Factoring Continuous Characters Defined on Subgroups of Products of Topological Groups †

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† To my colleague and friend M.J. Chasco, with great respect.

Abstract: This study is on the factorization properties of continuous homomorphisms defined on subgroups (or submonoids) of products of (para)topological groups (or monoids). A typical result is the following one: Let $D = \prod_{i \in I} D_i$ be a product of paratopological groups, S be a dense subgroup of D, and χ a continuous character of S. Then one can find a finite set $E \subset I$ and continuous characters χ_i of D_i, for $i \in E$, such that $\chi = \left(\prod_{i \in E} \chi_i \circ p_i \right) {\restriction} S$, where $p_i \colon D \to D_i$ is the projection.

1. Introduction

Factorization of continuous functions defined on (dense) subspaces of topological products has a long and illustrious history, with several new ideas and discoveries. The articles [1–5] provide an excellent overview of the methodologies employed in this area of research.

The current paper is a natural extension of [6,7], in which we investigated continuous homomorphisms of subgroups (submonoids) of topological group products (monoids). We proved in those articles that in many circumstances, a continuous homomorphism $f \colon S \to H$ of a submonoid (subgroup) S of a product $D = \prod i \in I D_i$ of topological monoids (groups) to a topological monoid (group) H enables a factorization in the form

$$f = g \circ p_J {\restriction} S, \tag{1}$$

where J is a "small" subset of the index set I, $p_J \colon D \to D_J = \prod_{i \in J} D_i$ is the projection, and $g \colon p_J(S) \to H$ is a continuous homomorphism. If one can find a finite (countable) set J for which (1) holds true, we say that f has a *finite (countable) type*. Most of the results in [6,7] present different conditions on S and/or H under which f has a countable or even finite type. Purely algebraic aspects of this study can be found in [8].

In this article we go further and try to decompose a given continuous homomorphism $f \colon S \to H$ into a product of 'coordinate' homomorphisms, as explained below.

It follows from the Pontryagin–van Kampen duality theory that every continuous homomorphism of a product $D = \prod_{i \in I} D_i$ of compact abelian groups to the circle group $\mathbb{T}$ (called *character*) has a finite type. Hence, every continuous character of D is a linear combination of finitely many continuous characters, each of which depends on exactly one coordinate. This fact remains valid in a considerably more general situation presented by S. Kaplan in [9]:

Proposition 1. *Let χ be a continuous character of a product $\Pi = \prod_{i \in I} G_i$ of (reflexive) topological abelian groups. Then one can find pairwise distinct indices $i_1, \ldots, i_n \in I$ and continuous characters $\chi_1, \ldots, \chi_n$ of the respective groups $G_{i_1}, \ldots, G_{i_n}$ such that the equality*

$$\chi(x) = \prod_{k=1}^{n} \chi_k(x_{i_k}) \tag{2}$$

holds for each $x \in \Pi$.

An examination of the argument offered in [9] demonstrates that the 'reflexive' can be omitted from the assumptions of Proposition 1. Thus, we may reformulate the conclusion of Proposition 1 by asserting that the dual group $\Pi^{\wedge}$ is algebraically isomorphic to the direct sum of the factors' duals, $\bigoplus_{i \in I} D_i^{\wedge}$. Our objective is to extend the conclusion of Proposition 1 to a much broader class of objects, such as subgroups or submonoids of Cartesian products of monoids or paratopological groups (see Theorem 2, Corollary 1, and Theorems 4–6).

An important property of the torus $\mathbb{T}$ is that it is an *NSS group*, which means that there exists an open neighborhood of the identity in $\mathbb{T}$ containing no nontrivial subgroups. Every Lie group is an NSS group. According to ([7], Theorem 3.11), every continuous homomorphism of an arbitrary subgroup of a product of topological monoids to a Lie group has a finite type. This is an essential ingredient in several arguments presented in Section 2.

In Section 3 we complement several results from ([7], Section 2) about the continuous character of a dense submonoid S of the *P-modification* of a product $D = \prod_{i \in I} D_i$ of *topologized* monoids. We show in Proposition 3 and Example 3 that if $\varphi \colon S \to H$ is a nontrivial continuous homomorphism of S to a topologized monoid of countable pseudocharacter, then the family $\mathcal{J}(\chi)$ of the subsets J of the index set I such that φ *depend on J* is often a filter on I, and this filter can have an empty intersection, even if $S = D$ and the product $D = \mathbb{Z}(2)^\omega$ is a compact metrizable topological group (hence the *P*-modification of D is a discrete group).

Notation and Auxiliary Results

Let $\mathbb{C}$ be the field of complex numbers with the usual Euclidean topology. The torus $\mathbb{T}$ is identified with the multiplicative subgroup $\{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{C}$.

A *semigroup* is a nonempty set S with a binary associative operation (called *multiplication*). A semigroup with an identity is called a *monoid*. Clearly a monoid has a unique identity.

A semigroup S with some topology is said to be a *semitopological semigroup* if multiplication in S is separately continuous. This is equivalent to saying that the left and right shifts in S are continuous. If multiplication in S is jointly continuous, we say that S is a *topological semigroup*. The concept of *topological monoid* is defined similarly.

Assume that G is a semigroup (monoid, group) with a topology. If the left shifts in G are continuous, then G is called a *left topological* semigroup (monoid, group). If both left and right shifts in G are continuous, then G is said to be a *semitopological* semigroup (monoid, group). Further, if G is a group and multiplication in G is jointly continuous, we say that G is a *paratopological* group. A paratopological group with continuous inversions is a *topological* group.

A *topologized* monoid (group) is a monoid (group) with an arbitrary topology that may have no relation to multiplication in the monoid (group). We say that a left topological monoid G has *open left shifts* if for every $x \in G$, the left shift λ_x of G defined by $y \mapsto x \cdot y$ for each $y \in G$ is an open mapping of G to itself.

The *character* of an arbitrary monoid G is a (not necessarily continuous) homomorphism of G to the torus $\mathbb{T}$. The continuity of a character, if it applies, will always be specified explicitly.

In the sequel we follow the notation of Proposition 1. For every $i \in I$, let p_i be the projection of Π onto the factor G_i. Then the conclusion of the proposition is equivalent to saying that $\chi = \prod_{k=1}^{n} \chi_k \circ p_{i_k}$. It is worth noting that the projections p_i are continuous open homomorphisms, so the characters $\chi_1, \ldots, \chi_n$ are 'automatically' continuous. This assertion follows from the next simple result, which shows that for finitely many factors, the conclusion of Proposition 1 remains valid, even if the factors are topologized monoids.

Lemma 1. *Let $G = G_1 \times \cdots \times G_n$ be a product of topologized monoids and χ be a continuous homomorphism of G to a topologized semigroup K. Then there exist homomorphisms $\chi_1, \ldots, \chi_n$ of the respective monoids $G_1, \ldots, G_n$ to K such that $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$, for each $x = (x_1, \ldots, x_n) \in G$. This representation of χ is unique and the homomorphisms $\chi_1, \ldots, \chi_n$ are continuous.*

Proof. For every $k = 1, \ldots, n$, let e_k be the identity of G_k and p_k be the projection of G onto the factor G_k. We define a homomorphism χ_k of G_k to K by $\chi_k(y) = \chi(e_1, \ldots, y, \ldots, e_n)$ for every $y \in G_k$, where y stands at the kth position in $(e_1, \ldots, y, \ldots, e_n)$. A direct verification shows that $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$, for each $x = (x_1, \ldots, x_n) \in G$.

Let $\psi_1, \ldots, \psi_n$ be homomorphisms of $G_1, \ldots, G_n$, respectively, to K, satisfying $\chi(x) = \psi_1(x_1) \cdots \psi_n(x_n)$, for each $x \in G$. We fix an integer k with $1 \leq k \leq n$, and for every $y \in G_k$, consider the element $\hat{y} = (e_1, \ldots, y, \ldots, e_n) \in G$, where y stands at the kth position in $\hat{y}$. Then $\chi_k(y) = \chi(\hat{y}) = \psi_k(y)$, so $\psi_k = \chi_k$ for each $k \leq n$, and hence, the representation $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$ is unique.

It follows from the continuity of the homomorphism χ and the equalities $\chi_k(y) = \chi(e_1, \ldots, y, \ldots, e_n)$, where $1 \leq k \leq n$ and $y \in G_k$, that $\chi_1, \ldots, \chi_n$ are continuous. $\square$

Let $X = \prod_{i \in I} X_i$ be the Tychonoff product of a family $\{X_i : i \in I\}$ of spaces and $a \in X$ be an arbitrary point. For every $i \in I$, the projection of X to the factor X_i is denoted by p_i. In addition, for every $x \in X$, we make

$$\mathrm{diff}(x, a) = \{i \in I : p_i(x) \neq p_i(a)\}.$$

Then

$$\Sigma X(a) = \{x \in X : |\mathrm{diff}(x, a)| \leq \omega\}$$

and

$$\sigma X(a) = \{x \in X : |\mathrm{diff}(x, a)| < \omega\}$$

are dense subspaces of X which are called, respectively, the Σ-*product* and σ-*product* of the family $\{X_i : i \in I\}$ with centers at a. If every X_i is a monoid (group), we will always choose a to be the identity e of X. In the latter case, $\Sigma X(e)$ and $\sigma X(e)$ are dense submonoids (subgroups) of the product monoid (group) X and we shorten $\Sigma X(e)$ and $\sigma X(e)$ to ΣX and σX, respectively.

Assume that Z is a nonempty subset of the product $X = \prod_{i \in I} X_i$ of a family $\{X_i : i \in I\}$ of sets and $f : Z \to Y$ is an arbitrary mapping. We say that f *depends on* J, for some $J \subset I$, if the equality $f(x) = f(y)$ holds for all $x, y \in Z$ with $p_J(x) = p_J(y)$, where $p_J : X \to \prod_{i \in J} X_i$ is the projection. It is clear that if f depends on J, then there exists a mapping g of $p_J(Z)$ to Y satisfying $f = g \circ p_J \upharpoonright Z$. Conversely, if there exists such a mapping g of $p_J(Z)$ to Y, then f depends on J.

Definition 1. *Assume that D_i is a monoid with identity e_i, where $i \in I$. For a nonempty subset J of I, we define a retraction r_J of $D = \prod_{i \in I} D_i$ by letting*

$$r_J(x)_i = \begin{cases} x_i & \text{if } i \in J; \\ e_i & \text{if } i \in I \setminus J, \end{cases}$$

for each element $x \in D$. A subset S of D is said to be retractable *if $r_J(S) \subset S$, for each $J \subset I$. If the inclusion $r_J(S) \subset S$ holds for each finite set $J \subset I$, we call S* finitely retractable.

The concept of finite retractability is used in Theorem 5.

Given a space X, we denote by PX the underlying set X with the topology whose base consists of all nonempty G_δ-sets in X. The space PX is usually referred to as the *P-modification* of X. If X is a (left) topological group (monoid), then PX with the same multiplication is also a (left) topological group (monoid).

The family of countable subsets of a given set I is denoted by $[I]^{\leq \omega}$.

2. Factoring Continuous Characters

In this section, we deal with not necessarily Hausdorff objects of topological algebra. Since a major proportion of the research articles and books on this subject treat the Hausdorffian case exclusively, we need to extend several well-known facts to non-Hausdorffian monoids and groups. We start with the following result that, informally, goes back to Graev's article ([10], pp. 52–53).

Lemma 2. *Let G be a topological group with identity e, N be the closure of the singleton $\{e\}$ in G, and $\pi \colon G \to G/N$ be the quotient homomorphism. For every continuous homomorphism $f \colon G \to H$ to a Hausdorff topological group H, there exists a unique homomorphism $g \colon G/N \to H$ satisfying $f = g \circ \pi$, and g is automatically continuous.*

Proof. Notice that N is a closed invariant subgroup of G, so the quotient topological group G/N is a T_1-space. Hence G/N is a Hausdorff. Denote by K the kernel of f. Since H is a Hausdorff, K is a closed subgroup of G. Hence, $\ker \pi = N \subset K = \ker f$. It now follows from ([11], Proposition 1.5.10) that there exists a homomorphism $g \colon G/N \to H$ satisfying $f = g \circ \pi$. Assume that a homomorphism $\tilde{g} \colon G/N \to H$ also satisfies $f = \tilde{g} \circ \pi$. If $y \in G/N$, we take an element $x \in G$ with $\pi(x) = y$. Then $g(y) = g(\pi(x)) = f(x)$, and similarly, $\tilde{g}(y) = \tilde{g}(\pi(x)) = f(x)$. Hence $\tilde{g}(y) = g(y)$ for each $y \in G/N$, so $\tilde{g} = g$. As π is open and continuous, we conclude that g is continuous. $\square$

The pair $(G/N, \pi)$ in Lemma 2 is called the *Hausdorff reflection* of G. Abusing terminology, we usually refer to G/N as the Hausdorff reflection of G, thereby omitting the quotient homomorphism π. We also denote G/N by $T_2(G)$.

Informally speaking, the following lemma states that the functor of the Hausdorff reflection in the category of topological groups and continuous homomorphisms describes arbitrary subgroups.

Lemma 3. *Let G be a topological group with identity e, N be the closure of the singleton $\{e\}$ in G, and $\pi \colon G \to G/N$ be the quotient homomorphism. Let S be an arbitrary subgroup of G and $N_S = S \cap N$. Then the quotient group $T_2(S) = S/N_S$ is topologically isomorphic to the subgroup $\pi(S)$ of $T_2(G) = G/N$ and the restriction of π to S is an open continuous homomorphism of S onto $\pi(S)$.*

Proof. It follows from the definition of π that every closed subset C of G satisfies $C = \pi^{-1}\pi(C)$. Therefore, if the subgroup S is closed in G then $N \subset S$, $S = \pi^{-1}\pi(S)$, and the restriction of π to S is an open continuous homomorphism of S onto the subgroup $\pi(S)$ of G/N. By the first isomorphism theorem, the groups $\pi(S)$ and S/N are topologically isomorphic.

In the general case, let K be the closure of S in G. Then K is a closed subgroup of G, $N \subset K$, and by the above argument, the groups $T_2(K) = K/N$ and $\pi(K) \subset T_2(G)$ are topologically isomorphic. Hence it suffices to verify that the group $T_2(S)$ is topologically isomorphic in relation to the subgroup $\pi(S)$ of K/N. To this end we show that the restriction of π to S is an open homomorphism onto the subgroup $\pi(S)$ of K/N. Let U be a nonempty open set in K and $V = U \cap S$. Since $K = \pi^{-1}\pi(K)$ and $N \subset K$, the set U satisfies the equality $U = \pi^{-1}\pi(U)$. Hence the set $\pi(U) \cap \pi(S) = \pi(U \cap S) = \pi(V)$ is open in $\pi(S)$. Thus, $\pi \!\restriction\! S$ is an open homomorphism of S onto $\pi(S)$ whose kernel is $S \cap N$, so the groups $T_2(S)$ and $\pi(S)$ are topologically isomorphic. $\square$

Let us recall that the *precompact Hausdorff reflection* of a given topological group G is a pair (H, φ_G), where H is a precompact Hausdorff topological group and $\varphi_G \colon G \to H$ is a continuous homomorphism, such that for every continuous homomorphism $g \colon G \to K$ to a Hausdorff precompact topological group K, there exists a continuous homomorphism

$h\colon H \to K$ satisfying $g = h \circ \varphi_G$. Every topological group G has a precompact Hausdorff reflection and this reflection is unique up to topological isomorphism [12]. The homomorphism φ_G is referred to as *universal* for G.

Lemma 4. *Let S be a dense subgroup of a topological group G and (H, φ_G) be the precompact Hausdorff reflection of G. Let $T = \varphi_G(S)$ and $\psi = \varphi_G{\restriction}S$. Then (T, ψ) is the precompact Hausdorff reflection of the group S.*

Proof. Since H is a precompact Hausdorff topological group, so is its dense subgroup T. Therefore it suffices to verify that the continuous onto homomorphism $\psi\colon S \to T$ is universal for S. Let $g\colon S \to K$ be a continuous homomorphism to a precompact Hausdorff group K. The completion of K, say, ϱK, is a compact Hausdorff topological group. Hence the group ϱK is complete. Since S is dense in G, g extends to a continuous homomorphism $g^*\colon G \to \varrho K$. By the universality of φ_G, there exists a continuous homomorphism $h^*\colon H \to \varrho K$ such that $g^* = h^* \circ \varphi_G$. Let h be the restriction of h^* to T. Then $g = g^*{\restriction}S = h^* \circ \varphi_G{\restriction}S = h^* \circ \psi = h \circ \psi$. This proves the universality of ψ for S. $\square$

A subgroup S of a topological abelian group G is said to be *dually embedded* in G if every continuous character of S extends to a continuous character of G. The next lemma is well known in the special case of Hausdorff topological groups ([13], Lemma 2.2).

Lemma 5. *Every subgroup S of a precompact topological abelian group G is dually embedded in G.*

Proof. Let e be the identity of G and N be the closure of the singleton $\{e\}$ in G. Additionally, let $p\colon G \to G/N$ be the quotient homomorphism. Since G is precompact, the pair $(G/N, p)$ is the precompact Hausdorff reflection of G. Let S be a subgroup of G. Denote by K the closure of S in G. It follows from the definition of N that $N \subset K$ and $K = p^{-1}p(K)$, so $K/N \cong p(K)$ and $(p(K), q)$ is the precompact Hausdorff reflection of K, where $q = p{\restriction}K$. Since S is dense in K, Lemma 4 implies that $(q(S), q{\restriction}S) = (p(S), p{\restriction}S)$ is the precompact Hausdorff reflection of S.

Let χ be a continuous character of S. There exists a continuous character λ of the subgroup $T = p(S)$ of the precompact Hausdorff group G/N such that $\chi = \lambda \circ p{\restriction}S$. By ([13], Lemma 2.2), T is dually embedded in the Hausdorff precompact abelian group G/N, so λ extends to a continuous character λ^* of G/N. Hence $\chi^* = \lambda^* \circ p$ is an extension of χ to a continuous character of G and S is dually embedded in the group G. $\square$

The following fact complements Lemma 5 in the non-abelian case.

Lemma 6. *Every dense subgroup S of an arbitrary topological group G is dually embedded in G.*

Proof. Let (H, φ_G) be the precompact Hausdorff reflection of the group G. We put $T = \varphi_G(S)$ and $\psi = \varphi_G{\restriction}S$. By Lemma 4, the pair (T, ψ) is the precompact Hausdorff reflection of S.

Let χ be a continuous character of S. Then there exists a continuous character χ_T of T such that $\chi = \chi_T \circ \psi$. Since the group H is precompact and Hausdorffian, it follows from ([13], Lemma 2.2) that T is dually embedded in H. Hence, χ_T extends to a continuous character λ of H. Thus, $\chi^* = \lambda \circ \varphi_G$ is a continuous character of G which extends χ. $\square$

Lemma 6 is not valid for *closed* subgroups of Hausdorff topological groups. In fact, even a compact subgroup of a separable metrizable topological abelian group can fail to be dually embedded ([11], Example 9.9.61).

According to Proposition 3.6.12 of [11], a continuous homomorphism of a dense subgroup S of a Hausdorff topological group G to a complete Hausdorffian topological group H extends to a continuous homomorphism of G to H. Below we generalize this fact by showing that it remains valid for dense subgroups of arbitrary paratopological

groups. Our argument makes use of the *topological group reflection* of a paratopological group (see [14]).

Theorem 1. *Let S be a dense subgroup of a paratopological group G and $f\colon S \to H$ be a continuous homomorphism of S to a complete Hausdorff topological group H. Then f extends to a continuous homomorphism $g\colon G \to H$.*

Proof. Let $i_G\colon G \to G_*$ be the identity mapping of G onto the topological group reflection G_* of G. It follows from ([14], Theorem 12) that the subgroup $T = i_G(S)$ of G_* is topologically isomorphic to the topological group reflection S_* of S, so we can identify the groups T and S_* algebraically and topologically.

Since H is a topological group, there exists a continuous homomorphism $f_*\colon T \to H$ satisfying $f = f_* \circ i_G \restriction S$. It follows from the continuity of i_G that T is a dense subgroup of G_*. However, the groups G_* and T may fail to be Hausdorffian.

To reduce our further argument to the case of Hausdorff groups, we denote by N the closure of the singleton $\{e_G\}$ in G_* and consider the quotient homomorphism $\pi\colon G_* \to G_*/N$. Then the quotient group G_*/N is the Hausdorff reflection of G_*. By Lemma 3, the subgroup $\pi(T)$ of G_*/N is the Hausdorff reflection of T and the homomorphism $\varphi = \pi \restriction T$ of T onto $\pi(T)$ is open and continuous. Since the group H is Hausdorffian, Lemma 2 implies the existence of a continuous homomorphism $f^*\colon \pi(T) \to H$ satisfying the equality $f_* = f^* \circ \varphi$. Notice that T is dense in G_* and $\pi(T)$ is dense in G_*/N. Therefore, by ([11], Corollary 3.6.17), f_* extends to a continuous homomorphism $g_*\colon G_*/N \to H$ (we use the completeness of H here).
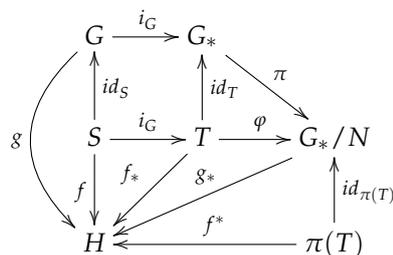
Then $g = g_* \circ \pi \circ i_G$ is a continuous homomorphism of G to H which extends f. This proves the theorem. □

We complement Theorem 1 in Proposition 2 by considering continuous homomorphisms defined on dense submonoids of topological monoids.

Example 1. Closed subgroups of completely regular paratopological groups need not be dually embedded. Hence Theorem 1 does not extend to closed subgroups of paratopological groups.

Proof. Let $\mathbb{S}$ be the Sorgenfrey line endowed with the usual topology and addition. Clearly $\mathbb{S}$ is a regular (even hereditarily normal) paratopological group. Additionally, let $\Delta = \{(x, -x) : x \in \mathbb{S}\}$ be the *second diagonal* of $\mathbb{S} \times \mathbb{S}$. It is well known and easy to verify that the subgroup Δ is discrete and closed. Hence every character of Δ is continuous and Δ can be identified with the real line $\mathbb{R}_d$ endowed with the discrete topology. On the one hand, an easy calculation shows that the family of characters of Δ has the cardinality $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$, where $\mathfrak{c} = 2^\omega$. On the other hand, the groups $\mathbb{S}$ and $\mathbb{S} \times \mathbb{S}$ are separable, so there are at most $\mathfrak{c}^\omega = \mathfrak{c}$ continuous characters of $\mathbb{S} \times \mathbb{S}$. Therefore, not every character of Δ extends to a continuous character of $\mathbb{S} \times \mathbb{S}$. In other words, Δ fails to be dually embedded in $\mathbb{S} \times \mathbb{S}$. It is also clear that not every character of Δ admits the representation described in Lemma 1 (or in Theorem 2 that follows). □

The next result is a considerable generalization of Proposition 1.

Theorem 2. *Let $D = \prod_{i \in I} D_i$ be a product of paratopological groups and S be a subgroup of D. Assume that for every finite set $F \subset I$, the subgroup $p_F(S)$ of $D_F = \prod_{i \in F} D_i$ is dually embedded in D_F, where $p_F : D \to D_F$ is the projection. Then for every continuous character χ of S, one can find a finite set $E \subset I$ and continuous characters χ_i of $p_i(S)$, for $i \in E$, such that $\chi = \left(\prod_{i \in E} \chi_i \circ p_i \right) {\upharpoonright} S$.*

Proof. By Corollary 3.12 in [7], one can find a finite set $E \subset I$ and a continuous character χ_E of $p_E(S)$ such that $\chi = \chi_E \circ p_E {\upharpoonright} S$, where $p_E : D \to \prod_{i \in E} D_i$ is the projection. By the assumptions of the theorem, $T = p_E(S)$ is a dually embedded subgroup of $D_E = \prod_{i \in E} D_i$. Hence χ_E extends to a continuous character ψ of D_E. According to Lemma 1, for every $i \in E$, there exists a continuous character ψ_i of G_i such that $\psi = \prod_{i \in E} \psi_i \circ q_i$, where $q_i : D_E \to D_i$ is the projection. Let $p_i : D \to D_i$ be the projection, for each $i \in E$. Since $p_i = q_i \circ p_E$ and $\chi = \psi \circ p_E {\upharpoonright} S$, we conclude that the required equality $\chi = \left(\prod_{i \in E} \chi_i \circ p_i \right) {\upharpoonright} S$ is valid. $\square$

Example 1 explains why in Theorem 2, we require the projections of a subgroup $S \subset D$ to finite subproducts to be dually embedded, though this does not exclude the possibility that the theorem be valid for *arbitrary* subgroups of products of (para)topological groups. Later, in Example 2, we will show that such a generalization of Theorem 2 is impossible, even if the factors of the product $D = \prod_{i \in I} D_i$ are topological groups.

By Theorem 1, a dense subgroup of a paratopological group is dually embedded. Hence the next corollary is immediate from Theorem 2.

Corollary 1. *Let $D = \prod_{i \in I} D_i$ be a product of paratopological groups, S be a dense subgroup of D, and χ be a continuous character of S. Then one can find a finite set $E \subset I$ and continuous characters χ_i of D_i, for $i \in E$, such that $\chi = \left(\prod_{i \in E} \chi_i \circ p_i \right) {\upharpoonright} S$, where $p_i : D \to D_i$ is the projection.*

The next example shows that the conditions on S for 'dual embedding' in Theorem 2 and 'dense' in Corollary 1 are essential.

Example 2. *There exist countably infinite, metrizable topological abelian groups G_1 and G_2, and a closed discrete subgroup Δ of the product $\Pi = G_1 \times G_2$ such that $p_1(\Delta) = G_1$, $p_2(\Delta) = G_2$, and the only continuous character of the group Π is the trivial one. Here p_1 and p_2 are projections of Π onto G_1 and G_2, respectively. In particular, the trivial character of Δ is the only one representable in the form described in Corollary 1.*

Proof. Let G be a countable, infinite Boolean group. Then G is the direct sum of countable copies of the group $\mathbb{Z}(2) = \{0, 1\}$, so G is as in item (2) of Lemma 0 in [15]. Therefore, Theorem' on page 22 of [15] implies that G admits a metrizable topological group topology τ_1 such that the only continuous character of $G_1 = (G, \tau_1)$ is the trivial one.

Our first observation is that the group G_1 is not precompact — otherwise continuous characters of G_1 would separate elements of G_1. Since every non-zero element of the countable group G_1 has order 2, one can apply ([16], Theorem 5.28) to find an open neighborhood U of zero e_1 in G_1 and a (necessarily discontinuous) automorphism f of the group G_1 such that $f(U) \cap U = \{e_1\}$. In other words, the group G_1 is *self-transversal*.

Let $\tau_2 = \{f(V) : V \in \tau_1\}$ be the image of the topology τ_1 under the automorphism f and $G_2 = (G, \tau_2)$. Then f is a topological isomorphism of G_1 onto G_2 and the only continuous character of G_2 is the trivial one. By Lemma 1 the product group $\Pi = G_1 \times G_2$ has the same property. Denote by Δ the subgroup $\{(x, x) : x \in G\}$ of the group Π. It is clear that $p_1(\Delta) = G_1$ and $p_2(\Delta) = G_2$. The set $O = U \times f(U)$ is open in Π and it follows from our choice of the set U that the intersection $O \cap \Delta$ contains only the identity element of $G_1 \times G_2$. Hence the subgroup Δ of Π is discrete and closed. It is clear that every character of Δ is continuous, and that the only character of Δ that can be expressed in the form presented in Corollary 1 is the trivial one. $\square$

Since the subgroup Δ of the group $G_1 \times G_2$ in Example 2 is discrete, we see that Corollary 1 is not valid for locally compact subgroups of products of topological groups. However, it is valid for *precompact* abelian subgroups of product groups.

First, we present a well-known result from [17] often called the *Comfort–Ross duality* for precompact topological abelian groups. We denote the family of all characters of an abstract group G to the torus $\mathbb{T}$ by $Hom(G, \mathbb{T})$. Clearly, the pointwise multiplication of characters in $Hom(G, \mathbb{T})$, $(\chi_1 \cdot \chi_2)(x) = \chi_1(x) \cdot \chi_2(x)$, makes it an abelian group.

Theorem 3. *For every abelian group G, there exists a natural (i.e., functorial) monotone bijection between the family of precompact topological group topologies on G and the subgroups of the group $Hom(G, \mathbb{T})$.*

'Monotone' in Theorem 3 means that a finer precompact topological group topology on G corresponds to a bigger subgroup of $Hom(G, \mathbb{T})$. For more details on this correspondence, see [17].

In the following theorem we do not impose any separation restrictions on the factors D_i:

Theorem 4. *Let C be a precompact abelian subgroup of a product $D = \prod_{i \in I} D_i$ of topological groups and χ be a continuous character of C. Then one can find a finite set $E \subset I$ and continuous characters χ_i of $p_i(C)$, for $i \in E$, such that $\chi = \left(\prod_{i \in E} \chi_i \circ p_i \right) \upharpoonright C$, where $p_i \colon D \to D_i$ is the projection.*

Proof. The projection $p_i(C)$ is a precompact abelian subgroup of the group D_i, for each $i \in I$. We can assume, therefore, that each factor $D_i = p_i(C)$ is a precompact abelian group. Then D is also a precompact topological abelian group. For every $i \in I$, let $D_i^\wedge$ be group of continuous characters of D_i. By ([17], Theorem 1.2), the topology of D_i is initial with respect to $D_i^\wedge$. Consider the family

$$\mathcal{A} = \{ \chi \circ p_i : i \in I, \ \chi \in D_i^\wedge \}.$$

Then each element of $\mathcal{A}$ is a continuous character of D, so $\mathcal{A} \subset D^\wedge$. Let H be the subgroup of $D^\wedge$ generated by $\mathcal{A}$. Every element χ of H has the form

$$\chi = \prod_{k=1}^n \chi_k \circ p_{i_k}, \tag{3}$$

where $i_1, \dots, i_n$ are pairwise distinct elements of I and $\chi_k \in D_{i_k}^\wedge$ for each $k = 1, \dots, n$. It is clear that the topology of D is initial with respect to H. Since C is a topological subgroup of D, the family of restrictions $H_C = \{ \chi \upharpoonright C : \chi \in H \}$ generates the topology of C. Notice that H_C is a subgroup of $C^\wedge \cap Hom(C, \mathbb{T})$, so Theorem 3 implies that $H_C = C^\wedge$. The latter equality, together with (3), implies the required conclusion. $\square$

Problem 1. *Does Theorem 4 extend to precompact subgroups of products of paratopological abelian groups?*

The main difficulty in solving Problem 1 is the fact that the topological group reflection of a subgroup C of a paratopological abelian group D can have a strictly finer topology than the topology of C inherited from D_*. In other words, Lemma 4 cannot be extended to paratopological groups. Even the very special case of Problem 1, where C is a precompact subgroup of the product of two (precompact) paratopological groups, is not clear.

The following result extends a well-known property of continuous homomorphisms of topological groups to a more general case when the domain of a homomorphism is a dense *submonoid* of a topological monoid with open shifts. First we recall the notions of Roelcke uniformity and Roelcke completeness in topological groups.

Let G be a topological group and $\mathcal{N}(e)$ be the family of open neighborhoods of the identity e in G. For every $U \in \mathcal{N}(e)$, the set

$$O_U = \{UxU : x \in G\}$$

is an open entourage of the diagonal in $G \times G$ and the family $\{O_U : U \in \mathcal{N}(e)\}$ constitutes a base for a compatible uniformity on G, say, $\mathcal{V}_G$, which is called the *Roelcke uniformity* of G (see [11], Section 1.8). If the uniform space $(G, \mathcal{V}_G)$ is complete, we say that the group G is *Roelcke-complete*.

Proposition 2. *Let S be a dense submonoid of a topological monoid D with open shifts. Then every continuous homomorphism $f : S \to K$ to a Roelcke-complete Hausdorff topological group K extends to a continuous homomorphism $f^* : D \to K$.*

Proof. Let $\mathcal{N}(e)$ be the family of open neighborhoods of the identity e in D. We denote by $\mathcal{Q}$ the *quasi-Roelcke* uniformity of D whose base consists of the sets

$$Q_V = \{(x,y) \in D \times D : Vx \cap yV \neq \varnothing \neq Vy \cap xV\},$$

where $V \in \mathcal{N}(e)$ (see [18]). It is easy to see that the topology of D generated by $\mathcal{Q}$ is weaker than the original topology of D. Additionally, let $\mathcal{V}_K$ be the Roelcke uniformity of the group K.

Consider a continuous homomorphism $f : S \to K$ to a Roelcke-complete Hausdorff topological group K with identity e_K. We claim that f is uniformly continuous considered as a mapping of $(S, \mathcal{Q}{\upharpoonright}S)$ to $(K, \mathcal{V}_K)$. To this end, take an arbitrary symmetric element $U \in \mathcal{N}(e_K)$ and choose an element $W \in \mathcal{N}(e_K)$ such that $W^2 \subset U$. Then $\overline{W} \subset U$. By the continuity of f, we can find an element $V \in \mathcal{N}(e)$ satisfying $f(V \cap S) \subset W$. We are yet to verify that $(f(x), f(y)) \in O_U$ whenever $(x,y) \in Q_V \cap S^2$, or equivalently, $(f \times f)(Q_V \cap S^2) \subset O_U$.

Let $(x,y) \in Q_V \cap S^2$. Then $Vx \cap yV \neq \varnothing$ and $Vy \cap xV \neq \varnothing$. Since S is dense in D and the sets Vx and yV are open in D, we can choose a point $z \in S \cap Vx \cap yV$. It follows from the continuity of shifts in D and the density of $S \cap V$ in V that for $z \in Vx \subset \overline{(S \cap V) \cdot x}$, the closure is taken in D. As $z \in S$, we see that z is in the closure of $(S \cap V) \cdot x$ in S. Hence $f(z) \in \overline{f(V \cap S) \cdot f(x)} = \overline{f(V \cap S)} \cdot f(x)$, by the continuity of f; the closure is taken in K. Since $\overline{f(V \cap S)} \subset \overline{W} \subset U$, the latter implies that $f(z) \in Uf(x)$. A similar argument, starting with $z \in yV$, shows that $f(z) \in f(y)U$. Thus $f(z) \in Uf(x) \cap f(y)U \neq \varnothing$, whence $f(y) \in Uf(x)U^{-1} = Uf(x)U$. This implies that $(f(x), f(y)) \in O_U$ and proves the uniform continuity of f as a mapping of $(S, \mathcal{Q}{\upharpoonright}S)$ to $(K, \mathcal{V}_K)$.

Since the space $(K, \mathcal{V}_K)$ is complete, f extends to a uniformly continuous mapping $f^* : (D, \mathcal{Q}) \to (K, \mathcal{V}_K)$. It follows from the density of S in D and the Hausdorffness of K that f^* is a homomorphism. $\square$

Corollary 2. *Let S be a dense submonoid of a topological monoid D with open shifts. Then every continuous homomorphism $f : S \to K$ to a locally compact topological group K extends to a continuous homomorphism $f^* : D \to K$.*

Proof. According to Proposition 2 it suffices to verify that every locally compact topological group K is Roelcke-complete. The latter fact is immediate since for every compact neighborhood U of the identity in K, every Cauchy filter ξ in the uniform space $(K, \mathcal{V}_K)$ has an element contained in the compact set UxU, for some $x \in K$. Hence ξ converges to an element of K and $(K, \mathcal{V}_K)$ is complete, where $\mathcal{V}_K$ is the Roelcke uniformity of K. $\square$

Now we apply Proposition 2 in a less obvious way.

Theorem 5. *Let S be a dense submonoid of a product $D = \prod_{i \in I} D_i$ of topological monoids with open shifts and $f : S \to K$ be a continuous homomorphism to a Lie group K. If S is either finitely*

retractable or open in D, then f extends to a continuous homomorphism $f^*: D \to K$. Hence, one can find a finite set $E \subset I$ and continuous homomorphisms $\chi_i: D_i \to K$ for $i \in E$, such that $f^*(x) = \prod_{i \in E} \chi_i(x_i)$ for each $x = (x_i)_{i \in I} \in D$.

Proof. Depending on whether S is finitely retractable or open, we apply, respectively, Theorem 2.12 or Theorem 3.8(b) of [7] to conclude that f depends on a finite set $E \subset I$. In either case, there exists a continuous homomorphism $g: p_E(S) \to K$ satisfying $f = g \circ p_E{\restriction}S$, where p_E is the projection of D to $D_E = \prod_{i \in E} D_i$. Then $p_E(S)$ is a dense submonoid of D_E and D_E is a topological monoid with open shifts, by ([7], Lemma 3.5). Hence we are entitled to apply Proposition 2 to the homomorphism g. Hence, there exists a continuous homomorphism $g^*: D_E \to K$ extending g. According to Lemma 1 we can find continuous homomorphisms $\chi_i: D_i \to K$ for $i \in E$ such that $g(y) = \prod_{i \in E} \chi_i(y_i)$, for each $y = (y_i)_{i \in E}$. Then $f^* = g^* \circ p_E$ is a continuous homomorphism of D to K extending f and satisfying $f^*(x) = \prod_{i \in E} \chi_i(x_i)$, for each $x \in D$. This implies the required equality for the homomorphism f. □

According to ([7], Theorem 5), every continuous homomorphism $f: S \to K$ of an arbitrary subgroup S of a product D of topological monoids to a Lie group K has a *finite type*, i.e., can be represented as the composition of the projection p_E of S to a finite subproduct D_E of D and a continuous homomorphism of $p_E(S)$ to K. Therefore, by arguing as in the proof of Theorem 5 and applying Proposition 2 we deduce the following:

Theorem 6. *Let $D = \prod_{i \in I} D_i$ be a product of topological monoids with open shifts, S be a dense subgroup of D, and $f: S \to K$ be a continuous homomorphism to a Lie group K. Then f extends to a continuous homomorphism $f^*: D \to K$, so one can find a finite set $E \subset I$ and continuous homomorphisms $\chi_i: D_i \to K$, for $i \in E$, such that $f^*(x) = \prod_{i \in E} \chi_i(x_i)$ for each $x = (x_i)_{i \in I} \in D$.*

3. More on Continuous Homomorphisms of *P*-Modifications of Products and Their Dense Submonoids

First we introduce notation which is used in this section and clarifies our aim.

Let $X = \prod_{i \in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of sets, Z be a subset of X, and $f: Z \to Y$ be a mapping. Denote by $\mathcal{J}(f)$ the family of all sets $J \subset I$ such that f depends on J. Our main concern is to determine the properties of the family $\mathcal{J}(f)$. For example, one can ask whether $\mathcal{J}(f)$ is a filter or whether it has minimal, by inclusion, elements, or even the smallest element. It has been shown by W. Comfort and I. Gotchev in [19–21] that the family $\mathcal{J}(f)$ can have quite a complicated set-theoretic structure, even if X is a Cartesian product of topological spaces and f is a continuous mapping to a space Y. It is worth mentioning that the thorough study of the family $\mathcal{J}(f)$ was motivated by a somewhat simpler question on whether $\mathcal{J}(f)$ had a countable element $J \subset I$. The reader can find an extensive bibliography related to this question in the aforementioned articles and in the earlier survey article [22] by M. Hušek.

It turns out that the intersection of the family $\mathcal{J}(f)$, denoted by J_f, admits a clear description in terms of f. We say that an index $i \in I$ is *f-essential* if there exist points $x, y \in Z$ such that $\mathrm{diff}(x, y) = \{i\}$ and $f(x) \neq f(y)$. Let E_f be the set of all f-essential indices in I. By Proposition 2.2 in [23], $J_f = E_f = \bigcap \mathcal{J}(f)$. In particular, the set J_f is empty if and only if no index $i \in I$ is f-essential.

Below we present a useful fact which is not valid for arbitrary dense subgroups of the topological group PD, the P-modifications of the product $D = \prod_{i \in I} D_i$ of topologized monoids D_i, not even if the factors D_i are finite discrete groups (see [6], Example 1).

Proposition 3. *Let $D = \prod_{i \in I} D_i$ be a Cartesian product of topologized monoids, S be a submonoid of D with $\Sigma D \subset S$, and $\varphi: PS \to H$ be a nontrivial continuous homomorphism of the P-modification of S to a topologized monoid H of countable pseudocharacter. Then the family*

$$\mathcal{J}(\varphi) = \{J \subset I : \varphi \text{ depends on } J\}$$

is a filter on the index set I.

Proof. Since the subspace PS of PD is a P-space, the homomorphism $\varphi\colon PS \to PH$ remains continuous (see, e.g., [6], Lemma 6). Notice that PH is a discrete space. Therefore, we can assume that H carries the discrete topology. By applying ([6], Proposition 2), we find a countable subset E of I and a continuous homomorphism φ_E of $p_E(S) \subset PD_E$ to H such that $\varphi = \varphi_E \circ p_E \upharpoonright S$, where $p_E\colon D \to D_E = \prod_{i \in E} D_i$ is the projection. It follows from $\Sigma D \subset S$ that $p_E(S) = D_E$. Hence $\bar{\varphi} = \varphi_E \circ p_E$ is a continuous homomorphism of PD to H. It follows from the definition of $\bar{\varphi}$ that this homomorphism depends on E. Furthermore, if $\bar{\varphi}$ depends on F, for some $F \subset I$, then so does φ. It is now clear that $\mathcal{J}(\bar{\varphi}) = \mathcal{J}(\varphi)$.

Therefore, we can assume without loss of generality that φ is a continuous character of $PD = S$. Assume that $J_1 \subset J_2 \subset I$ and $J_1 \in \mathcal{J}(\varphi)$. Then there exists a mapping $g\colon D_{J_1} = \prod_{i \in J_1} D_i \to H$ satisfying $\varphi = g \circ p_{J_1}$, where $p_{J_1}\colon PD \to PD_{J_1}$ is the projection. Clearly g is a homomorphism. Since the projection p_{J_1} is open, the homomorphism g is continuous. Therefore, g is a continuous homomorphism of PD_{J_1} to H. Let $p_{J_1}^{J_2}$ be the projection of D_{J_2} to D_{J_1}. Then $\varphi = g \circ p_{J_1} = g \circ p_{J_1}^{J_2} \circ p_{J_2} = f \circ p_{J_2}$, where $f = g \circ p_{J_1}^{J_2}$ is a continuous homomorphism of PD_{J_2}. Hence, φ depends on J_2 and $J_2 \in \mathcal{J}(\varphi)$.

Let J_1 and J_2 be arbitrary elements of $\mathcal{J}(\varphi)$. It is easy to see that $\ker p_{J_1} \subset \ker \varphi$ and $\ker p_{J_2} \subset \ker \varphi$. Put $J = J_1 \cap J_2$. Then

$$\ker p_J = \ker p_{J_1} \cdot \ker p_{J_2} \subset \ker \varphi \neq D.$$

In particular, $J \neq \varnothing$ (we identify $p_\varnothing$ with the constant mapping of D to the identity e_D of D). It follows from the inclusion $\ker p_J \subset \ker \varphi$ that there exists a homomorphism $h\colon D_J \to H$ satisfying $\varphi = h \circ p_J$ (see [24], Theorem 1.48 or [6], Lemma 2). We conclude that $J \in \mathcal{J}(\varphi)$.

To sum up, the family $\mathcal{J}(\varphi)$ is a filter. $\square$

The reader can find several results about continuous homomorphisms or characters defined on dense submonoids and subgroups of Cartesian (equivalently, *Tychonoff*) products in [6,7]. On many occasions, the conclusions there are stronger than the one in Proposition 3.

It is natural to ask whether the filter $\mathcal{J}(\varphi)$ in Proposition 3 contains a minimal by inclusion element. The next example answers this question in the negative, even if S is the P-modification of the compact metrizable group $\mathbb{Z}(2)^\omega$ (so S is discrete). Notice that the continuous characters of the compact group $\mathbb{Z}(2)^\omega$ are described in Proposition 1.

Example 3. Let the group $G = \mathbb{Z}(2)^\omega$ carry the discrete topology. There exist a non-trivial character χ of G and a decreasing sequence $\{J_n : n \in \omega\}$ of infinite subsets of ω with empty intersection such that χ depends on J_n, for each $n \in \omega$. Hence the filter $\mathcal{J}(\chi)$ does not have minimal elements.

Proof. Let $J_n = \omega \setminus \{0, 1, \dots, n\}$, for each $n \in \omega$. Denote by $\mathbf{1}$ the point of $\mathbb{Z}(2)^\omega$ all coordinates of which are equal to 1. Additionally, let

$$H_n = \{x \in \mathbb{Z}(2)^\omega : x(i) = 0 \text{ for each } i \in J_n\}.$$

Clearly, H_n is a subgroup of G and $H_n \subset H_{n+1}$, for each $n \in \omega$. Hence $H = \bigcup_{n=0}^\infty H_n$ is also a subgroup of G. Since $\mathbf{1} \notin H$, there exists a character χ of G such that $\chi(H) = \{1\}$ and $\chi(\mathbf{1}) = -1$. It is immediate from the definition that χ depends on J_n, for each $n \in \omega$. Since $\bigcap_{n=0}^\infty J_n = \varnothing$, the family $\mathcal{J}(\chi)$ has no smallest element. Taking into account that $\mathcal{J}(\chi)$ is a filter (see Proposition 3), we infer that it does not contain minimal elements either. $\square$

Since the subgroup H of G in the proof of Example 3 is dense in $G = \mathbb{Z}(2)^\omega$ provided the latter group is endowed with the usual Tychonoff product topology, the above character χ is discontinuous on the compact group $\mathbb{Z}(2)^\omega$. It turns out that considering the Tychonoff product topology improves the situation greatly—the family $\mathcal{J}(\chi)$ always has a finite

minimal (by inclusion) element, for each continuous character χ of an arbitrary subgroup G of a product of left topological groups. This conclusion can be recovered using techniques from [9] in the special case where G itself is a product of *topological* groups, but the reader can find a direct argument in the more general Proposition 2.1 of [7].

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