

Article

# Rough Approximation Operators on a Complete Orthomodular Lattice

Songsong Dai 

School of Electronics and Information Engineering, Taizhou University, Taizhou 318000, China; ssdai@tzc.edu.cn

**Abstract:** This paper studies rough approximation via join and meet on a complete orthomodular lattice. Different from Boolean algebra, the distributive law of join over meet does not hold in orthomodular lattices. Some properties of rough approximation rely on the distributive law. Furthermore, we study the relationship among the distributive law, rough approximation and orthomodular lattice-valued relation.

**Keywords:** rough set; complete orthomodular lattice; quantum logic; distributive law

## 1. Introduction

In 1982, Pawlak [1] proposed rough set theory as an excellent tool for incomplete information processing. Since then, a series of works describe rough sets relying on the mathematical structure of the set, such as logics [2–4], algebraic and topological structures [5,6]. Rough sets over different mathematical structures have different properties. In particular, rough sets over Boolean algebra [7,8], lattice effect algebra [9], residuated lattices [10–16] and other lattices [17–19] have been studied.

In 1936, Birkhoff and von Neumann [20] considered the orthomodular lattice as quantum logic for studying the algebraic structure of quantum mechanics. There are many extensions of Pawlak’s rough set, such as fuzzy rough sets [21,22], covering based rough sets [23–25], probabilistic rough sets [26,27], soft rough sets [28–31], Diophantine fuzzy rough sets [32–35], multi-granulation rough sets [36–38], hesitant fuzzy rough set [39] and so on. However, as far as I know, there is only a little literature addressing the rough sets and quantum logics together. In 2017, Hassan [40] considered a rough set classification method via quantum logic. In our previous work [41], we proposed rough set via join and meet on a complete orthomodular lattice (COL). Since orthomodular lattices are different from Boolean algebras, in particular, the distributive law not hold in orthomodular lattices, then we find many basic properties of rough sets rely on the distributive law. Obviously, the underlying laws of logic play an important part in the concept of rough approximations. However, sometimes the underlying laws of logic were taken for granted in rough approximations based on logics. In order to enhance the importance of the underlying laws of logic in rough set theory, a useful method is set up based on the equivalence between the underlying laws of logic and the basic properties of rough approximations. Note that Pawlak’s rough sets also rely on the equivalence relation. In this paper, we studied the relations among the distributive law, rough approximations and lattice-valued relation. Moreover, some topological structures of orthomodular lattice-valued rough approximations are investigated.

The paper is organized as follows: In Section 2, we recall definitions of orthomodular lattices and orthomodular lattice-valued rough approximations. In Section 3, we study the relationship among the distributive law, rough approximations and lattice-valued relations. In Section 4 is the conclusion.



**Citation:** Dai, S. Rough Approximation Operators on a Complete Orthomodular Lattice. *Axioms* **2021**, *10*, 164. <https://doi.org/10.3390/axioms10030164>

Academic Editor: Oscar Castillo

Received: 3 June 2021

Accepted: 26 July 2021

Published: 27 July 2021

**Publisher’s Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 2. Preliminaries

### 2.1. Orthomodular Lattices

First, we recall definitions of orthomodular lattices, for details see [42–48].

A COL  $L = \langle L, \leq, \wedge, \vee, \perp, 0, 1 \rangle$  is a complete bounded lattice and unary operator  $\perp$  has the following properties: for all  $\eta, \xi \in L$

- (1)  $\eta^\perp \vee \eta = 1, \eta^\perp \wedge \eta = 0;$
- (2)  $\eta^{\perp\perp} = \eta;$
- (3)  $\eta \leq \xi \Rightarrow \xi^\perp \leq \eta^\perp;$
- (4)  $\eta \geq \xi \Rightarrow \eta \wedge (\eta^\perp \vee \xi) = \xi.$

The property (4) is the orthomodular law, denoted by (OL). It is weaker than the distributive law, denoted by DL, which is not valid in orthomodular lattice.

An orthomodular lattice-valued set (*l*-valued set for short) is a mapping  $E : U \rightarrow L$ , where  $U$  is a finite universe. For any  $a \in L$ ,  $\hat{a}$  is the constant *l*-valued set, i.e.,  $\hat{a}(x) = a, \forall x \in U$ . Similarly, an orthomodular lattice-valued relation (*l*-valued relation for short) on  $U$  is a mapping  $E : U \times U \rightarrow L$ . We say  $R$  is serial, if  $\bigvee_{y \in U} R(x, y) = 1$  for all  $x \in U$ . We say  $R$  is  $\wedge$ -transitive, if  $R(\alpha, \gamma) \geq \bigvee_{\beta \in U} R(\alpha, \beta) \wedge R(\beta, \gamma)$  for all  $\alpha, \beta, \gamma \in U$ .

### 2.2. Rough Approximations on a COL

Then, we recall the rough approximations on a COL [41].

**Definition 1** ([41]). Let  $L$  be a COL and  $R$  be an  $L$ -valued relation on a finite universe  $U$ . With each  $L$ -valued set  $E$  on  $U$ , the lower approximation operator (LAO) and the upper approximation operator (UAO) of  $E$  are defined, respectively, as follows:

$$L_R(E)(\mu) = \bigwedge_{v \in U} (R(\mu, v)^\perp \vee E(v)), \quad \mu \in U \tag{1}$$

and

$$T_R(E)(\mu) = \bigvee_{v \in U} (R(\mu, v) \wedge E(v)), \quad \mu \in U. \tag{2}$$

The pair  $(L_R(E), T_R(E))$  is a  $L$ -valued rough set of  $E$  relative to COL  $L$ .

**Example 1.** Let  $C$  be the set of complex numbers. In the complex Hilbert space  $\otimes^2 C^2, |00\rangle, |01\rangle, |10\rangle$  and  $|11\rangle$  represent its orthonormal base.  $\rho_{ij} = \text{span}\{|ij\rangle\}$  is denoted the subspace spanned by  $|ij\rangle, i, j = 0, 1$ . For any closed subspace  $G$  and  $H$  of  $\otimes^2 C^2, G \leq H$  if the subspace  $G$  is included in  $H, \wedge$  is intersection of subspaces,  $\vee$  is union of subspaces,  $G^\perp$  is the orthogonal space of  $G, 0$  is the zero subspace and  $1$  is  $\otimes^2 C^2$ . Then  $L_2 = \langle L_2, \leq, \wedge, \vee, \perp, 0, 1 \rangle$  is a COL, where  $L_2$  is the set of all closed subspaces of  $\otimes^2 C^2$ , more details see [43,47].

We define an COL-valued rough approximation whose truth value is a closed subspace of  $\otimes^2 C^2$ . Let  $U = \{p, q\}, R(p, p) = R(q, q) = 1$  and  $R(p, q) = R(q, p) = v_{01}$ . Moreover, define  $E : U \rightarrow L_2$  as  $E(p) = v_{00}, E(q) = v_{11}$ . Then

- $T_R(E)(p) = (R(p, p) \wedge E(p)) \vee (R(p, q) \wedge E(q)) = (1 \wedge v_{00}) \vee (v_{01} \wedge v_{11}) = v_{00}$
- $T_R(E)(q) = (R(q, p) \wedge E(p)) \vee (R(q, q) \wedge E(q)) = (v_{01} \wedge v_{00}) \vee (1 \wedge v_{11}) = v_{11}$
- $L_R(E)(p) = (R(p, p)^\perp \vee E(p)) \wedge (R(p, q)^\perp \vee E(q)) = (1^\perp \vee v_{00}) \wedge (v_{01}^\perp \vee v_{11}) = 0$
- $L_R(E)(q) = (R(q, p)^\perp \vee E(p)) \wedge (R(q, q)^\perp \vee E(q)) = (v_{01}^\perp \vee v_{00}) \wedge (1^\perp \vee v_{11}) = 0$

**Example 2.** Consider the smallest OL which is not a Boolean algebra, called MO2 [49], as shown in Figure 1. Let the universe  $U = \{u_1, u_2, u_3\}$ . Define a  $L$ -valued set

$$E = \frac{a}{u_1} + \frac{a^\perp}{u_2} + \frac{b}{u_3} \tag{3}$$

and a  $L$ -valued relation  $R$  on  $MO2$  in Table 1. Thus, we have

$$\begin{aligned}
 L_R(E)(\mu_1) &= (R(\mu_1, \mu_1)^\perp \vee E(\mu_1)) \wedge (R(\mu_1, \mu_2)^\perp \vee E(\mu_2)) \wedge (R(\mu_1, \mu_3)^\perp \vee E(\mu_3)) \\
 &= (1^\perp \vee a) \wedge (0^\perp \vee a^\perp) \wedge (0^\perp \vee b) \\
 &= a \wedge 1 \wedge 1 = a \\
 L_R(E)(\mu_2) &= (R(\mu_2, \mu_1)^\perp \vee E(\mu_1)) \wedge (R(\mu_2, \mu_2)^\perp \vee E(\mu_2)) \wedge (R(\mu_2, \mu_3)^\perp \vee E(\mu_3)) \\
 &= (0^\perp \vee a) \wedge (a^\perp \vee a^\perp) \wedge (0^\perp \vee b) \\
 &= 1 \wedge a^\perp \wedge 1 = a^\perp \\
 L_R(E)(\mu_3) &= (R(\mu_3, \mu_1)^\perp \vee E(\mu_1)) \wedge (R(\mu_3, \mu_2)^\perp \vee E(\mu_2)) \wedge (R(\mu_3, \mu_3)^\perp \vee E(\mu_3)) \\
 &= (0^\perp \vee a) \wedge (0^\perp \vee a^\perp) \wedge (b^\perp \vee b) \\
 &= 1 \wedge 1 \wedge 1 = 1 \\
 T_R(E)(\mu_1) &= (R(\mu_1, \mu_1) \wedge E(\mu_1)) \vee (R(\mu_1, \mu_2) \wedge E(\mu_2)) \vee (R(\mu_1, \mu_3) \wedge E(\mu_3)) \\
 &= (1 \wedge a) \vee (0 \wedge a^\perp) \vee (0 \wedge b) \\
 &= a \wedge 0 \wedge 0 = a \\
 T_R(E)(\mu_2) &= (R(\mu_2, \mu_1) \wedge E(\mu_1)) \vee (R(\mu_2, \mu_2) \wedge E(\mu_2)) \vee (R(\mu_2, \mu_3) \wedge E(\mu_3)) \\
 &= (0 \wedge a) \vee (a \wedge a^\perp) \vee (0 \wedge b) \\
 &= 0 \wedge 0 \wedge 0 = 0 \\
 T_R(E)(\mu_3) &= (R(\mu_3, \mu_1) \wedge E(\mu_1)) \vee (R(\mu_3, \mu_2) \wedge E(\mu_2)) \vee (R(\mu_3, \mu_3) \wedge E(\mu_3)) \\
 &= (0 \wedge a) \vee (0 \wedge a^\perp) \vee (b \wedge b) \\
 &= 0 \wedge 0 \wedge b = b.
 \end{aligned}$$

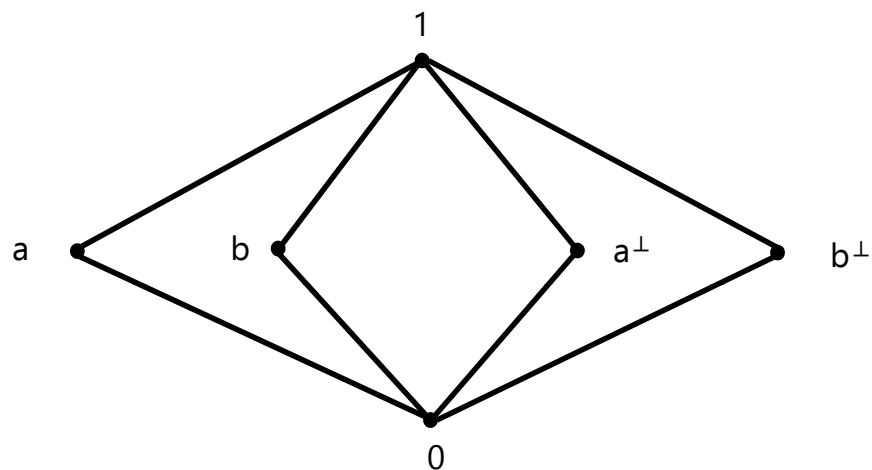


Figure 1. Orthomodular lattice  $MO2$  [49].

Table 1. The  $L$ -valued relation  $R$  in Example 2.

$R$	$u_1$	$u_2$	$u_3$
$u_1$	1	0	0
$u_2$	0	a	0
$u_3$	0	0	b

### 3. Relation among the Distributive Law, Rough Approximations and Lattice-Valued Relations

First, we give some relation between distributive law and rough approximations.

**Proposition 1.** *The following three statements are equivalent:*

- (1)  $L$  satisfies DL.
- (2)  $T_R(E \cup F) \equiv (T_R(E) \cup T_R(F))$ .
- (3)  $L_R(E \cap F) \equiv (L_R(E) \cap L_R(F))$ .

**Proof.** (1)  $\Rightarrow$  (2): By using the distributive law of meet over join, we have

$$\begin{aligned}
 T_R(E \vee F)(x) &= \bigvee_{y \in U} (R(x, y) \wedge (E \vee F)(y)) \\
 &= \bigvee_{y \in U} (R(x, y) \wedge (E(y) \vee F(y))) \\
 &= \bigvee_{y \in U} ((R(x, y) \wedge E(y)) \vee (R(x, y) \wedge F(y))) \\
 &= (T_R(E) \vee T_R(F))(x).
 \end{aligned}
 \tag{4}$$

(2)  $\Rightarrow$  (1): Our purpose is to show  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ ,  $\forall a, b, c \in L$ . Put  $R(x, y_1) = a$ , and  $R(x, y) = 0$  for other  $y \in U$ ;  $E(y_1) = b$ ,  $F(y_1) = c$ , and  $E(y) = F(y) = 0$  for other  $y \in U$ . Then, we have

$$\begin{aligned}
 T_R(E \vee F)(x) &= \bigvee_{y \in U} (R(x, y) \wedge (E \vee F)(y)) \\
 &= R(x, y_1) \wedge (E \vee F)(y_1) \\
 &= a \wedge (b \vee c).
 \end{aligned}
 \tag{5}$$

and

$$\begin{aligned}
 (T_R(E) \vee T_R(F))(x) &= T_R(E)(x) \vee T_R(F)(x) \\
 &= \left( \bigvee_{y \in U} (R(x, y) \wedge E(y)) \right) \vee \left( \bigvee_{y \in U} (R(x, y) \wedge F(y)) \right) \\
 &= (R(x, y_1) \wedge E(y_1)) \vee (R(x, y_1) \wedge F(y_1)) \\
 &= (a \wedge b) \vee (a \wedge c)
 \end{aligned}
 \tag{6}$$

Therefore, we obtain  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  from (2).

(1)  $\Rightarrow$  (3): By using the distributive law of join over meet,

$$\begin{aligned}
 L_R(F \wedge F)(x) &= \bigwedge_{y \in U} (R(x, y)^\perp \vee (E \wedge F)(y)) \\
 &= \bigwedge_{y \in U} (R(x, y)^\perp \vee (E(y) \wedge F(y))) \\
 &= \bigwedge_{y \in U} ((R(x, y)^\perp \vee E(y)) \wedge (R(x, y)^\perp \vee F(y))) \\
 &= (L_R(E) \wedge L_R(F))(x).
 \end{aligned}
 \tag{7}$$

(3)  $\Rightarrow$  (1): Similarly, our purpose is to show  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ ,  $\forall a, b, c \in L$ . Let  $R(x, y_1)^\perp = a$ , and  $R(x, y)^\perp = 1$  for other  $y \in U$ ;  $E(y_1) = b$ ,  $F(y_1) = c$ , and  $E(y) = F(y) = 1$  for other  $y \in U$ . Then,

$$\begin{aligned} L_R(E \wedge F)(x) &= \bigwedge_{y \in U} (R(x, y)^\perp \vee (E \wedge F)(y)) \\ &= R(x, y_1)^\perp \vee (E(y_1) \wedge F(y_1)) \\ &= a \vee (b \wedge c). \end{aligned} \tag{8}$$

and

$$\begin{aligned} (L_R(E) \wedge L_R(F))(x) &= L_R(E)(x) \wedge L_R(F)(x) \\ &= \left( \bigwedge_{y \in U} (R(x, y)^\perp \vee E(y)) \right) \wedge \left( \bigwedge_{y \in U} (R(x, y)^\perp \vee F(y)) \right) \\ &= \left( R(x, y_1)^\perp \vee E(y_1) \right) \wedge \left( R(x, y_1)^\perp \vee F(y_1) \right) \\ &= (a \vee b) \wedge (a \vee c) \end{aligned} \tag{9}$$

Therefore, we obtain  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$  from (3).  $\square$

The following results are also based on the distributive law of join over meet.

**Proposition 2.** *If  $L$  satisfies DL, then the following three statements are equivalent*

- (1)  $R$  is serial, i.e.,  $\forall y \in U R(x, y) = 1$  for all  $x \in U$ .
- (2)  $T_R(\hat{a}) \equiv \hat{a}$ , for any  $a \in L$ .
- (3)  $L_R(\hat{a}) \equiv \hat{a}$ , for any  $a \in L$ .

**Proof.** (1)  $\Rightarrow$  (2): By using the distributive law of join over meet, we have

$$\begin{aligned} T_R(\hat{a})(x) &= \bigvee_{y \in U} (R(x, y) \wedge \hat{a}(y)) \\ &= \bigvee_{y \in U} (R(x, y) \wedge a) \\ &= a \wedge \bigvee_{y \in U} R(x, y) \\ &= a \wedge 1 \\ &= a. \end{aligned} \tag{10}$$

(2)  $\Rightarrow$  (1): Take  $a = 1$ ; then it follows from the proof of necessity and  $T_R(\hat{1})(x) = 1$  for every  $x \in X$  that  $\forall y \in U R(x, y) = 1$  holds for every  $x \in U$ . Hence  $R$  is serial.

Similarly, we can prove (1)  $\Leftrightarrow$  (3).  $\square$

Now, we study the relationship among the distributive law, rough approximation and COL-valued relation.

**Proposition 3.** *If two of the following statements hold, then the third statement holds:*

- (1)  $L$  satisfies DL.
- (2)  $R$  is  $\wedge$ -transitive, i.e.,  $R(\alpha, \gamma) \geq \bigvee_{\beta \in U} R(\alpha, \beta) \wedge R(\beta, \gamma)$  holds for all  $\alpha, \beta, \gamma \in U$ .
- (3)  $T_R(T_R(E)) \subseteq T_R(E)$ .

**Proof.** (1) + (2)  $\Rightarrow$  (3): By using the distributive law of join over meet, we have

$$\begin{aligned}
 T_R(T_R(E))(x) &= \bigvee_{y \in U} (R(x, y) \wedge T_R(E)(y)) \\
 &= \bigvee_{y \in U} (R(x, y) \wedge (\bigvee_{z \in U} (R(y, z) \wedge E(z)))) \\
 &= \bigvee_{z \in U} (\bigvee_{y \in U} (R(x, y) \wedge R(y, z)) \wedge E(z)) \tag{11} \\
 &\leq \bigvee_{z \in U} (R(x, z) \wedge E(z)) \\
 &= T_R(E)(x).
 \end{aligned}$$

(1) + (3)  $\Rightarrow$  (2): Assume that  $R$  is not  $\wedge$ -transitive. It follows that for some  $x_0, z_0 \in U$ ,  $\bigvee_{y \in U} (R(x_0, y) \wedge R(y, z_0)) \leq R(x_0, z_0)$  is not hold.

Let  $E(z_0) = 1$  and  $E(y) = 0$  for other  $y \in U$ . Then, we have

$$T_R(T_R(E))(x_0) = \bigvee_{y \in U} (R(x_0, y) \wedge R(y, z_0)) \tag{12}$$

and

$$T_R(E)(x_0) = R(x_0, z_0). \tag{13}$$

Therefore, it follows from (3) that  $\bigvee_{y \in U} (R(x_0, y) \wedge R(y, z_0)) \leq R(x_0, z_0)$ .

(2) + (3)  $\Rightarrow$  (1): Given  $a, b, c \in L$ , let  $U = x, y, z$ ,  $R \in L^{U \times U}$  which  $R(x, y) = R(x, z) = a$ ,  $R(y, z) = R(z, z) = b$ ,  $R(z, y) = R(y, y) = c$  and others are  $o$ , and  $E \in L^U$  which  $E(y) = c$ ,  $E(z) = b$ ,  $E(x) = 0$ . It easy to see  $R$  is  $\wedge$ -transitive. Then, we have

$$T_R(E)(x) = (R(x, y) \wedge E(y)) \vee (R(x, z) \wedge E(z)) = (a \wedge c) \vee (a \wedge b), \tag{14}$$

$$T_R(E)(y) = (R(y, y) \wedge E(y)) \vee (R(y, z) \wedge E(z)) = b \vee c, \tag{15}$$

$$T_R(E)(z) = (R(z, z) \wedge E(z)) \vee (R(z, y) \wedge E(y)) = b \vee c, \tag{16}$$

and

$$\begin{aligned}
 T_R(T_R(E))(x) &= (R(x, y) \wedge T_R(E)(y)) \vee (R(x, z) \wedge T_R(E)(z)) \\
 &= (a \wedge (b \vee c)) \vee (a \wedge (b \vee c)) \\
 &= a \wedge (b \vee c).
 \end{aligned} \tag{17}$$

Therefore, by  $T_R(T_R(E))(x) \leq T_R(E)(x)$  we obtain  $(a \wedge (b \vee c)) \leq ((a \wedge c) \vee (a \wedge b))$ . Since  $(a \wedge (b \vee c)) \geq ((a \wedge b) \vee (a \wedge c))$  always holds. Thus  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ .

□

**Proposition 4.** If two of the following statements hold, then the third statement holds:

- (1)  $L$  satisfies DL.
- (2)  $R$  is  $\wedge$ -transitive.
- (3)  $L_R(E) \subseteq L_R(L_R(E))$ .

**Proof.** Similar to Proposition 3. □

Propositions 3 and 4 give some basic properties of rough approximations that do not only rely on the binary relation but also on the distributive law.

**Definition 2 ([50]).** Let  $U$  be a non-empty set, a function in:  $L^U \rightarrow L^U$  is an  $l$ -valued interior operator if and only if (iff) for all  $E, F \in L^U$  it satisfies:

- (1)  $in(\hat{a}) = \hat{a}$ ;

- (2)  $in(E) \subseteq E$ ;
- (3)  $in(E \cap F) = in(E) \cap in(F)$ ;
- (4)  $in(in(E)) = in(E)$ ;

**Definition 3** ([50]). Let  $U$  be a non-empty set, a function  $cl: L^U \rightarrow L^U$  is an  $l$ -valued closure operator iff for all  $E, F \in L^U$  it satisfies:

- (1)  $cl(\hat{a}) = \hat{a}$ ;
- (2)  $E \subseteq cl(E)$ ;
- (3)  $cl(E \cup F) = cl(E) \cup cl(F)$ ;
- (4)  $cl(cl(E)) = cl(E)$ ;

**Proposition 5.** If two of the following statements hold, then the third statement holds:

- (1)  $L$  satisfies DL.
- (2)  $R$  is serial and  $\wedge$ -transitive.
- (3)  $L_R$  is an  $l$ -valued interior operator.

**Proof.** Immediate from Proposition 1 of [41] and Propositions 1, 2 and 4.  $\square$

Note that DL is also a condition of that lower rough approximation is an interior operator.

**Proposition 6.** If two of the following statements hold, then the third statement holds:

- (1)  $L$  satisfies DL.
- (2)  $R$  is serial and  $\wedge$ -transitive.
- (3)  $T_R$  is an  $l$ -valued closure operator.

**Proof.** Immediate from Proposition 1 of [41] and Propositions 1, 2 and 3.  $\square$

Similar to Propositions 3 and 4, Propositions 5 and 6 also show that some topology properties of rough approximations that do not only rely on the binary relation but also on the distributive law.

**Example 3.** Consider a  $L$ -valued relation  $R$  on MO2 in Table 2.

**Table 2.** The  $L$ -valued relation  $R$  in Example 3.

<b>R</b>	<b><math>u_1</math></b>	<b><math>u_2</math></b>	<b><math>u_3</math></b>
$u_1$	1	0	0
$u_2$	0	1	0
$u_3$	0	0	1

Then  $L_R$  and  $T_R$ , respectively, are  $l$ -valued interior operator and  $l$ -valued closure operator for any  $l$ -valued set  $E : U \rightarrow MO2$ .

Moreover, if we use the following definitions of  $l$ -valued interior operator and  $l$ -valued closure operator which are weaker than Definitions 2 and 3, respectively.

**Definition 4.** Let  $U$  be a non-empty set, a function  $in: L^U \rightarrow L^U$  is an  $l$ -valued interior operator iff for all  $E, F \in L^U$  it satisfies:

- (1)  $in(\hat{0}) = \hat{0}$ ;
- (2)  $in(E) \subseteq E$ ;
- (3)  $in(E \cap F) = in(E) \cap in(F)$ ;
- (4)  $in(in(E)) = in(E)$ ;

**Definition 5.** Let  $U$  be a non-empty set, a function  $cl: L^U \rightarrow L^U$  is an  $l$ -valued closure operator iff for all  $E, F \in L^U$  it satisfies:

- (1)  $cl(\hat{1}) = \hat{1}$ ;
- (2)  $E \subseteq cl(E)$ ;
- (3)  $cl(E \cup F) = cl(E) \cup cl(F)$ ;
- (4)  $cl(cl(E)) = cl(E)$ ;

Then, we have the following results.

**Proposition 7.** *If two of the following statements hold, then the third statement holds:*

- (1)  $L$  satisfies DL.
- (2)  $R$  is  $\wedge$ -transitive.
- (3)  $L_R$  is an  $l$ -valued interior operator.

**Proof.** Immediate from Proposition 1 of [41] and Propositions 1 and 4.  $\square$

**Proposition 8.** *If two of the following statements hold, then the third statement holds:*

- (1)  $L$  satisfies DL.
- (2)  $R$  is  $\wedge$ -transitive.
- (3)  $T_R$  is an  $l$ -valued closure operator.

**Proof.** Immediate from Proposition 1 of [41] and Propositions 1 and 3.  $\square$

**Example 4.** Consider a  $L$ -valued relation  $R$  on  $MO_2$  in Table 3.

**Table 3.** The  $L$ -valued relation  $R$  in Example 4.

$R$	$u_1$	$u_2$	$u_3$
$u_1$	a	0	0
$u_2$	0	a	0
$u_3$	0	0	a

Then  $L_R$  is an  $l$ -valued interior operator of Definition 4 but is not an  $l$ -valued interior operator of Definition 2, and  $T_R$  is an  $l$ -valued closure operator of Definition 5 but is not an  $l$ -valued closure operator of Definition 3.

#### 4. Conclusions

In this paper, we studied COL-valued rough approximation. Some properties of rough approximations rely on DL of  $\vee$  over  $\wedge$  and binary relation. Obviously, the distributive law plays an important part in the operation of rough approximations. Many basic properties of rough approximations do not only rely on the binary relation but also on the distributive law (see Propositions 2–4). Moreover, some topology properties of rough approximations do not only rely on the binary relation but also on the distributive law (see Propositions 5–8).

Orthomodular lattices can be viewed as a sharp quantum structure. There is another concrete or standard quantum logic, called unsharp quantum logics which do not satisfy the non-contradiction principle [49]. Proceeding from this angle, we can study the rough approximations based on unsharp quantum logics as future work.

**Funding:** This research was funded by the National Science Foundation of China under Grant No. 62006168 and Zhejiang Provincial Natural Science Foundation of China under Grant No. LQ21A010001.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.



## Abbreviations

The following abbreviations are used in this manuscript:

COL	complete orthomodular lattice
OL	orthomodular law
DL	distributive law
LAO	lower approximation operator
UAO	upper approximation operator
iff	if and only if

## References

- Pawlak, Z. Rough sets. *Int. J. Comput. Inf. Sci.* **1982**, *11*, 341–356. [[CrossRef](#)]
- Rauszer, C. An equivalence between theory of functional dependence and a fragment of intuitionistic logic. *Bull. Pol. Acad. Sci. Math.* **1985**, *33*, 571–579.
- Vakarelov, D. A modal logic for similarity relations in Pawlak knowledge representation systems. *Fundam. Inform.* **1991**, *15*, 61–79. [[CrossRef](#)]
- Vakarelov, D. Modal logics for knowledge representation systems. *Theor. Comput. Sci.* **1991**, *90*, 433–456.
- Yao, Y.Y. Constructive and algebraic methods of the theory of rough sets. *Inf. Sci.* **1998**, *109*, 21–47. [[CrossRef](#)]
- Pei, Z.; Pei, D.; Zheng, L. Topology vs. generalized rough sets. *Int. J. Approx. Reason.* **2011**, *52*, 231–239. [[CrossRef](#)]
- Pawlak, Z.; Skowron, A. Rough sets and boolean reasoning. *Inf. Sci.* **2007**, *177*, 41–73. [[CrossRef](#)]
- Qi, G.; Liu, W. Rough operations on Boolean algebras. *Inf. Sci.* **2005**, *173*, 49–63. [[CrossRef](#)]
- Hua, X. Fuzzy rough set based on lattice effect algebra. *J. Intell. Fuzzy Syst.* **2019**, *36*, 29–36. [[CrossRef](#)]
- Radzikowska, A.M.; Kerre, E.E. Fuzzy rough sets based on residuated lattices. *Trans. Rough Sets Lect. Notes Comput. Sci.* **2004**, *3135*, 278–296.
- Bao, Y.; Yang, H.; She, Y. Using one axiom to characterize L-fuzzy rough approximation operators based on residuated lattices. *Fuzzy Sets Syst.* **2018**, *336*, 87–115. [[CrossRef](#)]
- Yin, Y.Q.; Huang, X.K. Fuzzy Roughness in hyperrings based on a complete residuated lattice. *Int. J. Fuzzy Syst.* **2010**, *13*, 185–194.
- Yin, Y.Q.; Zhan, J.M.; Corsini, P. Fuzzy roughness of n-ary hypergroups based on a complete residuated lattice. *Neural Comput. Appl.* **2011**, *20*, 41–57. [[CrossRef](#)]
- Wang, C.Y.; Hu, B.Q. Fuzzy rough sets based on generalized residuated lattices. *Inf. Sci.* **2013**, *248*, 31–49. [[CrossRef](#)]
- She, Y.H.; Wang, G.J. An axiomatic approach of fuzzy rough sets based on residuated lattices. *Comput. Math. Appl.* **2009**, *58*, 189–201. [[CrossRef](#)]
- Jiang, H.; Zhan, J.; Chen, D. Covering-based variable precision L-fuzzy rough sets based on residuated lattices and corresponding applications. *Int. J. Mach. Learn. Cybern.* **2021**, *12*, 2407–2429. [[CrossRef](#)]
- Chen, D.; Zhang, W.; Yeung, D.; Tsang, E.C.C. Rough approximations on a complete completely distributive lattice with applications to generalized rough sets. *Inf. Sci.* **2006**, *176*, 1829–1848.
- Liu, G.L. Generalized rough sets over fuzzy lattices. *Inf. Sci.* **2008**, *178*, 1651–1662. [[CrossRef](#)]
- Ma, L.W. Two fuzzy covering rough set models and their generalizations over fuzzy lattices. *Fuzzy Sets Syst.* **2016**, *294*, 1–17. [[CrossRef](#)]
- Birkhoff, G.; von Neumann, J. The logic of quantum mechanics. *Ann. Math.* **1936**, *37*, 823–843. [[CrossRef](#)]
- Li, T.; Leung, Y.; Zhang, W. Generalized fuzzy rough approximation operators based on fuzzy coverings. *Int. J. Approx. Reason.* **2008**, *48*, 836–856. [[CrossRef](#)]
- Dubois, D.; Prade, H. Rough fuzzy sets and fuzzy rough sets. *Int. J. Gen. Syst.* **1990**, *17*, 191–209. [[CrossRef](#)]
- Yang, B.; Hu, B.Q. Matrix representations and interdependency on L-fuzzy covering-based approximation operators. *Int. J. Approx. Reason.* **2018**, *96*, 57–77. [[CrossRef](#)]
- Li, L.Q.; Jin, Q.; Hu, K.; Zhao, F.F. The axiomatic characterizations on L-fuzzy covering-based approximation operators. *Int. J. Gen. Syst.* **2017**, *46*, 332–353. [[CrossRef](#)]
- Yao, Y.; Yao, B. Covering based rough set approximations. *Inf. Sci.* **2012**, *200*, 91–107. [[CrossRef](#)]
- Yao, Y. The superiority of three-way decisions in probabilistic rough set models. *Inf. Sci.* **2011**, *180*, 1080–1096. [[CrossRef](#)]
- Yao, Y. Three-way decisions with probabilistic rough sets. *Inf. Sci.* **2010**, *180*, 341–352. [[CrossRef](#)]
- Ali, M.I. A note on soft sets, rough soft sets and fuzzy soft sets. *Appl. Soft Comput.* **2011**, *11*, 3329–3332.
- Feng, F.; Li, C.; Davvaz, B.; Ali, M.I. Soft sets combined with fuzzy sets and rough sets: A tentative approach. *Soft Comput.* **2010**, *14*, 899–911. [[CrossRef](#)]
- Kanwal, R.S.; Shabir, M. Rough approximation of a fuzzy set in semigroups based on soft relations. *Comput. Appl. Math.* **2019**, *38*, 1–23. [[CrossRef](#)]
- Riaz, M.; Karaaslan, F.; Nawaz, I.; Sohail, M. Soft multi-rough set topology with applications to multi-criteria decision-making problems. *Soft Comput.* **2021**, *25*, 799–815. [[CrossRef](#)]
- Riaz, M.; Hashmi, M.R.; Kalsoom, H.; Pamucar, D.; Chu, Y.M. Linear Diophantine fuzzy soft rough sets for the selection of sustainable material handling equipment. *Symmetry* **2020**, *12*, 1215. [[CrossRef](#)]

33. Ayub, S.; Shabir, M.; Riaz, M.; Aslam, M.; Chinram, R. Linear Diophantine Fuzzy Relations and Their Algebraic Properties with Decision Making. *Symmetry* **2021**, *13*, 945. [[CrossRef](#)]
34. Riaz, M.; Hashmi, M.R.; Pamucar, D.; Chu, Y.M. Spherical linear Diophantine fuzzy sets with modeling uncertainties in MCDM. *Comput. Modeling Eng. Sci.* **2021**, *126*, 1125–1164. [[CrossRef](#)]
35. Riaz, M.; Hashmi, M.R. Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems. *J. Intell. Fuzzy Syst.* **2019**, *37*, 5417–5439. [[CrossRef](#)]
36. Sun, B.; Ma, W.; Xiao, X. Three-way group decision making based on multigranulation fuzzy decision-theoretic rough set over two universes. *Int. J. Approx. Reason.* **2017**, *81*, 87–102. [[CrossRef](#)]
37. Qian, Y.; Zhang, H.; Sang, Y.; Liang, J. Multigranulation decision theoretic rough sets. *Int. J. Approx. Reason.* **2014**, *55*, 225–237. [[CrossRef](#)]
38. Kong, Q.; Xu, W. The comparative study of covering rough sets and multi-granulation rough sets. *Soft Comput.* **2019**, *23*, 3237–3251. [[CrossRef](#)]
39. Ma, W.; Lei, W.; Sun, B. Three-way group decisions based on multigranulation hesitant fuzzy decision-theoretic rough set over two universes. *J. Intell. Fuzzy Syst.* **2020**, *38*, 2165–2179. [[CrossRef](#)]
40. Hassan, Y.F. Rough set classification based on quantum logic. *J. Exp. Theor. Artif. Intell.* **2017**, *29*, 1325–1336. [[CrossRef](#)]
41. Dai, S. Topological characterizations of rough set theory based on quantum logic. *New Math. Nat. Comput.* **2021**. [[CrossRef](#)]
42. Pták, P.; Pulmannová, S. *Orthomodular Structures as Quantum Logics*; Kluwer: Dordrecht, The Netherlands, 1991.
43. Mittelstaedt, P. *Quantum Logic*; D. Reidel: Dordrecht, The Netherlands, 1978.
44. Ying, M.S. Automata theory based on quantum logic (I). *Int. J. Theor. Phys.* **2000**, *39*, 985–995. [[CrossRef](#)]
45. Ying, M.S. Automata theory based on quantum logic (II). *Int. J. Theor. Phys.* **2000**, *39*, 2545–2557. [[CrossRef](#)]
46. Ying, M.S. A theory of computation based on quantum logic(I). *Theor. Comput. Sci.* **2005**, *344*, 134–207. [[CrossRef](#)]
47. Qiu, D.W. Notes on automata theory based on quantum logic. *Sci. China Ser. F Inf. Sci.* **2007**, *50*, 154–169. [[CrossRef](#)]
48. Dai, S. A note on implication operators of quantum logic. *Quantum Mach. Intell.* **2020**, *2*, 15. [[CrossRef](#)]
49. Chiara, M.L.D.; Giuntini, R.; Greechie, R. *Reasoning in Quantum Theory: Sharp and Unsharp Quantum Logics*; Springer: Berlin/Heidelberg, Germany, 2004.
50. Davey, B.A.; Priestley, H.A. *Introduction to Lattices and Order*; Cambridge University Press: Cambridge, UK, 2002.