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A Tseng-Type Algorithm with Self-Adaptive Techniques for Solving the Split Problem of Fixed Points and Pseudomonotone Variational Inequalities in Hilbert Spaces

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Abstract: In this paper, we survey the split problem of fixed points of two pseudocontractive operators and variational inequalities of two pseudomonotone operators in Hilbert spaces. We present a Tseng-type iterative algorithm for solving the split problem by using self-adaptive techniques. Under certain assumptions, we show that the proposed algorithm converges weakly to a solution of the split problem. An application is included.

Keywords: split problem; fixed point; variational inequality; pseudomonotone; pseudocontractive operator

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1. Introduction

In this paper, we survey the variational inequality (in short, VI(C,ϕ)) of seeking an element $p^\dagger \in C$ such that

$$\langle \phi(p^\dagger), x - p^\dagger \rangle, \text{ for all } x \in C, \tag{1}$$

where C is a nonempty closed convex set in a real Hilbert space H, $\langle \cdot, \cdot \rangle$ means the inner product of H, and $\phi : H \to H$ is a nonlinear operator. Denote by Sol(C,ϕ) the solution set of variational inequality (1).

A host of problems such as optimization problem, saddle point, equilibrium problem, fixed point problem can be converted into the form of variational inequality (1), see [1–12]. Many numerical algorithms have been proposed and developed for solving (1) and related problems, see [13–25] and the references therein. Generally speaking, ϕ should satisfy the following assumptions

- ϕ is strongly monotone, i.e., there exists a positive constant γ such that

$$\langle \phi(u) - \phi(\hat{u}), u - \hat{u} \rangle \geq \gamma \|u - \hat{u}\|^2, \text{ for all } u, \hat{u} \in H. \tag{2}$$

- ϕ is Lipschitz continuous, i.e., there exists a positive constant κ such that

$$\|\phi(u) - \phi(\hat{u})\| \leq \kappa \|u - \hat{u}\|, \text{ for all } u, \hat{u} \in H. \tag{3}$$

In order to abate the restriction (2), Korpelevich's extragradient algorithm ([26]) was proposed in 1976

$$\begin{cases} y^k = \text{proj}_C[x^k - \tau\phi(x^k)], \\ x^{k+1} = \text{proj}_C[x^k - \tau\phi(y^k)], \end{cases} \tag{4}$$

where proj_C denotes the orthogonal projection from H onto C and the step-size τ is in $(0, \frac{1}{\kappa})$.

Extragradient algorithm (4) guarantees the convergence of the sequence $\{x^k\}$ provided ϕ is monotone. Extragradient algorithm and its variant have been investigated extensively, see [27–31]. However, we have to compute (i) twice proj_C at two different points and (ii) two values $\phi(x^k)$ and $\phi(y^k)$. Two important modifications of extragradient algorithm have been made. One is proposed in [32] by Censor, Gibali and Reich and another is the following remarkable algorithm proposed in [33] by Tseng

$$\begin{cases} y^k = \text{proj}_C[x^k - \gamma\phi(x^k)], \\ x^{k+1} = y^k + \gamma[\phi(x^k) - \phi(y^k)], \end{cases} \tag{5}$$

where $\gamma \in (0, \frac{1}{\kappa})$.

On the other hand, if ϕ is not Lipschitz continuous or its Lipschitz constant κ is difficult to estimate, then algorithms (4) and (5) are invalid. To avoid this obstacle, Iusem [34] used a self-adaptive technique without prior knowledge of Lipschitz constant κ of ϕ for solving (1). Some related works on self-adaptive methods for solving (1), please refer to [35–38].

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed and convex subsets of H_1 and H_2, respectively. Let $S\colon C \to C$, $T\colon Q \to Q$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be four nonlinear operators. We consider the classical split problem which is to find a point $x^* \in C$ such that

$$x^* \in \text{Fix}(S) \cap \text{Sol}(C, f) \text{ and } Ax^* \in \text{Fix}(T) \cap \text{Sol}(Q, g), \tag{6}$$

where $\text{Fix}(S) := \{u^\dagger | u^\dagger = Su^\dagger\}$ and $\text{Fix}(T) := \{v^\dagger | v^\dagger = Tv^\dagger\}$ are the fixed point sets of S and T, respectively.

The solution set of (6) is denoted by Γ, i.e.,

$$\Gamma = \{x^* \in \text{Fix}(S) \cap \text{Sol}(C, f), Ax^* \in \text{Fix}(T) \cap \text{Sol}(Q, g)\}.$$

Let f and g be the null operators in C and Q, respectively. Then, the split problem (6) becomes to the split fixed point problem studied in [39,40] which is to find an element point $x^* \in C$ such that

$$x^* \in \text{Fix}(S) \text{ and } Ax^* \in \text{Fix}(T). \tag{7}$$

Let S and T be the identity operators in C and Q, respectively. Then, the split problem (6) becomes to the split variational inequality problem studied in [41] which is to find an element $x^* \in C$ such that

$$x^* \in \text{Sol}(C, f) \text{ and } Ax^* \in \text{Sol}(Q, g). \tag{8}$$

The solution set of (8) is denoted by Γ_1, i.e.,

$$\Gamma = \{x^* \in \text{Sol}(C, f), Ax^* \in \text{Sol}(Q, g)\}.$$

The split problems (6)–(8) have a common prototype that is the split feasibility ([42]) problem of finding a point x^* such that

$$x^* \in C \text{ and } Ax^* \in Q. \tag{9}$$

The split problems have emerged their powerful applications in image recovery and signal processing, control theory, biomedical engineering and geophysics. Some iterative algorithms for solving the split problems have been studied and extended by many scholars, see [43–47].

Motivated by the work in this direction, in this paper, we further survey the split problem (6) in which S and T are two pseudocontractive operators and f and g are two pseudomonotone operators. We present a Tseng-type iterative algorithm for solving the split problem (6) by using self-adaptive techniques. Under certain conditions, we show that the proposed algorithm converges weakly to a solution of the split problem (6).

2. Preliminaries

Let H be a real Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle$ and the induced norm defined by $x \mapsto \|x\| := \sqrt{(x,x)}$. For any $x, x^\dagger \in H$ and constant $\eta \in \mathbb{R}$, we have

$$\|\eta x + (1-\eta)x^\dagger\|^2 = \eta\|x\|^2 + (1-\eta)\|x^\dagger\|^2 - \eta(1-\eta)\|x - x^\dagger\|^2. \tag{10}$$

The symbol "$\rightharpoonup$" denotes the weak convergence and the symbol "$\to$" denotes the strong convergence. Use $\omega_w(u^k)$ to denote the set of all weak cluster points of the sequence $\{u^k\}$, namely, $\omega_w(u^k) = \{u^\dagger : \text{there exists } \{u^{k_i}\} \subset \{u^k\} \text{ such that } u^{k_i} \rightharpoonup u^\dagger \text{ as } i \to \infty\}$.

Recall that an operator $\phi : H \to H$ is said to be

- Pseudomonotone, if

$$\langle \phi(\tilde{x}), x - \tilde{x}\rangle \geq 0 \text{ implies } \langle \phi(x), x - \tilde{x}\rangle \geq 0, \ \forall x, \tilde{x} \in H.$$

- Weakly sequentially continuous, if $H \ni u^k \rightharpoonup \tilde{u}$ implies that $\phi(u^k) \rightharpoonup \phi(\tilde{u})$.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that an operator $S\colon C \to C$ is said to be pseudocontractive if

$$\|S(x) - S(x^\dagger)\|^2 \leq \|x - x^\dagger\|^2 + \|(I-S)x - (I-S)x^\dagger\|^2, \text{ for all } x, x^\dagger \in C.$$

For given $u^\dagger \in H$, there exists a unique point in C, denoted by $\text{proj}_C[u^\dagger]$ such that

$$\|u^\dagger - \text{proj}_C[u^\dagger]\| \leq \|x - u^\dagger\|, \quad \text{for all } x \in C.$$

It is known that proj_C is firmly nonexpansive, that is, proj_C satisfies

$$\|\text{proj}_C[x^*] - \text{proj}_C[x^\dagger]\|^2 \leq \langle \text{proj}_C[x^*] - \text{proj}_C[x^\dagger], x^* - x^\dagger\rangle, \quad \text{for all } x^*, x^\dagger \in H.$$

It is obvious that proj_C is nonexpansive, i.e., $\|\text{proj}_C[x^*] - \text{proj}_C[x^\dagger]\| \leq \|x^* - x^\dagger\|$ for all $x^*, x^\dagger \in H$. Moreover, proj_C satisfies the following inequality ([48])

$$\langle x^* - \text{proj}_C[x^*], x^\dagger - \text{proj}_C[x^*]\rangle \leq 0, \ x^* \in H \text{ and for all } x^\dagger \in C. \tag{11}$$

Lemma 1 ([49])**.** *Let C be a nonempty, convex and closed subset of a Hilbert space H. Assume that the operator $S\colon C \to C$ is pseudocontractive and κ-Lipschitz continuous. Then, for all $\tilde{u} \in C$ and $u^\dagger \in \text{Fix}(S)$, we have*

$$\|u^\dagger - S((1-\alpha)\tilde{u} + \alpha S(\tilde{u}))\|^2 \leq \|\tilde{u} - u^\dagger\|^2 + (1-\alpha)\|\tilde{u} - S((1-\alpha)\tilde{u} + \alpha S(\tilde{u}))\|^2,$$

where α is a constant in $(0, \frac{1}{\sqrt{1+\kappa^2}+1})$.

Lemma 2 ([50])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a continuous and pseudomonotone operator. Then $p^\dagger \in \text{Sol}(C, f)$ iff $p^\dagger$ solves the following variational inequality*

$$\langle f(u), u - p^\dagger\rangle \geq 0, \ \textit{for all } u \in C.$$

Lemma 3 ([51])**.** *Let C be a nonempty, convex and closed subset of a Hilbert space H. Let the operator $S\colon C \to C$ be continuous pseudocontractive. Then, S is demiclosed, i.e., $u^k \rightharpoonup \tilde{u}$ and $S(u^k) \to u^\dagger$ as $k \to +\infty$ imply that $S(\tilde{u}) = u^\dagger$.*

Lemma 4 ([52]). *Let Γ be a nonempty closed convex subset of a real Hilbert space H. Let $\{x^k\} \subset H$ be a sequence. If the following assumptions are satisfied*

(i) $\forall x^* \in \Gamma$, $\lim_{k\to+\infty} \|x^k - x^*\|$ *exists;*

(ii) $w_\omega(x^k) \subset \Gamma$.

Then the sequence $\{x^k\}$ converges weakly to some point in Γ.

3. Main Results

In this section, we present our main results.

Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$ and $Q \subset H_2$ be two nonempty closed convex sets. Let $S\colon C \to C$, $T\colon Q \to Q$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be four nonlinear operators. Let $A\colon C \to Q$ be a bounded linear operator with its adjoint A^*.

Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{\zeta_k\}$ and $\{\lambda_k\}$ be four real number sequences. Let ϑ, δ, ω, μ and ε be five constants. Let γ_0 and τ_0 be two positive constants.

Next, we introduce an iterative algorithm for solving the split problem (6).

In order to demonstrate the convergence analysis of Algorithm 1, we add some conditions on the operators and the parameters.

Algorithm 1: Select an initial point $x^0 \in C$. Set $k = 0$.

Step 1. Assume that the present iterate x^k and the step-sizes γ_k and τ_k are given. Compute

$$\begin{cases} v^k = (1-\beta_k)x^k + \beta_k S[(1-\alpha_k)x^k + \alpha_k S(x^k)], & (12) \\ y^k = \mathrm{proj}_C[v^k - \gamma_k f(v^k)], & (13) \\ u^k = (1-\vartheta)v^k + \vartheta y^k + \vartheta\gamma_k[f(v^k) - f(y^k)], & (14) \\ w^k = \mathrm{proj}_Q[Au^k - \tau_k g(Au^k)], & (15) \\ t^k = (1-\delta)Au^k + \delta w^k + \delta\tau_k[g(Au^k) - g(w^k)], & (16) \\ q^k = (1-\zeta_k)t^k + \zeta_k T[(1-\lambda_k)t^k + \lambda_k T(t^k)]. & (17) \end{cases}$$

Step 2. Compute the next iterate x^{k+1} by the following form

$$x^{k+1} = \mathrm{proj}_C[u^k + \varepsilon A^*(q^k - Au^k)]. \tag{18}$$

Step 3. Increase k by 1 and go back to Step 1. Meanwhile, update

$$\gamma_{k+1} = \begin{cases} \min\left\{\gamma_k, \frac{\omega\|y^k - v^k\|}{\|f(y^k) - f(v^k)\|}\right\}, & f(y^k) \neq f(v^k), \\ \gamma_k, & else. \end{cases} \tag{19}$$

and

$$\tau_{k+1} = \begin{cases} \min\left\{\tau_k, \frac{\mu\|w^k - Au^k\|}{\|g(w^k) - g(Au^k)\|}\right\}, & g(w^k) \neq g(Au^k), \\ \tau_k, & else. \end{cases} \tag{20}$$

Suppose that

(c1): S and T are two pseudocontractive operators with Lipschitz constants L_1 and L_2, respectively;

(c2): the operator f is pseudomonotone on H_1, weakly sequentially continuous and κ_1-Lipschitz continuous on C;

(c3): the operator g is pseudomonotone on H_2, weakly sequentially continuous and κ_2-Lipschitz continuous on Q.

(r1): $L_1 > 1$ and $0 < \underline{\beta} < \beta_k < \overline{\beta} < \alpha_k < \overline{\alpha} < \frac{1}{\sqrt{1+L_1^2}+1} (\forall k \geq 0)$;

(r2): $L_2 > 1$ and $0 < \underline{\zeta} < \zeta_k < \overline{\zeta} < \lambda_k < \overline{\lambda} < \frac{1}{\sqrt{1+L_2^2}+1} (\forall k \geq 0)$;
(r3): $\vartheta \in (0,1]$, $\delta \in (0,1]$, $\omega \in (0,1)$, $\mu \in (0,1)$ and $\varepsilon \in (0, 1/\|A\|^2)$.

Remark 1. *According to (19), the sequence $\{\gamma_k\}$ is monotonically decreasing. Moreover, by the κ_1-Lipschitz continuity of f, we obtain that $\frac{\omega\|y^k - v^k\|}{\|f(y^k) - f(v^k)\|} \geq \frac{\omega}{\kappa_1}$. Thus, $\{\gamma_k\}$ has a lower bound $\min\{\gamma_0, \frac{\omega}{\kappa_1}\}$. Therefore, the limit $\lim_{k\to+\infty} \gamma_k$ exists. Similarly, the sequence $\{\tau_k\}$ is monotonically decreasing and has a lower bound $\min\{\tau_0, \frac{\mu}{\kappa_2}\}$. So, the limit $\lim_{k\to+\infty} \tau_k$ exists.*

Now, we prove our main theorem.

Theorem 1. *Suppose that $\Gamma \neq \emptyset$. Then the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to some point $p \in \Gamma$.*

Proof. Let $x^* \in \Gamma$. Then, $x^* \in \mathrm{Fix}(S) \cap \mathrm{Sol}(C, f)$ and $Ax^* \in \mathrm{Fix}(T) \cap \mathrm{Sol}(Q, g)$. By (10) and (12), we have

$$\begin{aligned} \|v^k - x^*\|^2 &= \|(1-\beta_k)(x^k - x^*) + \beta_k(S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^*)\|^2 \\ &= (1-\beta_k)\|x^k - x^*\|^2 + \beta_k\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^*\|^2 \\ &\quad - (1-\beta_k)\beta_k\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\|^2. \end{aligned} \tag{21}$$

Using Lemma 1, we obtain

$$\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^*\|^2 \leq \|x^k - x^*\|^2 + (1-\alpha_k)\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\|^2. \tag{22}$$

Combining (21) and (22), we derive

$$\begin{aligned} \|v^k - x^*\|^2 &\leq (1-\beta_k)\|x^k - x^*\|^2 + \beta_k(1-\alpha_k)\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\|^2 \\ &\quad + \beta_k\|x^k - x^*\|^2 - (1-\beta_k)\beta_k\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\|^2 \\ &= \|x^k - x^*\|^2 - \beta_k(\alpha_k - \beta_k)\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\|^2 \\ (\text{by condition (r1)}) &\leq \|x^k - x^*\|^2. \end{aligned} \tag{23}$$

Similarly, according to (10), Lemma 1 and (17), we have the following estimate

$$\begin{aligned} \|q^k - Ax^*\|^2 &\leq \|t^k - Ax^*\|^2 - (\lambda_k - \zeta_k)\zeta_k\|T[(1-\lambda_k)t^k + \lambda_k T(t^k)] - t^k\|^2 \\ (\text{by condition (r2)}) &\leq \|t^k - Ax^*\|^2. \end{aligned} \tag{24}$$

Applying the inequality (11) to (13), we obtain

$$\langle y^k - v^k + \gamma_k f(v^k), y^k - x^* \rangle \leq 0. \tag{25}$$

Since $x^* \in \mathrm{Sol}(C, f)$ and $y^k \in C$, $\langle f(x^*), y^k - x^* \rangle \geq 0$. This together with the pseudomonotonicity of f implies that

$$\langle f(y^k), y^k - x^* \rangle \geq 0. \tag{26}$$

Based on (25) and (26), we get

$$\langle y^k - v^k, y^k - x^* \rangle + \gamma_k \langle f(v^k) - f(y^k), y^k - x^* \rangle \leq 0.$$

It follows that

$$\frac{1}{2}(\|y^k - v^k\|^2 + \|y^k - x^*\|^2 - \|v^k - x^*\|^2) + \gamma_k \langle f(v^k) - f(y^k), y^k - x^* \rangle \leq 0,$$

which yields

$$\|y^k - x^*\|^2 \leq \|v^k - x^*\|^2 - 2\gamma_k \langle f(v^k) - f(y^k), y^k - x^* \rangle - \|y^k - v^k\|^2. \tag{27}$$

By (14), we have

$$\begin{aligned} \|u^k - x^*\|^2 &= \|(1-\vartheta)(v^k - x^*) + \vartheta(y^k - x^*) + \vartheta\gamma_k[f(v^k) - f(y^k)]\|^2 \\ &= \|(1-\vartheta)(v^k - x^*) + \vartheta(y^k - x^*)\|^2 + \vartheta^2\gamma_k^2\|f(v^k) - f(y^k)\|^2 \\ &\quad + 2\vartheta(1-\vartheta)\gamma_k\langle v^k - x^*, f(v^k) - f(y^k)\rangle \\ &\quad + 2\vartheta^2\gamma_k\langle y^k - x^*, f(v^k) - f(y^k)\rangle. \end{aligned} \tag{28}$$

From (10), we obtain

$$\begin{aligned} \|(1-\vartheta)(v^k - x^*) + \vartheta(y^k - x^*)\|^2 = (1-\vartheta)\|v^k - x^*\|^2 + \vartheta\|y^k - x^*\|^2 \\ - (1-\vartheta)\vartheta\|v^k - y^k\|^2. \end{aligned} \tag{29}$$

Substituting (27) and (29) into (28), we deduce

$$\begin{aligned} \|u^k - x^*\|^2 &\leq (1-\vartheta)\|v^k - x^*\|^2 + \vartheta\|v^k - x^*\|^2 - 2\vartheta\gamma_k\langle f(v^k) - f(y^k), y^k - x^*\rangle - \vartheta\|y^k - v^k\|^2 \\ &\quad - (1-\vartheta)\vartheta\|v^k - y^k\|^2 + \vartheta^2\gamma_k^2\|f(v^k) - f(y^k)\|^2 + 2\vartheta(1-\vartheta)\gamma_k\langle v^k - x^*, f(v^k) - f(y^k)\rangle \\ &\quad + 2\vartheta^2\gamma_k\langle y^k - x^*, f(v^k) - f(y^k)\rangle \\ &= \|v^k - x^*\|^2 - (2-\vartheta)\vartheta\|v^k - y^k\|^2 + \vartheta^2\gamma_k^2\|f(v^k) - f(y^k)\|^2 \\ &\quad - 2\vartheta(1-\vartheta)\gamma_k\langle f(v^k) - f(y^k), y^k - v^k\rangle \\ &\leq \|v^k - x^*\|^2 - (2-\vartheta)\vartheta\|v^k - y^k\|^2 + \vartheta^2\gamma_k^2\|f(v^k) - f(y^k)\|^2 \\ &\quad + 2\vartheta(1-\vartheta)\gamma_k\|f(v^k) - f(y^k)\|\|y^k - v^k\|. \end{aligned} \tag{30}$$

Thanks to (19), $\|f(v^k) - f(y^k)\| \leq \frac{\omega\|y^k - v^k\|}{\gamma_{k+1}}$. It follows from (30) that

$$\begin{aligned} \|u^k - x^*\|^2 &\leq \|v^k - x^*\|^2 - (2-\vartheta)\vartheta\|v^k - y^k\|^2 + \vartheta^2\omega^2\frac{\gamma_k^2}{\gamma_{k+1}^2}\|y^k - v^k\| \\ &\quad + 2\vartheta(1-\vartheta)\omega\frac{\gamma_k}{\gamma_{k+1}}\|y^k - v^k\|^2 \\ &= \|v^k - x^*\|^2 - \vartheta\left[2 - \vartheta - \vartheta\omega^2\frac{\gamma_k^2}{\gamma_{k+1}^2} - 2(1-\vartheta)\omega\frac{\gamma_k}{\gamma_{k+1}}\right]\|y^k - v^k\|^2. \end{aligned} \tag{31}$$

By Remark 1, $\lim_{k\to+\infty}\frac{\gamma_k}{\gamma_{k+1}} = 1$. So,

$$\lim_{k\to+\infty}\left[2 - \vartheta - \vartheta\omega^2\frac{\gamma_k^2}{\gamma_{k+1}^2} - 2(1-\vartheta)\omega\frac{\gamma_k}{\gamma_{k+1}}\right] = 2 - \vartheta - \vartheta\omega^2 - 2(1-\vartheta)\omega = -\vartheta(\omega - 1)(\omega + \frac{2-\vartheta}{\vartheta}) > 0.$$

Then, there exists $\sigma > 0$ and m_1 such that $2 - \vartheta - \vartheta\omega^2\frac{\gamma_k^2}{\gamma_{k+1}^2} - 2(1-\vartheta)\omega\frac{\gamma_k}{\gamma_{k+1}} \geq \sigma$ when $k \geq m_1$. In combination with (31), we get

$$\|u^k - x^*\|^2 \leq \|v^k - x^*\|^2 - \sigma\vartheta\|y^k - v^k\|^2.$$

This together with (23) implies that

$$\begin{aligned} \|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \beta_k(\alpha_k - \beta_k)\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\|^2 \\ - \sigma\vartheta\|y^k - v^k\|^2. \end{aligned} \tag{32}$$

By the property (11) of proj_Q and (15), we have

$$\langle w^k - Au^k + \tau_k g(Au^k), w^k - Ax^* \rangle \leq 0. \tag{33}$$

Since $Ax^* \in \mathrm{Sol}(Q, g)$ and $w^k \in Q$, $\langle g(Ax^*), w^k - Ax^* \rangle \geq 0$. By the pseudomonotonicity of g, we obtain

$$\langle g(w^k), w^k - Ax^* \rangle \geq 0. \tag{34}$$

Taking into account (33) and (34), we obtain

$$\langle w^k - Au^k, w^k - Ax^* \rangle + \tau_k \langle g(Au^k) - g(w^k), w^k - Ax^* \rangle \leq 0,$$

which is equivalent to

$$\frac{1}{2}(\|w^k - Au^k\|^2 + \|w^k - Ax^*\|^2 - \|Au^k - Ax^*\|^2) + \tau_k \langle g(Au^k) - g(w^k), w^k - Ax^* \rangle \leq 0.$$

It follows that

$$\|w^k - Ax^*\|^2 \leq \|Au^k - Ax^*\|^2 - 2\tau_k \langle g(Au^k) - g(w^k), w^k - Ax^* \rangle - \|w^k - Au^k\|^2. \tag{35}$$

From (14), we receive

$$\begin{aligned} \|t^k - Ax^*\|^2 &= \|(1-\delta)(Au^k - Ax^*) + \delta(w^k - Ax^*) + \delta\tau_k[g(Au^k) - g(w^k)]\|^2 \\ &= \|(1-\delta)(Au^k - Ax^*) + \delta(w^k - Ax^*)\|^2 + \delta^2\tau_k^2\|g(Au^k) - g(w^k)\|^2 \\ &\quad + 2\delta(1-\delta)\tau_k\langle Au^k - Ax^*, g(Au^k) - g(w^k)\rangle \\ &\quad + 2\delta^2\tau_k\langle w^k - Ax^*, g(Au^k) - g(w^k)\rangle. \end{aligned} \tag{36}$$

By virtue of (10), we achieve

$$\begin{aligned} \|(1-\delta)(Au^k - Ax^*) + \delta(w^k - Ax^*)\|^2 = (1-\delta)\|Au^k - Ax^*\|^2 + \delta\|w^k - Ax^*\|^2 \\ - (1-\delta)\delta\|Au^k - w^k\|^2. \end{aligned} \tag{37}$$

Substituting (35) and (37) into (36), we obtain

$$\begin{aligned} \|t^k - Ax^*\|^2 &\leq \|Au^k - Ax^*\|^2 - (2-\delta)\delta\|Au^k - w^k\|^2 + \delta^2\tau_k^2\|g(Au^k) - g(w^k)\|^2 \\ &\quad - 2\delta(1-\delta)\tau_k\langle w^k - Au^k, g(Au^k) - g(w^k)\rangle \\ &\leq \|Au^k - Ax^*\|^2 - (2-\delta)\delta\|Au^k - w^k\|^2 + \delta^2\tau_k^2\|g(Au^k) - g(w^k)\|^2 \\ &\quad + 2\delta(1-\delta)\tau_k\|w^k - Au^k\|\|g(Au^k) - g(w^k)\|. \end{aligned} \tag{38}$$

Duo to (20), we have

$$\|g(Au^k) - g(w^k)\| \leq \frac{\mu\|Au^k - w^k\|}{\tau_{k+1}}.$$

This together with (38) implies that

$$\begin{aligned} \|t^k - Ax^*\|^2 &\leq \|Au^k - Ax^*\| - (2-\delta)\delta\|Au^k - w^k\|^2 + \delta^2\mu^2\frac{\tau_k^2}{\tau_{k+1}^2}\|w^k - Au^k\| \\ &\quad + 2\delta(1-\delta)\mu\frac{\tau_k}{\tau_{k+1}}\|Au^k - w^k\|^2 \\ &= \|Au^k - Ax^*\| - \delta\left[2 - \delta - \delta\mu^2\frac{\tau_k^2}{\tau_{k+1}^2} - 2(1-\delta)\mu\frac{\tau_k}{\tau_{k+1}}\right]\|Au^k - w^k\|^2. \end{aligned} \tag{39}$$

By Remark 1, $\lim_{k\to+\infty}\frac{\tau_k}{\tau_{k+1}}=1$ and hence

$$\lim_{k\to\infty}\left[2-\delta-\delta\mu^2\frac{\tau_k^2}{\tau_{k+1}^2}-2(1-\delta)\mu\frac{\tau_k}{\tau_{k+1}}\right]=2-\delta-\delta\mu^2-2(1-\delta)\mu>0.$$

So, there exists $\varrho>0$ and m_2 such that

$$2-\delta-\delta\mu^2\frac{\tau_k^2}{\tau_{k+1}^2}-2(1-\delta)\mu\frac{\tau_k}{\tau_{k+1}}\geq\varrho,$$

when $k\geq m_2$.

In the light of (39), we have

$$\|t^k-Ax^*\|^2\leq\|Au^k-Ax^*\|-\varrho\delta\|w^k-Au^k\|^2. \tag{40}$$

Owing to (24) and (40), we get

$$\begin{aligned}\|q^k-Ax^*\|^2\leq{}&\|Au^k-Ax^*\|-(\lambda_k-\zeta_k)\zeta_k\|T[(1-\lambda_k)t^k+\lambda_kT(t^k)]-t^k\|^2\\&-\varrho\delta\|w^k-Au^k\|^2.\end{aligned} \tag{41}$$

Observe that

$$\begin{aligned}\langle u^k-x^*,A^*(q^k-Au^k)\rangle&=\langle Au^k-Ax^*,q^k-Au^k\rangle\\&=\frac{1}{2}[\|q^k-Ax^*\|^2-\|Au^k-Ax^*\|^2]-\frac{1}{2}\|q^k-Au^k\|^2.\end{aligned} \tag{42}$$

Combining (41) and (42), we acquire

$$\begin{aligned}\langle u^k-x^*,A^*(q^k-Au^k)\rangle\leq{}&-\frac{1}{2}\varrho\delta\|w^k-Au^k\|^2-\frac{1}{2}\|q^k-Au^k\|^2\\&-\frac{1}{2}(\lambda_k-\zeta_k)\zeta_k\|T[(1-\lambda_k)t^k+\lambda_kT(t^k)]-t^k\|^2.\end{aligned} \tag{43}$$

In view of (18), we have

$$\begin{aligned}\|x^{k+1}-x^*\|^2&=\|\mathrm{proj}_C[u^k+\varepsilon A^*(q^k-Au^k)]-\mathrm{proj}_C[x^*]\|^2\\&\leq\|u^k-x^*+\varepsilon A^*(q^k-Au^k)\|^2\\&=\|u^k-x^*\|^2+\|\varepsilon A^*(q^k-Au^k)\|^2+2\varepsilon\langle A^*(q^k-Au^k),u^k-x^*\rangle.\end{aligned}$$

It follows from (32) and (43) that

$$\begin{aligned}\|x^{k+1}-x^*\|^2\leq{}&\|u^k-x^*\|^2+\varepsilon^2\|A\|^2\|q^k-Au^k\|^2-\varepsilon\varrho\delta\|w^k-Au^k\|^2\\&-\varepsilon\|q^k-Au^k\|^2-\varepsilon(\lambda_k-\zeta_k)\zeta_k\|T[(1-\lambda_k)t^k+\lambda_kT(t^k)]-t^k\|^2\\={}&\|u^k-x^*\|^2-\varepsilon(1-\varepsilon\|A\|^2)\|q^k-Au^k\|^2-\varepsilon\varrho\delta\|w^k-Au^k\|^2\\&-\varepsilon(\lambda_k-\zeta_k)\zeta_k\|T[(1-\lambda_k)t^k+\lambda_kT(t^k)]-t^k\|^2\\\leq{}&\|x^k-x^*\|^2-\varepsilon\varrho\delta\|w^k-Au^k\|^2-\varepsilon(1-\varepsilon\|A\|^2)\|q^k-Au^k\|^2\\&-\beta_k(\alpha_k-\beta_k)\|S[(1-\alpha_k)x^k+\alpha_kS(x^k)]-x^k\|^2-\sigma\vartheta\|y^k-v^k\|^2\\&-\varepsilon(\lambda_k-\zeta_k)\zeta_k\|T[(1-\lambda_k)t^k+\lambda_kT(t^k)]-t^k\|^2\\\leq{}&\|x^k-x^*\|^2,\end{aligned} \tag{44}$$

which implies that $\lim_{k\to+\infty}\|x^k-x^*\|$ exists. Since $\|x^{k+1}-x^*\|\leq\|u^k-x^*\|\leq\|v^k-x^*\|\leq\|x^k-x^*\|$, we deduce

$$\lim_{k\to+\infty}\|u^k - x^*\| = \lim_{k\to+\infty}\|v^k - x^*\| = \lim_{k\to+\infty}\|x^k - x^*\|. \tag{45}$$

So, the sequences $\{x^k\}$, $\{u^k\}$ and $\{v^k\}$ are all bounded.

By virtue of (44), we derive

$$\begin{aligned}
&\beta_k(\alpha_k - \beta_k)\|S[(1-\alpha_k)x^k + \alpha_k Sx^k] - x^k\|^2 + \sigma\vartheta\|y^k - v^k\|^2 + \varepsilon\varrho\delta\|w^k - Au^k\|^2 \\
&\quad + \varepsilon(1-\varepsilon\|A\|^2)\|q^k - Au^k\|^2 + \varepsilon(\lambda_k - \zeta_k)\zeta_k\|T[(1-\lambda_k)t^k + \lambda_k Tt^k] - t^k\|^2 \\
&\le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \to 0,
\end{aligned}$$

which implies that

$$\begin{cases}
\lim_{k\to+\infty}\|q^k - Au^k\| = 0, & (46)\\
\lim_{k\to+\infty}\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\| = 0, & (47)\\
\lim_{k\to+\infty}\|T[(1-\lambda_k)t^k + \lambda_k T(t^k)] - t^k\| = 0, & (48)\\
\lim_{k\to+\infty}\|y^k - v^k\| = 0, & (49)\\
\lim_{k\to+\infty}\|w^k - Au^k\| = 0. & (50)
\end{cases}$$

By the L_1-Lipschitz continuity of S, we have

$$\begin{aligned}
\|S(x^k) - x^k\| &\le \|S(x^k) - S[(1-\alpha_k)x^k + \alpha_k S(x^k)]\| + \|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\| \\
&\le L_1\alpha_k\|S(x^k) - x^k\| + \|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\|.
\end{aligned}$$

It follows that

$$\|S(x^k) - x^k\| \le \frac{1}{1 - L_1\alpha_k}\|S[(1-\alpha_k)x^k + \alpha_k S(x^k)] - x^k\|.$$

This together with (47) implies that

$$\lim_{k\to+\infty}\|S(x^k) - x^k\| = 0. \tag{51}$$

From (12) and (47), we conclude that $\|x^k - v^k\| \to 0$.

Next, we show that $\omega_w(x^k) \subset \Gamma$. Pick any $p^\dagger \in \omega_w(x^k)$. Then, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $x^{k_i} \rightharpoonup p^\dagger$ as $i \to +\infty$. In addition, $y^{k_i} \rightharpoonup p^\dagger$ and $v^{k_i} \rightharpoonup p^\dagger$ as $i \to +\infty$.

First, we prove that $p^\dagger \in \mathrm{Sol}(C, f)$. In view of (11) and $y^{k_i} = \mathrm{proj}_C[v^{k_i} - \gamma_{k_i} f(v^{k_i})]$, we achieve

$$\langle y^{k_i} - v^{k_i} + \gamma_{k_i} f(v^{k_i}), y^{k_i} - u\rangle \le 0, \quad \text{for all } u \in C.$$

It follows that

$$\frac{1}{\gamma_{k_i}}\langle v^{k_i} - y^{k_i}, u - y^{k_i}\rangle + \langle f(v^{k_i}), y^{k_i} - v^{k_i}\rangle \le \langle f(v^{k_i}), u - v^{k_i}\rangle, \quad \text{for all } u \in C. \tag{52}$$

Noting that from (49), we have $\lim_{i\to+\infty}\|v^{k_i} - y^{k_i}\| = 0$. Meanwhile, $\{y^{k_i}\}$ and $\{f(v^{k_i})\}$ are bounded. Then, by (52), we deduce

$$\liminf_{i\to+\infty}\langle f(v^{k_i}), u - v^{k_i}\rangle \ge 0, \quad \text{for all } u \in C. \tag{53}$$

Let $\{\epsilon_j\}$ be a positive real numbers sequence satisfying $\lim_{j\to+\infty}\epsilon_j = 0$. On account of (53), for each ϵ_j, there exists the smallest positive integer n_j such that

$$\langle f(v^{k_{i_j}}), u - v^{k_{i_j}} \rangle + \epsilon_j \geq 0, \text{ for all } j \geq n_j. \tag{54}$$

Moreover, for each $j > 0$, $f(v^{k_{i_j}}) \neq 0$. Setting $\varphi(v^{k_{i_j}}) = \frac{f(v^{k_{i_j}})}{\|f(v^{k_{i_j}})\|^2}$, we have $\langle f(v^{k_{i_j}}), \varphi(v^{k_{i_j}}) \rangle = 1$. From (54), we have

$$\langle f(v^{k_{i_j}}), u + \epsilon_j \varphi(v^{k_{i_j}}) - v^{k_{i_j}} \rangle \geq 0.$$

By the pseudomonotonicity of f, we get

$$\langle f(u + \epsilon_j \varphi(v^{k_{i_j}})), u + \epsilon_j \varphi(v^{k_{i_j}}) - v^{k_{i_j}} \rangle \geq 0,$$

which implies that

$$\begin{aligned} \langle f(u), u - v^{k_{i_j}} \rangle \geq &\langle f(u) - f(u + \epsilon_j \varphi(v^{k_{i_j}})), u + \epsilon_j \varphi(v^{k_{i_j}}) - v^{k_{i_j}} \rangle \\ &+ \langle f(u), -\epsilon_j \varphi(v^{k_{i_j}}) \rangle. \end{aligned} \tag{55}$$

Because of $f(v^{k_{i_j}}) \rightharpoonup f(p^\dagger)$, we have

$$\liminf_{j \to +\infty} \|f(v^{k_{i_j}})\| \geq \|f(p^\dagger)\| > 0.$$

Then,

$$\lim_{j \to +\infty} \|\epsilon_j \varphi(v^{k_{i_j}})\| = \lim_{j \to +\infty} \frac{\epsilon_j}{\|f(v^{k_{i_j}})\|} = 0.$$

This together with (55) implies that

$$\langle f(u), u - p^\dagger \rangle \geq 0. \tag{56}$$

By Lemma 2 and (56), we conclude that $p^\dagger \in \mathrm{Sol}(C, f)$.

On the other hand, by (51), $\|Sx^{k_i} - x^{k_i}\| \to 0$ as $i \to +\infty$. This together with $x^{k_i} \rightharpoonup p^\dagger$ and Lemma 3 implies that $p^\dagger \in \mathrm{Fix}(S)$. Therefore, $p^\dagger \in \mathrm{Fix}(S) \cap \mathrm{Sol}(C, f)$.

Next, we show that $Ap^\dagger \in \mathrm{Fix}(T) \cap \mathrm{Sol}(Q, g)$. Observe that

$$\begin{aligned} \|T(t^k) - t^k\| &\leq \|T(t^k) - T[(1-\lambda_k)t^k + \lambda_k T(t^k)]\| + \|T[(1-\lambda_k)t^k + \lambda_k T(t^k)] - t^k\| \\ &\leq L_2 \lambda_k \|T(t^k) - t^k\| + \|T[(1-\lambda_k)t^k + \lambda_k T(t^k)] - t^k\|. \end{aligned}$$

It follows that

$$\|T(t^k) - t^k\| \leq \frac{1}{1 - L_2 \lambda_k} \|T[(1-\lambda_k)t^k + \lambda_k T(t^k)] - t^k\|.$$

This together with (48) implies that

$$\lim_{k \to +\infty} \|T(t^k) - t^k\| = 0. \tag{57}$$

From (14), $u^{k_i} \rightharpoonup p^\dagger$ as $i \to +\infty$. Thanks to (17) and (48), we have $q^{k_i} - t^{k_i} \to 0$ as $i \to +\infty$. Combining with (46), we deduce that $t^{k_i} \rightharpoonup Ap^\dagger$. Applying Lemma 3 to (57), we obtain that $Ap^\dagger \in \mathrm{Fix}(T)$.

Next, we show that $Ap^\dagger \in \mathrm{Sol}(Q, g)$. In view of (10) and $w^{k_i} = proj_Q[Au^{k_i} - \tau_{k_i} g(Au^{k_i})]$, we achieve

$$\langle w^{k_i} - Au^{k_i} + \tau_{k_i} g(Au^{k_i}), w^{k_i} - v \rangle \leq 0, \text{ for all } v \in Q.$$

It follows that

$$\frac{1}{\tau_{k_i}}\langle w^{k_i} - Au^{k_i}, w^{k_i} - v\rangle + \langle g(Au^{k_i}), w^{k_i} - Au^{k_i}\rangle \le \langle g(Au^{k_i}), v - Au^{k_i}\rangle, \ \text{for all } v \in Q. \tag{58}$$

Noting that from (r3), we have $\lim_{i\to+\infty}\|w^{k_i} - Au^{k_i}\| = 0$. Then, by (58), we deduce

$$\liminf_{i\to+\infty}\langle g(Au^{k_i}), v - Au^{k_i}\rangle \ge 0, \ \text{for all } v \in Q. \tag{59}$$

Choose a positive real numbers sequence $\{v_j\}$ such that $\lim_{j\to+\infty} v_j = 0$. In terms of (59), for each v_j, there exists the smallest positive integer m_j such that

$$\langle g(Au^{k_i}), v - Au^{k_i}\rangle + v_j \ge 0, \ \text{ for all } j \ge m_j. \tag{60}$$

Moreover, for each $j > 0$, $g(Au^{k_i}) \neq 0$. Setting $\psi(u^{k_i}) = \frac{g(Au^{k_i})}{\|g(Au^{k_i})\|^2}$, we have $\langle g(Au^{k_i}), \psi(u^{k_i})\rangle = 1$. From (60), we have

$$\langle g(A(u^{k_{i_j}})), v + v_j\psi(u^{k_{i_j}}) - Au^{k_{i_j}}\rangle \ge 0.$$

By the pseudomonotonicity of g, we get

$$\langle g(v + v_j\psi(u^{k_{i_j}})), v + v_j\psi(u^{k_{i_j}}) - Au^{k_{i_j}}\rangle \ge 0,$$

which implies that

$$\begin{aligned}\langle g(v), v - Au^{k_{i_j}}\rangle &\ge \langle g(v) - g(v + v_j\psi(u^{k_{i_j}})), v + v_j\psi(u^{k_{i_j}}) - Au^{k_{i_j}}\rangle \\ &\quad + \langle g(v), -v_j\psi(u^{k_{i_j}})\rangle.\end{aligned} \tag{61}$$

Because of $g(A(u^{k_{i_j}})) \rightharpoonup g(Ap^\dagger)$, we have

$$\liminf_{j\to+\infty}\|g(A(u^{k_{i_j}}))\| \ge \|g(Ap^\dagger)\| > 0.$$

Then,

$$\lim_{j\to+\infty}\|v_j\psi(u^{k_{i_j}}))\| = \lim_{j\to\infty}\frac{v_j}{\|g(Au^{k_i})\|} = 0.$$

This together with (61) implies that

$$\langle g(v), v - Ap^\dagger\rangle \ge 0, \ \text{for all } v \in Q. \tag{62}$$

By Lemma 2 and (62), we conclude that $Ap^\dagger \in \mathrm{Sol}(Q, g)$. So, $p \in \Gamma$ and $\omega_w(x^k) \subset \Gamma$.

Finally, we show that the entire sequence $\{x^k\}$ converges weakly to $p^\dagger$. As a matter of fact, we have the following facts:

(i) $\forall x^* \in \Gamma$, $\lim_{k\to+\infty}\|x^k - x^*\|$ exists;
(ii) $w_\omega(x^k) \subset \Gamma$;
(iii) $p^\dagger \in w_\omega(x^k)$.

Thus, by Lemma 4, we deduce that the sequence $\{x^k\}$ weakly converges to $p^\dagger \in \Gamma$. This completes the proof. □

Corollary 1. *Suppose that $\Gamma_1 \neq \emptyset$. Then the sequence $\{x^k\}$ generated by Algorithm 2 converges weakly to some point $p_1 \in \Gamma_1$.*

Algorithm 2: Select an initial point $x^0 \in C$. Set $k = 0$.

Step 1. Assume that the present iterate x^k and the step-sizes γ_k and τ_k are given. Compute

$$\begin{cases} y^k = \mathrm{proj}_C[x^k - \gamma_k f(x^k)], & (63) \\ u^k = (1-\vartheta)x^k + \vartheta y^k + \vartheta\gamma_k[f(x^k) - f(y^k)], & (64) \\ w^k = \mathrm{proj}_Q[Au^k - \tau_k g(Au^k)], & (65) \\ t^k = (1-\delta)Au^k + \delta w^k + \delta\tau_k[g(Au^k) - g(w^k)]. & (66) \end{cases}$$

Step 2. Compute the next iterate x^{k+1} by the following form

$$x^{k+1} = \mathrm{proj}_C[u^k + \varepsilon A^*(t^k - Au^k)]. \tag{67}$$

Step 3. Increase k by 1 and go back to Step 1. Meanwhile, update

$$\gamma_{k+1} = \begin{cases} \min\left\{\gamma_k, \frac{\omega\|y^k - x^k\|}{\|f(y^k) - f(x^k)\|}\right\}, & f(y^k) \neq f(x^k), \\ \gamma_k, & else. \end{cases} \tag{68}$$

$$\tau_{k+1} = \begin{cases} \min\left\{\tau_k, \frac{\mu\|w^k - Au^k\|}{\|g(w^k) - g(Au^k)\|}\right\}, & g(w^k) \neq g(Au^k), \\ \tau_k, & else. \end{cases} \tag{69}$$

4. Application to Split Pseudoconvex Optimization Problems and Fixed Point Problems

In this section, we apply Algorithm 1 to solve split pseudoconvex optimization problems and fixed point problems.

Let $\mathbb{R}^n$ be the Euclidean space. Let C be a closed convex set in $\mathbb{R}^n$. Recall that a differentiable function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be pseudoconvex on C if for every pair of distinct points $x, y \in C$,

$$\nabla F(x)^T(y - x) \geq 0 \text{ implies } F(y) \geq F(x).$$

Now, we consider the following optimization problem

$$\min F(x) \text{ subject to } x \in C, \tag{70}$$

where $F(x)$ is pseudoconvex and twice continuously differentiable.

Denote by $\mathrm{SOP}(C, F)$ the solution set of optimization problem (70).

The following lemma reveals the relationship between the variational inequality and the pseudoconvex optimization problem.

Lemma 5 ([53])**.** *Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable and pseudoconvex on C. Then $x^* \in C$ satisfies*

$$\nabla F(x^*)^T(x - x^*) \geq 0 \text{ for all } x \in C$$

if and only if x^* is a minimum of $F(x)$ in C.

Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be two Euclidean spaces. Let $C \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ be two nonempty closed convex sets. Let A be a given $m \times n$ real matrix. Let $S : C \to C$ and $T : Q \to Q$ be two pseudocontractive operators with Lipschitz constants L_1 and L_2, respectively. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with κ_1-Lipschitz continuous gradient which is also pseudoconvex on C. Let $G : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function with κ_2-Lipschitz continuous gradient which is also pseudoconvex on Q.

Consider the following split problem of finding a point $x^* \in C$ such that

$$x^* \in \mathrm{Fix}(S) \cap \mathrm{SOP}(C, F) \text{ and } Ax^* \in \mathrm{Fix}(T) \cap \mathrm{SOP}(Q, G). \tag{71}$$

The solution set of (71) is denoted by Γ_2, i.e.,

$$\Gamma_2 = \{x^* \in \mathrm{Fix}(S) \cap \mathrm{SOP}(C, F), Ax^* \in \mathrm{Fix}(T) \cap \mathrm{SOP}(Q, G)\}.$$

Next, we introduce an iterative algorithm for solving the split problem (71).

Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{\zeta_k\}$ and $\{\lambda_k\}$ be four real number sequences. Let ϑ, δ, ω, μ and ε be five constants. Let γ_0 and τ_0 be two positive constants.

Theorem 2. *Suppose that $\Gamma_2 \neq \emptyset$ and the conditions (r1)–(r3) hold. Then the sequence $\{x^k\}$ generated by Algorithm 3 converges to some point $p \in \Gamma_2$.*

Algorithm 3: Select an initial point $x^0 \in C$. Set $k = 0$.

Step 1. Assume that the present iterate x^k and the step-sizes γ_k and τ_k are given. Compute

$$\begin{cases} v^k = (1-\beta_k)x^k + \beta_k S[(1-\alpha_k)x^k + \alpha_k S(x^k)], \\ y^k = \mathrm{proj}_C[v^k - \gamma_k \nabla F(v^k)], \\ u^k = (1-\vartheta)v^k + \vartheta y^k + \vartheta\gamma_k[\nabla F(v^k) - \nabla F(y^k)], \\ w^k = \mathrm{proj}_Q[Au^k - \tau_k \nabla G(Au^k)], \\ t^k = (1-\delta)Au^k + \delta w^k + \delta\tau_k[\nabla G(Au^k) - \nabla G(w^k)], \\ q^k = (1-\zeta_k)t^k + \zeta_k T[(1-\lambda_k)t^k + \lambda_k T(t^k)]. \end{cases}$$

Step 2. Compute the next iterate x^{k+1} by the following form

$$x^{k+1} = \mathrm{proj}_C[u^k + \varepsilon A^*(q^k - Au^k)].$$

Step 3. Increase k by 1 and go back to Step 1. Meanwhile, update

$$\gamma_{k+1} = \begin{cases} \min\left\{\gamma_k, \frac{\omega\|y^k - v^k\|}{\|\nabla F(y^k) - \nabla F(v^k)\|}\right\}, & \nabla F(y^k) \neq \nabla F(v^k), \\ \gamma_k, & else. \end{cases}$$

and

$$\tau_{k+1} = \begin{cases} \min\left\{\tau_k, \frac{\mu\|w^k - Au^k\|}{\|\nabla G(w^k) - \nabla G(Au^k)\|}\right\}, & \nabla G(w^k) \neq \nabla G(Au^k), \\ \tau_k, & else. \end{cases}$$

5. Concluding Remarks

In this paper, we survey iterative methods for solving the split problem of fixed points of two pseudocontractive operators and variational inequalities of two pseudomonotone operators in Hilbert spaces. By using self-adaptive techniques, we construct a Tseng-type iterative algorithm for solving this split problem. We prove that the proposed Tseng-type iterative algorithm converges weakly to a solution of the split problem under some additional conditions imposed the operators and the parameters. Finally, we apply our algorithm to solve split pseudoconvex optimization problems and fixed point problems.

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