tag{21a}$$

$$z_k^{\alpha}(t) = \alpha_k(\eta_t)E_k(t, \eta_t)x(t), \tag{21b}$$

where $\alpha_k(\eta_t) \in (0, \infty)$ and $\Delta_k^\alpha(t, \cdot, i)$ is defined by

$$\Delta_k^\alpha(t, z_k^\alpha(t), i) \triangleq \alpha_k(i)\Delta_k(t, \alpha_k^{-1}(i)z_k, i) \tag{22}$$

for all $(t, z_k, i) \in \mathbb{R}_+ \times \mathbb{R}^{\nu_k} \times \mathcal{N}$.

From (3) and (22), one sees that for any vector of scaling parameters $\alpha = (\alpha_k(i))_{(k,i)\in\underline{r}\times\mathcal{N}} \in (0, \infty)^{\underline{r}\times\mathcal{N}}$, we have that $\Delta_k^\alpha(\cdot, \cdot, i)$ satisfy the assumption of Hypothesis 2, if $\Delta_k(\cdot, \cdot, i)$ satisfy Hypothesis 2.

Moreover, from (8) it follows that

$$|||\underline{\Delta}_k^\alpha(\cdot, \cdot)||| = |||\underline{\Delta}_k(\cdot, \cdot)|||, \tag{23}$$

where $\underline{\Delta}_k^\alpha(\cdot, \cdot) = \left(\Delta_k^\alpha(\cdot, \cdot, 1), \Delta_k^\alpha(\cdot, \cdot, 2), \cdots, \Delta_k^\alpha(\cdot, \cdot, N)\right)$.

We set $\mathbf{\Delta}^\alpha(\cdot, \cdot) = (\underline{\Delta}_1^\alpha(\cdot, \cdot), \underline{\Delta}_2^\alpha(\cdot, \cdot), \cdots, \underline{\Delta}_N^\alpha(\cdot, \cdot))$. From (10) and (23), we obtain that

$$\|\mathbf{\Delta}^\alpha(\cdot, \cdot)\| = \|\mathbf{\Delta}(\cdot, \cdot).\| \tag{24}$$

Let us consider the BLDE,

$$\dot{\mathbf{X}}(t) + \mathcal{L}^*(t)[\mathbf{X}(t)] + \mathcal{E}^\alpha(t) = 0 \tag{25a}$$

$$\mathcal{E}^\alpha(t) = (\mathcal{E}^\alpha(t, 1), \cdots, \mathcal{E}^\alpha(t, N)) \tag{25b}$$

$$\mathcal{E}^\alpha(t, i) = \sum_{k=1}^{r} \alpha_k^2(i)E_k^T(t, i)E_k(t, i); \tag{25c}$$

$\mathcal{L}^*(t)$ being the operator valued function defined by (19).

Specializing the statement of Proposition 2 to the case of BLDE (25), we obtain:

Proposition 3. *Suppose that the following conditions are satisfied:*

(a) *The assumption* $(\mathbf{H}_1)$ *is fulfilled;*

(b) *The nominal system (4) is ESMS–C;*

Under these conditions, for each vector of scaling parameters $\alpha = \{\alpha_k(i)\}_{(k,i)\in\underline{r}\times\mathcal{N}}$, $\alpha_k(i) > 0$, *the BLDE (25) has a unique solution* $\mathbf{X}^\alpha(\cdot) = (X^\alpha(\cdot, 1), \cdots X^\alpha(\cdot, N))$, *which is bounded on* $\mathbb{R}_+$ *and* $X^\alpha(t, i) \geqslant 0$, $\forall(t, i) \in \mathbb{R}_+ \times \mathcal{N}$. *Moreover, the dependence with respect to the scaling parameters* $\alpha_k(i)$ *of the solution* $\mathbf{X}^\alpha(\cdot)$ *is described by*

$$\mathbf{X}^\alpha(t) = \sum_{l=1}^{r}\sum_{i=1}^{N} \alpha_l^2(i)\mathbf{X}_{li}(t). \tag{26}$$

$t \in \mathbb{R}_+$, *where for each* $(l, i) \in \underline{r} \times \mathcal{N}$, $\mathbf{X}_{li}(\cdot) = (X_{li}(\cdot, 1), \cdots, X_{li}(\cdot, N))$ *is the unique bounded solution of the following BLDE:*

$$\dot{\mathbf{X}}_{li}(t) + \mathcal{L}^*(t)[\mathbf{X}_{li}(t)] + \mathcal{E}_{li}(t) = 0 \tag{27a}$$

$$\mathcal{E}_{li}(t) = (\mathcal{E}_{li}(t, 1), \cdots, \mathcal{E}_{li}(t, N)) \tag{27b}$$

$$\mathcal{E}_{li}(t, j) = \begin{cases} 0 \text{ if } j \neq i \\ E_l^T(t, i)E_l(t, i) \text{ if } j = i. \end{cases} \tag{27c}$$

4. The Main Results

4.1. A Lower Bound of the Stability Radius

In the sequel, we shall use the notation $(0, \infty)^{\underline{r}\times\mathcal{N}}$ and the set of the vectors $\alpha = (\alpha_{ki})_{(k,i)\in\underline{r}\times\mathcal{N}}$, $\alpha_{ki} > 0$.

The next Theorem provides a lower bound of the stability radius in terms of the unique bounded and positive semi-definite solutions of the BLDE (25).

Theorem 4. *Assume:*

(a) *The assumptions* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold true;*

(b) *The nominal system* (4) *is ESMS–C.*

Let $\sigma > 0$ be given. If there exists $\alpha \in (0, \infty)^{\underline{r} \times \mathcal{N}}$ such that

$$\max_{k \in \underline{r}} \max_{j \in \mathcal{N}} \sup_{t \in \mathbb{R}_+} \left\{ \left(\frac{\sigma}{\alpha_{kj}} \right)^2 \left\| D_k^T(t, j) X^\alpha(t, j) D_k(t, j) \right\| \right\} < 1, \tag{28}$$

where $\mathbf{X}^\alpha(\cdot) = (X^\alpha(\cdot, 1), \cdots, X^\alpha(\cdot, N))$ *is the unique bounded solution of the BLDE* (25) *associated to the parameter* α, *then*

$$\rho_{\mathcal{D}}\big(\mathbf{A}(\cdot), Q \big| (D_k(\cdot, i), E_k(\cdot, i)), (k, i) \in \underline{r} \times \mathcal{N}\big) \geqslant \sigma. \tag{29}$$

Proof. Let $\mathbf{\Delta}(\cdot, \cdot) = (\underline{\Delta}_1(\cdot, \cdot), \cdots, \underline{\Delta}_r(\cdot, \cdot)) \in \mathcal{D}$ be an arbitrary admissible uncertainty satisfying

$$\|\mathbf{\Delta}(\cdot, \cdot)\| \leqslant \sigma. \tag{30}$$

We show that the corresponding system $\mathcal{G}_\mathbf{\Delta}$ is GESMS–C. Invoking Proposition 3, we deduce that, under the considered assumptions, the BLDE (25) has a unique bounded on $\mathbb{R}_+$ solution $\mathbf{X}^\alpha(\cdot)$, and this solution is positive semidefinite, that is, $\mathbf{X}^\alpha(t) \in \mathcal{S}_n^{N+}$, $\forall t \in \mathbb{R}_+$. Applying an Itô type formula (see, for example, Theorem 1.10.2 from [4]) in the case of the function $V(t, x, i) \triangleq x^T X^\alpha(t, i) x$, along the trajectories of the equivalent version (20) of the system $\mathcal{G}_\mathbf{\Delta}$, we obtain

$$\mathbb{E}\big[V(t, x(t), \eta_t) | \eta_{t_0} = i\big] = \mathbb{E}\big[V(t_0, x_0, \eta_{t_0}) | \eta_{t_0} = i\big]$$
$$+ \int_{t_0}^t \mathbb{E}\Big[x^T(s)\Big(\dot{X}^\alpha(s, \eta_s) + \mathcal{L}(s)[X^\alpha(s)](\eta_s)\Big)x(s) | \eta_{t_0} = i\Big] ds \tag{31}$$
$$+ \sum_{k=1}^r \int_{t_0}^t \mathbb{E}\left[\frac{1}{\alpha_k^2(\eta_s)} \big(\Delta_k^\alpha(s, z_k^\alpha(s), \eta_s)\big)^T D_k^T(s, \eta_s) X^\alpha(s, \eta_s) D_k(s, \eta_s) \Delta_k^\alpha(s, z_k^\alpha(s), \eta_s) | \eta_{t_0} = i\right] ds.$$

$\forall t \geqslant t_0 \geqslant 0$, $i \in \mathcal{N}$, $x_0 \in \mathcal{X}_{t_0}$ and $x(s) = x(s; t_0, x_0)$ is the solution of (1) or, equivalently, of its version (21). Employing (21b) and (25), we rewrite the first integral from the right hand side of (31) as

$$\int_{t_0}^t \mathbb{E}\Big[x^T(s)\Big(\dot{X}^\alpha(s, \eta_s) + \mathcal{L}(s)[X^\alpha(s)](\eta_s)\Big)x(s) | \eta_{t_0} = i\Big] ds = -\sum_{k=1}^r \int_{t_0}^t \mathbb{E}\Big[|z_k^\alpha(s)|^2 | \eta_{t_0} = i\Big] ds. \tag{32}$$

On the other hand, (9) and (10) allow us to obtain

$$\int_{t_0}^t \mathbb{E}\left[\frac{1}{\alpha_k^2(\eta_s)} \big(\Delta_k^\alpha(s, z_k^\alpha(s), \eta_s)\big)^T D_k^T(s, \eta_s) X^\alpha(s, \eta_s) D_k(s, \eta_s) \Delta_k^\alpha(s, z_k^\alpha(s), \eta_s) | \eta_{t_0} = i\right] ds$$
$$\leqslant \int_{t_0}^t \mathbb{E}\left[\frac{1}{\alpha_k^2(\eta_s)} \| D_k^T(s, \eta_s) X^\alpha(s, \eta_s) D_k(s, \eta_s) \| |\Delta_k^\alpha(s, z_k^\alpha(s), \eta_s)|^2 | \eta_{t_0} = i\right] ds \tag{33}$$
$$\leqslant \max_{j \in \mathcal{N}} \sup_{s \in \mathbb{R}_+} \left(\frac{\|\mathbf{\Delta}(\cdot, \cdot)\|}{\alpha_k(j)} \right)^2 \cdot \| D_k^T(s, j) X^\alpha(s, j) D_k(s, j) \| \cdot \int_{t_0}^t \mathbb{E}\Big[|z_k^\alpha(s)|^2 | \eta_{t_0} = i\Big] ds.$$

Substituting (32) and (33) in (31), we obtain (from (24) and (30)):

$$
\mathbb{E}\big[V(t, x(t), \eta_t)\big|\eta_{t_0} = i\big] \leqslant \mathbb{E}\big[V(t_0, x_0, i)\big|\eta_{t_0} = i\big]
$$
$$
+ \left\{ \sum_{k=1}^{r} \max_{j \in \mathcal{N}} \sup_{s \in \mathbb{R}_+} \left[\left(\frac{\sigma}{\alpha_k(j)} \right)^2 \|D_k^T(s, j) X^\alpha(s, j) D_k(s, j)\| \right] - 1 \right\}
$$
$$
\times \int_{t_0}^{t} \mathbb{E}\big[|z_k^\alpha(s)|^2\big|\eta_{t_0} = i\big] ds.
$$

Setting $\xi = \max\limits_{k \in \underline{r}} \max\limits_{j \in \mathcal{N}} \sup\limits_{s \in \mathbb{R}_+} \left\{ \left(\frac{\sigma}{\alpha_k(j)} \right)^2 \|D_k^T(s, j) X^\alpha(s, j) D_k(s, j)\| \right\}$, we get

$$
(1 - \xi) \sum_{k=1}^{r} \int_{t_0}^{t} \mathbb{E}\big[|z_k^\alpha(s)|^2\big|\eta_{t_0} = i\big] ds \leqslant \mathbb{E}\big[x_0^T X^\alpha(t_0, i) x_0\big|\eta_{t_0} = i\big], \tag{34}
$$

$\forall t \geqslant t_0 \geqslant 0$, $i \in \mathcal{N}$, $x_0 \in \mathcal{X}_{t_0}$.

Since $\mathbf{X}^\alpha(\cdot)$ is unique bounded on $\mathbb{R}_+$ and a positive semi-definite solution of the BLDE (25), we deduce that there exists $\mu > 0$ such that $0 \leqslant X^\alpha(t, i) \leqslant \mu \mathbb{I}_n$; $\forall (t, i) \in \mathbb{R}_+ \times \mathcal{N}$. This inequality, together with (28) and (34), allow us to infer that:

$$
0 \leqslant \sum_{k=1}^{r} \int_{t_0}^{t} \mathbb{E}\big[|z_k^\alpha(s)|^2\big|\eta_{t_0} = i\big] ds \leqslant c_1 \mathbb{E}\big[|x_0|^2\big|\eta_{t_0} = i\big],
$$

where $c_1 = \mu(1 - \xi)^{-1} > 0$ does not depend upon t, t_0, i, x_0.

Letting $t \to \infty$, we obtain

$$
0 \leqslant \sum_{k=1}^{r} \int_{t_0}^{\infty} \mathbb{E}\big[|z_k^\alpha(s)|^2\big|\eta_{t_0} = i\big] ds \leqslant c_1 \mathbb{E}\big[|x_0|^2\big|\eta_{t_0} = i\big], \tag{35}
$$

$\forall t_0 \geqslant 0$, $i \in \mathcal{N}$, $x_0 \in \mathcal{X}_{t_0}$.

Further, from (9), (10), (24), (30) and (35), we deduce that

$$
\int_{t_0}^{\infty} \mathbb{E}\big[|\Delta_k(s, z_k(s), \eta_s)|^2\big|\eta_{t_0} = i\big] ds \leqslant c_2 \mathbb{E}\big[|x_0|^2\big|\eta_{t_0} = i\big], \tag{36}
$$

$\forall t_0 \geqslant 0$, $i \in \mathcal{N}$, $x_0 \in \mathcal{X}_{t_0}$ and $c_2 = \sigma c_1$ does not depend upon t_0, x_0, i.

Applying Theorem 3.6.1 (ii) from [4] in the case of system (1), taking $f_0(t) = 0$, $f_k(t) = D_k(t, \eta_t)\Delta_k(t, z_k(t), \eta_t)$, $1 \leqslant k \leqslant r$, one gets

$$
\int_{t_0}^{\infty} \mathbb{E}\big[|x(t; t_0, x_0)|^2\big|\eta_{t_0} = i\big] dt \leqslant \tilde{c} \mathbb{E}\big[|x_0|^2\big|\eta_{t_0} = i\big],
$$

$\forall t_0 \geqslant 0$, $i \in \mathcal{N}$, $x_0 \in \mathcal{X}_{t_0}$, $\tilde{c} > 0$ being a constant. Thus, we have shown that, for any admissible uncertainty $\mathbf{\Delta}(\cdot, \cdot)$ satisfying (30), the corresponding system $\mathcal{G}_{\mathbf{\Delta}}$ is GSSM–C. Finally, applying Theorem 8.3.7 from [4], we obtain that the zero solution of $\mathcal{G}_{\mathbf{\Delta}}$ is GESMS–C. This means that (29) is satisfied. The proof is complete. □

The next result provides a lower bound of the stability radius of the nominal system (4) with respect to the stochastic structured uncertainties of type (5).

Theorem 5. *Assume that the assumptions of Theorem 4 are fulfilled. Then,*

$$
\rho_{\mathcal{D}}\big(\mathbf{A}(\cdot), Q\big|(D_k(\cdot, i), E_k(\cdot, i)), (k, i) \in \underline{r} \times \mathcal{N}\big) \geqslant \tilde{\rho}, \tag{37}
$$

where

$$\tilde{\rho} \triangleq \sup_{\alpha \in (0,\infty)^{\underline{r} \times \mathcal{N}}} \left[\max_{(k,j) \in \underline{r} \times \mathcal{N}} \sup_{t \in \mathbb{R}_+} \alpha_k^{-2}(j) \| D_k^T(t,j) X^\alpha(t,j) D_k(t,j) \| \right]^{-\frac{1}{2}}, \tag{38}$$

$\mathbf{X}^\alpha(\cdot) = (X^\alpha(\cdot, 1), \cdots, X^\alpha(\cdot, N))$ *being the unique bounded and positive semi-definite solution of the BLDE (25) associated to the vector of scaling parameters* $\alpha = (\alpha_k(j))_{(k,j) \in \underline{r} \times \mathcal{N}}$.

Proof. Let us assume by contradiction that (37) is not true. Hence, $\rho_{\mathcal{D}} \big(\mathbf{A}(\cdot), Q \big| (D_k(\cdot, i), E_k(\cdot, i)), (k,i) \in \underline{r} \times \mathcal{N} \big) < \tilde{\rho}$. Let $\sigma > 0$ be such that

$$\rho_{\mathcal{D}} \big(\mathbf{A}(\cdot), Q \big| (D_k(\cdot, i), E_k(\cdot, i)), (k,i) \in \underline{r} \times \mathcal{N} \big) < \sigma < \tilde{\rho}. \tag{39}$$

From (38), we deduce that there exists $\hat{\alpha} = (\hat{\alpha}_k(j))_{(k,j) \in \underline{r} \times \mathcal{N}}$, $\hat{\alpha}_k(j) > 0$ such that

$$\left(\max_{(k,j) \in \underline{r} \times \mathcal{N}} \sup_{t \in \mathbb{R}_+} \hat{\alpha}_k^{-2}(j) \| D_k^T(t,j) X^{\hat{\alpha}}(t,j) D_k(t,j) \| \right)^{-\frac{1}{2}} > \sigma,$$

or equivalently

$$\max_{k \in \underline{r}} \max_{j \in \mathcal{N}} \sup_{t \in \mathbb{R}_+} \left(\frac{\sigma}{\hat{\alpha}_k(j)} \right)^2 \| D_k^T(t,j) X^{\hat{\alpha}}(t,j) D_k(t,j) \| < 1,$$

which is just (28) with α replaced by $\hat{\alpha}$.

From Theorem 4, we obtain that $\rho_{\mathcal{D}} \big(\mathbf{A}(\cdot), Q \big| (D_k(\cdot, i), E_k(\cdot, i)), (k,i) \in \underline{r} \times \mathcal{N} \big) \geqslant \sigma$, which contradicts (39). So, (37) holds true. This completes the proof. □

Besides BLDE (25), we may consider the following two kinds of backward Lyapunov differential inequalities (BLDIs) on $\mathcal{S}_n^N$:

$$\dot{\mathbf{Y}}(t) + \mathcal{L}^*(t)[\mathbf{Y}(t)] + \mathcal{E}^\alpha(t) \leq 0, \ t \in \mathbb{R}_+ \tag{40}$$

and

$$\dot{\mathbf{Y}}(t) + \mathcal{L}^*(t)[\mathbf{Y}(t)] + \mathcal{E}^\alpha(t) \ll 0, \ t \in \mathbb{R}_+. \tag{41}$$

We recall that if $\mathbf{H}(\cdot) : [a, \ b] \subset \mathbb{R}_+ \to \mathcal{S}_n^N$, we shall say that $\mathbf{H}(\cdot)$ is uniform negative and we shall write $\mathbf{H}(t) \ll 0$, $t \in [a, \ b]$, if there exists $\epsilon > 0$ such that $H(t,i) \leqslant -\epsilon \mathbb{I}_n$ (i.e., $H(t,i) + \epsilon \mathbb{I}_n$ is negative semi-definite), $\forall (t,i) \in [a, \ b] \times \mathcal{N}$. We will say that $\mathbf{H}(\cdot)$ is uniform positive on $[a, \ b]$ and we shall write $\mathbf{H}(t) \gg 0$, $t \in [a, \ b]$ iff $-\mathbf{H}(\cdot)$ is uniform negative on $[a, \ b]$.

Regarding the solutions of the BLDIs (40), (41), we have the following Lemma:

Lemma 6. *Under the assumptions Hypotheses 1b and 1c the following hold:*

(i)　the nominal system (4) is ESMS–C if and only if the BLDI (41) has at least a uniform positive and bounded solution on $\mathbb{R}_+$;

(ii)　if the nominal system (4) is ESMS–C, then any bounded solution $\mathbf{Y}^\alpha(\cdot)$ *of the BLDIs (40) and (41) satisfies:* $\mathbf{Y}^\alpha(t) \geq \mathbf{X}^\alpha(t)$, $\forall t \in \mathbb{R}_+$, *where* $\mathbf{X}^\alpha(\cdot)$ *is the unique bounded on* $\mathbb{R}_+$ *solution of the BLDE (25).*

Proof.

(i)　It follows from applying Theorem 3.3.4 from [4] in the case of the nominal system (4).

(ii)　Is obtained by subtracting (25) from (40) and using Proposition 2 to conclude that $\mathbf{Y}^\alpha(t) - \mathbf{X}^\alpha(t) \geq 0$.
　　□

Now, we provide a lower bound of the stability radius of the nominal system (4) in terms of the solutions of the BLDIs (40) and (41).

Corollary 7.

(a)   *Assume that the assumptions of Theorem 4 are fulfilled. Let $\sigma > 0$ be given. If there exists a vector of scaling parameters $\alpha \in (0, \infty)^{\underline{r} \times \mathcal{N}}$ and a bounded solution $\mathbf{Y}^\alpha(\cdot)$ of the corresponding BLDI (40), satisfying the condition*

$$\max_{k \in \underline{r}} \max_{j \in \mathcal{N}} \sup_{t \in \mathbb{R}_+} \left(\frac{\sigma}{\alpha_k(j)} \right)^2 \| D_k^T(t, j) Y^\alpha(t, j) D_k(t, j) \| < 1, \tag{42}$$

then

$$\rho_{\mathcal{D}}\big(\mathbf{A}(\cdot), Q \big| (D_k(\cdot, i), E_k(\cdot, i)), (k, i) \in \underline{r} \times \mathcal{N}\big) \geqslant \sigma. \tag{43}$$

(b)   *Assume that the assumptions of Hypotheses 1a–1c is fulfilled. If there exists a vector of scaling parameters $\alpha \in (0, \infty)^{\underline{r} \times \mathcal{N}}$ and a bounded and uniform positive on $\mathbb{R}_+$ solution $\mathbf{Y}^\alpha(\cdot)$ of the BLDI (41) satisfying a condition of type (42), then*

   (i)   *the nominal system (4) is ESMS–C;*
   (ii)   *the stability radius of the nominal system (4) with respect to the stochastic disturbance of type (5) satisfy a condition of type (43).*

Proof.

(a)   Based on Lemma 6 (ii), we deduce that (28) holds if (42) is true, because the spectral norm on the space $\mathcal{S}_n^N$ is monotone with respect to the convex cone $\mathcal{S}_n^{N+}$. The conclusion follows from applying Theorem 4.
(b)   The fact that the nominal system is ESMS–C is obtained from Lemma 6 (i). The part (ii) is obtained in the same way as in the proof of (a) from above. Thus the proof ends.

   $\square$

4.2. Robust Stabilization via a State Feedback

   Let us consider the controlled system,

$$\dot{x}(t) = A(t, \eta_t) x(t) + B(t, \eta_t) u(t), \tag{44}$$

where $x(t) \in \mathbb{R}^n$ are the states of the system at time t and $u(t) \in \mathbb{R}^{n_u}$ is the vector of the control parameters. If the system (44) is subject to the parametric uncertainties of type (5), it takes the form

$$dx(t) = A(t, \eta_t) x(t) dt + \sum_{k=1}^{r} D_k(t, \eta_t) \Delta_k(t, z_k(t), \eta_t) dw_k(t) + B(t, \eta_t) u(t) dt \tag{45a}$$

$$z_k(t) = E_k(t, \eta_t) x(t), \tag{45b}$$

$k \in \underline{r}$. In both (44) and (45) $t \to B(t, i) : \mathbb{R}_+ \to \mathbb{R}^{n \times n_u}$, $i \in \mathcal{N}$ are bounded and continuous matrix valued functions. The other matrix valued functions involved in (44) and (45) are satisfying the assumptions of Hypotheses 1a–1c and Hypothesis 2.

   Our aim is to find conditions that guarantee the existence of a control of type:

$$u(t) = F(t, \eta_t) x(t), \tag{46}$$

which stabilizes the nominal system (44) together with all perturbed systems of type (45), corresponding to the admissible uncertainties $\mathbf{\Delta}(\cdot, \cdot) \in \mathcal{D}$ satisfying $\|\mathbf{\Delta}(\cdot, \cdot)\| < \sigma$, where $\sigma > 0$ is a prescribed level of robustness of the stabilization achieved by the designed control.

   An answer to the problem stated here is given in the following theorem:

Theorem 8. *Let $\sigma > 0$ be given. Assume:*

(a)   *There exist $\mathcal{C}^1$-matrix valued functions $t \to Z(t, i) : \mathbb{R}_+ \to \mathcal{S}_n$, which are bounded with a bounded derivative;*

(b)   *There exist bounded and continuous matrix valued functions $t \to K(t,i) : \mathbb{R}_+ \to \mathbb{R}^{n_u \times n}$;*

(c)   *There exist positive scalars $\zeta_k(i)$, $\xi_k(i)$, $\epsilon(i)$, $k \in \underline{r}$, $i \in \mathcal{N}$, satisfying the following system of linear matrix inequalities (LMIs):*

$$
\begin{pmatrix}
\Theta_1(t, Z(t,i), K(t,i), i) & Z(t,i)\Xi(t,i) & Z(t,i)\Gamma(t,i) \\
\star & -\Theta_2(\zeta_1(i), \cdots, \zeta_r(i)) & 0 \\
\star & \star & -\Theta_3(Z_{-i}(t))
\end{pmatrix} + \epsilon(i)\mathbb{I}_{\tilde{n}} \leqslant 0 \quad \text{(47a)}
$$

$$
\begin{pmatrix}
\zeta_k(i)\mathbb{I}_{m_k} & \sigma\zeta_k(i)D_k^T(t,i) \\
\star & Z(t,i)
\end{pmatrix} - \xi_k(i)\mathbb{I}_{n+m_k} \geqslant 0 \quad \text{(47b)}
$$

$t \in \mathbb{R}_+$, $k \in \underline{r}$, $i \in \mathcal{N}$, where we denoted

$$
\begin{aligned}
\Theta_1(t, Z(t,i), K(t,i), i) &= -\dot{Z}(t,i) + A(t,i)Z(t,i) + Z(t,i)A^T(t,i) + B(t,i)K(t,i) \\
&\quad + K^T(t,i)B(t,i) + q_{ii}\mathbb{I}_n \in \mathcal{S}_n
\end{aligned} \quad \text{(48a)}
$$

$$
\Theta_2(\zeta_1(i), \cdots, \zeta_r(i)) = diag[\zeta_1(i)\mathbb{I}_{\nu_1}, \zeta_2(i)\mathbb{I}_{\nu_2}, \cdots, \zeta_r(i)\mathbb{I}_{\nu_r}] \in \mathcal{S}_\nu \quad \text{(48b)}
$$

$$
\Theta_3(Z_{-i}(t)) = diag[Z(t,1), \cdots, Z(t,i-1), Z(t,i+1), \cdots, Z(t,N)] \in \mathcal{S}_{n(N-1)} \quad \text{(48c)}
$$

$$
\Xi(t,i) = \left(E_1^T(t,i)\ E_2^T(t,i)\ \cdots\ E_r^T(t,i) \right) \in \mathbb{R}^{n \times \nu} \quad \text{(48d)}
$$

$$
\Gamma(t,i) = \left(q_{i1}^{\frac{1}{2}}\mathbb{I}_n\ \cdots\ q_{ii-1}^{\frac{1}{2}}\mathbb{I}_n\ q_{ii+1}^{\frac{1}{2}}\mathbb{I}_n\ \cdots\ q_{iN}^{\frac{1}{2}}\mathbb{I}_n \right) \in \mathbb{R}^{n \times n(N-1)} \quad \text{(48e)}
$$

$\nu = \nu_1 + \cdots + \nu_r$, $\tilde{n} = \nu + nN$.

Under these conditions, the control (46), with the gain matrix defined by $F(t,i) = K(t,i)Z^{-1}(t,i)$ robustly stabilizes the nominal system (44). The stability radius of the closed-loop nominal system,

$$
\dot{x}(t) = (A(t, \eta_t) + B(t, \eta_t)F(t, \eta_t))x(t), \quad \text{(49)}
$$

satisfies

$$
\rho_{\mathcal{D}}\left(\mathbf{A}(\cdot) + \mathbf{B}(\cdot)\mathbf{F}(\cdot), Q | (D_k(\cdot,i), E_k(\cdot,i))_{(k,i) \in \underline{r} \times \mathcal{N}} \right) \geqslant \sigma. \quad \text{(50)}
$$

Proof. Pre and post multiplying (47a) by $\mathcal{U}(t,i) = \text{diag}[Z^{-1}(t,i), \mathbb{I}_\nu, \mathbb{I}_{n(N-1)}]$ and using the Schur complement technique, we obtain that $Y(t,i) = Z^{-1}(t,i)$ solves the following BLDI

$$
\dot{Y}(t,i) + (A(t,i) + B(t,i)F(t,i))^T Y(t,i) + Y(t,i)(A(t,i) + B(t,i)F(t,i)) + \sum_{j=1}^N q_{ij}Y(t,j)
$$

$$
+ \sum_{k=1}^r \zeta_k^{-1}(i)E_k^T(t,i)E_k(t,i) + \frac{\epsilon(i)}{2}\mathbb{I}_n \leqslant 0. \quad \text{(51)}
$$

At the same time, the Schur complement technique used in the case of (47b) allows us to deduce that:

$$
\mathbb{I}_{m_k} - \sigma^2\zeta_k(i)D_k^T(t,i)Y(t,i)D_k(t,i) - \hat{\xi}_k(i)\mathbb{I}_{m_k} \geqslant 0, \quad \text{(52)}
$$

where we have denoted $\hat{\xi}_k(i) = \frac{\xi_k(i)}{\zeta_k(i)}$. Rewriting (52) in the equivalent form,

$$
D_k^T(t,i)Y(t,i)D_k(t,i) \leqslant \frac{1 - \hat{\xi}_k(i)}{\sigma^2\zeta_k(i)}\mathbb{I}_{m_k},
$$

we may obtain, via the definition of the spectral norm of a symmetric and positive semidefinite matrix, that:

$$
\|D_k^T(t,i)Y(t,i)D_k(t,i)\| \leqslant \frac{1 - \hat{\xi}_k(i)}{\sigma^2\zeta_k(i)}
$$

for all $t \in \mathbb{R}_+, (k, i) \in \underline{r} \times \mathcal{N}$. The previous inequality yields

$$\sigma^2 \zeta_k(i) \| D_k^T(t,i) Y(t,i) D_k(t,i) \| < 1 - \frac{1}{2} \check{\xi}$$

where $\check{\xi} = min_{(k,i) \in \underline{r} \times \mathcal{N}} \, hat\xi_k(i)$. This allows us to infer that

$$\max_{k \in \underline{r}} \max_{i \in \mathcal{N}} \sup_{t \in \mathbb{R}_+} \sigma^2 \zeta_k(i) \| D_k^T(t,i) Y(t,i) D_k(t,i) \| < 1. \tag{53}$$

One sees that if we take $\alpha_k(i) := \zeta_k^{-\frac{1}{2}}(i)$, then (51) is of type (41), while (53) is of type (42). Hence, by applying Corollary 7 (b) in the case of the nominal system (49), we deduce that this system is ESMS–C and its stability radius satisfies (50). Thus the proof is completed. $\quad\square$

Remark 4.

(a)　*It is worth mentioning that if the matrix valued functions, which are involved as coefficients of (47), are periodic functions, and if (47) has a solution $\mathbf{Z}(\cdot) = (Z(\cdot, 1), \cdots, Z(\cdot, N))$ and $\mathbf{K}(\cdot) = (K(\cdot, 1), \cdots, K(\cdot, N))$ with $\mathbf{Z}(t) \gg 0$, $t \in \mathbb{R}_+$, then (47) also has a solution $\tilde{\mathbf{Z}}(\cdot)$, $\tilde{\mathbf{K}}(\cdot)$, which is periodic with the same period as the coefficients of (47);*

(b)　*Since the constant functions can be regarded as periodic functions of an arbitrary period, one obtains that, if the coefficients of (47) do not depend upon t, then (47) has a constant solution $\mathbf{Z} = (Z(1), \cdots, Z(N))$ and $\mathbf{K} = (K(1), \cdots, K(N))$ with $Z(i) > 0$, if it is solvable. Hence, without loss of generality, in the periodic case to test the solvability of (47) we look for a periodic solution with the same period as the coefficients. Moreover, if the coefficients of (47) do not depend upon t, then we test its solvability, looking for constant solutions.*

5. Numerical Experiments

In this section, we will address the problem of robust stabilization described in Section 4.2. We will consider the time-invariant case. This will simplify the presentation and we believe that the conclusions from this section still hold in the time-varying case. The main objective here is to show how the extra degree of freedom provided by the *scaling* technique allows the improvement of the estimation of the lower bound of the stability radius when compared to a *non-scaling* one. Let σ^* and σ_α^* be the lower bounds corresponding to the non-scaling and the scaling paradigms, respectively. σ^* is obtained by solving the following maximization problem under LMIs constraints:

$$\begin{cases} \max\limits_{Z(i) \in \mathcal{S}_n^+, \, K(i) \in \mathbb{R}^{n_u \times n}} \sigma \\ \text{s.t.} \\ \begin{pmatrix} \Theta_1(Z(i), K(i), i) & Z(i)\Xi(i) & Z(i)\Gamma(i) \\ \star & -\Theta_2(1, \cdots, 1) & \mathbf{0} \\ \star & \star & -\Theta_3(Z_{-i}) \end{pmatrix} < 0 \\ \begin{pmatrix} \mathbb{I}_{m_k} & \sigma D_k^T(i) \\ \star & Z(i) \end{pmatrix} > 0, \end{cases} \tag{54}$$

$i \in \mathcal{N}$, while σ_α^* is computed using Theorem 8.

We have randomly generated 100 numerical examples and computed for each example the parameter $\rho^* = \frac{\sigma_\alpha^*}{\sigma^*}$. The obtained results are illustrated by Figure 1. As one expected, one clearly sees the advantage of the scaling technique paradigm.
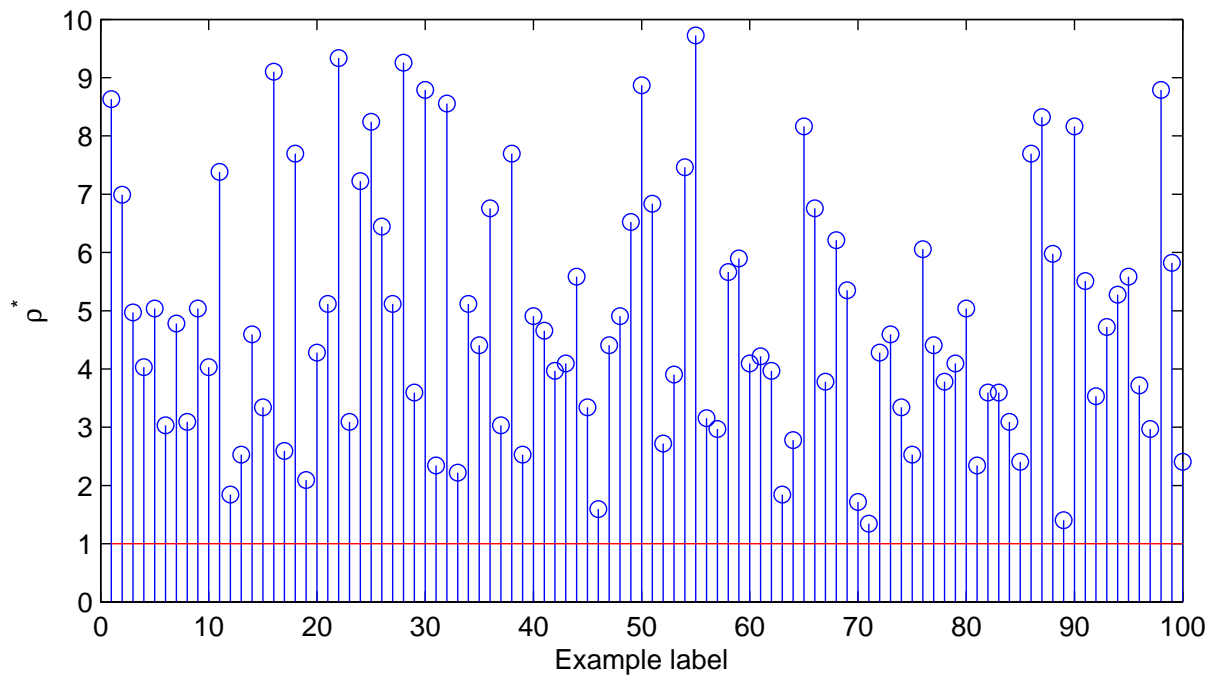
Figure 1. Plot of the quantity $\rho^* = \frac{\sigma_a^*}{\sigma^*}$.

6. Conclusions

In this paper, we have addressed the problem of a robust stability analysis of a class of continuous-time Markovian jump linear systems subject to block-diagonal stochastic parameter perturbations. The multi-perturbations case has been efficiently tackled by using scaling techniques. As a robustness metric, we used the concept of stability radius and we obtained an estimation of its lower bound. A first characterization of this lower bound is obtained in terms of solutions of adequately defined parameterized backward Lyapunov differential equations. Then, a second characterization was obtained in terms of solutions of a class of parameterized backward Lyapunov differential inequalities. Based on this result, we solved, in a second step, a state-feedback robust stabilization problem under a convex optimization paradigm.

Our ongoing efforts are devoted, on one hand, to the problem of output–feedback robust stabilization of the considered class of stochastic systems. In this case, only output measurements are available instead of the whole state variables. The resulting robust controller will be in a dynamical/static output–feedback form. Some recent works on the deterministic framework [19–21] offer some potential directions to be explored and adapted to our setting. On the other hand, we are interested by the generalization of our results to a more general setting. More specifically, we would like to consider a class of MJLSs with the state space of the underlying Markov chain being countably infinite or being some local compact topological space.

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