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On Uniform Stability with Growth Rates of Stochastic Skew-Evolution Semiflows in Banach Spaces

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Abstract: The main purpose of this paper is to study a more general concept of uniform stability in mean in which the uniform behavior in the classical sense is replaced by a weaker requirement with respect to some probability measure. This concept includes, as particular cases, the concepts of uniform exponential stability in mean and uniform polynomial stability in mean. Extending techniques employed in the deterministic case, we obtain variants of some results for the general cases of uniform stability in mean for stochastic skew-evolution semiflows in Banach spaces.

Keywords: stochastic skew-evolution semiflows; stochastic evolution cocycle; growth rate; uniform stability in mean

MSC: 34D05; 93D20; 37L55

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1. Introduction

During the last decades, considerable attention has been devoted to the problem of asymptotic behaviors of nonautonomous differential equations in Banach spaces. Many results can be carried out not only for differential equations and evolution operators but also for skew-evolution semiflows. The notion of skew-evolution semiflow was introduced in [1] and includes some particular cases of many well-known concepts in dynamical system theory, such as C_0-semigroups, evolution operators and skew-product semiflows.

In this paper, we consider the case of stochastic skew-evolution semiflows studied in [2]. We note that the stochastic cocycles studied in [3] are particular cases of the concept below.

Several important examples of stochastic evolution semiflows give rise to stochastic evolution equations, and the reader can refer to the monographs by [4,5].

The main purpose of the present paper is to study a general concept of uniform asymptotic stability in mean, which we call "uniform h-stability in mean" where $h : \mathbb{R}_+ \to [1, \infty)$ is a growth rate (i.e., h is nondecreasing and bijective).

In particular cases, we obtain the concepts of uniform exponential stability in mean and uniform polynomial stability in mean for stochastic skew-evolution semiflows in Banach spaces.

Our approach is based on the extension of some techniques used in the deterministic case by many authors, and here, we only mention the papers [6–16].

Thus, we obtain three types of characterizations for each stability in mean concept considered in our study. Connections between these concepts are given. For some other approaches to the study of uniform exponential stability in mean and uniform polynomial stability in mean, we refer to [17–20].

The paper is organized as follows. In Section 2, we review some preliminaries on stochastic skew-evolution semiflows, which will be used in the paper. A general concept of uniform asymptotic stability in mean for stochastic skew-evolution semiflows is defined. In particular, the results of the concept of uniform exponential stability in mean and uniform polynomial stability in mean. In Section 3, connections between these concepts are presented. In Section 4, we state and prove the main results of our paper. Thus, we obtain three types of characterizations for each stability in mean concept considered in our study.

2. Preliminaries

Let $(\Omega, \mathcal{B}, P)$ be a probability space. Denote $\Delta = \{(t,s) : t \geq s \geq 0\}$. We also denote by X a real or complex Banach space and by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X.

Definition 1. *A measurable random field $\varphi : \Delta \times \Omega \to \Omega$ is called a stochastic evolution semiflow on Ω if*

- *(s1) $\varphi(t,t,\omega) = \omega$ for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$,*
- *(s2) $\varphi(t,s,\varphi(s,t_0,\omega)) = \varphi(t,t_0,\omega)$ for all $t \geq s \geq t_0 \geq 0$ and all $\omega \in \Omega$.*

Definition 2. *A map $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is called a stochastic evolution cocycle associated with a stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$ if*

- *(c1) $\Phi(t,t,\omega) = I$ (the identity operator on X) for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$,*
- *(c2) $\Phi(t,s,\varphi(s,t_0,\omega))\Phi(s,t_0,\omega) = \Phi(t,t_0,\omega)$ for all $t \geq s \geq t_0 \geq 0$ and all $\omega \in \Omega$.*

In this case, the pair (Φ,φ) is called a stochastic skew-evolution semiflow on $X_1 = \Omega \times X$

Example 1. *If $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is a stochastic evolution cocycle associated with the stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$ and $g : \mathbf{R}_+ \to [1,\infty)$ is a growth rate, then*

$$\varphi_h : \Delta \times \Omega \to \Omega, \quad \varphi_h(t,s,\omega) = \varphi(h^{-1}(e^t), h^{-1}(e^s), \omega)$$

is a stochastic evolution semiflow on Ω and

$$\Phi_h : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_h(t,s,\omega) = \Phi(h^{-1}(e^t), h^{-1}(e^s), \omega)$$

is a stochastic evolution cocycle associated with the stochastic evolution semiflow φ_h.

Example 2. *If $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ is a stochastic evolution cocycle associated with the stochastic evolution semiflow $\varphi : \Delta \times \Omega \to \Omega$ and $h : \mathbf{R}_+ \to [1,\infty)$ is a growth rate, then*

$$\psi_h : \Delta \times \Omega \to \Omega, \quad \psi_h(t,s,\omega) = \varphi(h^{-1}(t+1), h^{-1}(s+1), \omega)$$

is a stochastic evolution semiflow on Ω and

$$\Psi_h : \Delta \times \Omega \to \mathcal{B}(X), \quad \Psi_h(t,s,\omega) = \Phi(h^{-1}(t+1), h^{-1}(s+1), \omega)$$

is a stochastic evolution cocycle associated with the stochastic evolution semiflow ψ_h.

Remark 1. *Other examples of stochastic skew-evolution semiflows are given in [20].*

We denote by $\mathcal{L}(\Omega, X, P)$ the Banach space of all Bochner measurable functions $f : \Omega \to X$ with

$$\int_\Omega \|f(\omega)\| dP(\omega) < \infty$$

We also denote $Y = \mathbb{R}_+ \times \mathcal{L}(\Omega, X, P)$ and $Z = \Delta \times \mathcal{L}(\Omega, X, P)$.

3. Stability Concepts in Mean

Let (Φ,φ) be a stochastic skew-evolution semiflow on $X_1 = \Omega \times X$ and $h : \mathbf{R}_+ \to [1,\infty)$ a growth rate.

The main stability in mean concept studied in this paper is introduced by

Definition 3. *The stochastic skew-evolution semiflow* (Φ, φ) *is uniformly h-stable in mean (u.h.s.m.) if there exist* $N > 1$ *and* $\nu > 0$ *with*

$$h(t)^{\nu} \int_{\Omega} \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \leq N h(s)^{\nu} \int_{\Omega} \|x(\omega)\| dP(\omega)$$

for all $(t,s,x) \in Z$.

As particular cases, we have

- *(i) if* $h(t) = e^t$, *we obtain the uniform exponential stability in mean (u.e.s.m.) concept.*
- *(ii) if* $h(t) = t + 1$, *it results in the concept of uniform polynomial stability (u.p.s.m.).*

Another concept used is given by

Definition 4. *The stochastic skew-evolution semiflow* (Φ, φ) *has uniform h-stable growth in mean (u.h.g.m.) if there are* $M > 1$ *and* $\omega > 0$ *with*

$$h(s)^{\omega} \int_{\Omega} \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \leq M h(t)^{\omega} \int_{\Omega} \|x(\omega)\| dP(\omega)$$

for all $(t,s,X) \in Z$.

As particular cases, we have

- *(i) for* $h(t) = e^t$, *we obtain the concept of uniform exponential growth in mean (u.e.g.m.).*
- *(ii) for* $h(t) = t + 1$, *it results in the the concept of uniform polynomial growth in mean (u.p.s.m.).*

Remark 2. *The connections between stability and growth in mean concepts are given in the next diagram*

$$u.h.s.m. \quad \Rightarrow \quad u.g.s.m.$$

and

$$\begin{array}{ccc} u.e.s.m. & \Rightarrow & u.p.s.m. \\ \Downarrow & & \Downarrow \\ u.e.g.m. & \Leftarrow & u.p.g.m. \end{array}$$

The next theorem presents the connection between uniform h-stability in mean and uniform exponential stability in mean.

Theorem 1. *The stochastic skew-evolution semiflow* (Φ, φ) *is uniformly h-stable in mean if and only if the stochastic skew-evolution semiflow* (Φ_h, φ_h), *where*

$$\Phi_h : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_h(t,s,w) \stackrel{def}{=} \Phi(h^{-1}(e^t), h^{-1}(e^s), \omega)$$

and

$$\varphi_h : \Delta \times \Omega \to \mathcal{B}(X), \quad \varphi_h(t,s,w) \stackrel{def}{=} \varphi(h^{-1}(e^t), h^{-1}(e^s), \omega)$$

is uniformly exponentially stable in mean.

Proof. *Necessity.* If (Φ, φ) is uniformly h-stable in mean and $(t,s,\omega) \in \Delta \times \Omega$ then there are the constants $N > 1$ and $\nu > 0$ such that

$$\begin{aligned} e^{\nu t}\int_{\Omega}\|\Phi_h(t,s,\omega)x(\omega)\|dP(\omega) &= e^{\nu t}\int_{\Omega}\|\Phi(h^{-1}(e^t),h^{-1}(e^s),\omega)\|dP(\omega) \leq \\ &\leq N\left(\frac{h(h^{-1}(e^s))}{h(h^{-1}(e^t))}\right)^{\nu}\int_{\Omega}\|x(\omega)\|dP(\omega) = \\ &= Ne^{\nu t}\int_{\Omega}\|x(\omega)\|dP(\omega). \end{aligned}$$

and hence, (Φ_h, φ_h) is uniformly exponentially stable in mean.

Sufficiency. If (Φ_h, φ_h) is uniformly exponentially stable in mean and $(t,s,\omega) \in \Delta \times \Omega$ then there are the constants $N > 1$ and $\nu > 0$ such that

$$\begin{aligned} h(t)^{\nu}\int_{\Omega}\|\Phi(t,s,\omega)x(\omega)\|dP(\omega) &= h(t)^{\nu}\int_{\Omega}\|\Phi(h^{-1}(h(t)),h^{-1}(h(s)),\omega)x(\omega)\|dP(\omega) = \\ &= h(t)^{\nu}\int_{\Omega}\|\Phi(h^{-1}(e^{\ln h(t)})),h^{-1}(e^{\ln h(s)}),\omega)x(\omega)\|dP(\omega) \leq \\ &\leq Nh(t)^{\nu}e^{-\nu(\ln h(t)-\ln h(s))}\int_{\Omega}\|x(\omega)\|dP(\omega) = \\ &= Nh(s)^{\nu}\int_{\Omega}\|x(\omega)\|dP(\omega). \end{aligned}$$

and hence (Φ, φ) is uniformly h-stable in mean. □

The connection between uniform polynomial stability in mean and uniform exponential stability in mean is given by

Corollary 1. *The stochastic skew-evolution semiflow (Φ, φ) is uniformly polynomial stable in mean if and only if the stochastic skew-evolution semiflow (Φ_1, φ_1), where*

$$\Phi_1 : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_1(t,s,w) \overset{def}{=} \Phi(e^t - 1, e^s - 1, \omega)$$

and

$$\varphi_1 : \Delta \times \Omega \to \mathcal{B}(X), \quad \varphi_1(t,s,w) \overset{def}{=} \varphi(e^t - 1, e^s - 1, \omega)$$

is uniformly exponentially stable in mean.

Proof. It yields from Theorem 1 for $h(t) = t + 1$. □

The next theorem presents the connection between uniform h-stability in mean and uniform polynomial stability in mean.

Theorem 2. *The stochastic skew-evolution semiflow (Φ, φ) is uniformly h-stable in mean if and only if the stochastic skew-evolution semiflow (Ψ_h, ψ_h), where*

$$\Psi_h : \Delta \times \Omega \to \mathcal{B}(X), \quad \Psi_h(t,s,w) \overset{def}{=} \Phi(h^{-1}(t+1), h^{-1}(s+1), \omega)$$

and

$$\psi_h : \Delta \times \Omega \to \mathcal{B}(X), \quad \psi_h(t,s,w) \overset{def}{=} \varphi(h^{-1}(t+1), h^{-1}(s+1), \omega)$$

is uniformly polynomially stable in mean.

Proof. It is similar to the proof of Theorem 1. □

As a particular case, we obtain

Corollary 2. *The stochastic skew-evolution semiflow* (Φ, φ) *is uniformly exponentially stable in mean if and only if the stochastic skew-evolution semiflow* (Φ_2, φ_2)*, where*

$$\Phi_2 : \Delta \times \Omega \to \mathcal{B}(X), \quad \Phi_2(t,s,w) \stackrel{def}{=} \Phi(\ln(t+1), \ln(s+1)\omega)$$

and

$$\varphi_2 : \Delta \times \Omega \to \mathcal{B}(X), \quad \varphi_2(t,s,w) \stackrel{def}{=} \varphi(\ln(t+1), \ln(s+1)\omega)$$

is uniformly polynomially stable in mean.

Proof. It follows from Theorem 2 for $h(t) = e^t$. □

4. The Main Results

A first characterization of uniform exponential stability in mean is given by

Theorem 3. *If the stochastic skew-evolution* (Φ, φ) *has uniform exponential growth in mean, then* (Φ, φ) *is uniformly exponentially stable in mean if and only if there are* $r > 1$ *and* $c \in (0,1)$ *such that*

$$\int_\Omega \|\Phi(r+s,s,\omega)x(\omega)\|dP(\omega) \le c\int_\Omega \|x(\omega)\|dP(\omega)$$

for all $(s,x) \in Y$.

Proof. *Necessity.* If (Φ, φ) is *u.e.s.m.* then so are the constants $N > 1$ and $\nu > 0$ with

$$r \stackrel{def}{=} 1 + \frac{\ln N}{\nu} > 1$$

We have

$$\begin{aligned} \int_\Omega \|\Phi(r+s,s,\omega)x(\omega)\|dP(\omega) &\le Ne^{-\nu r}\int_\Omega \|x(\omega)\|dP(\omega) = \\ &= c\int_\Omega \|x(\omega)\|dP(\omega) \end{aligned}$$

for all $(s,x) \in Y$ where $c \stackrel{def}{=} Ne^{-\nu r} \in (0,1)$ because $e^{\nu r} > e^{\ln N} = N$

Sufficiency. If $(t,s) \in \Delta$, then there exists $n \in \mathbb{N}$ and $r_1 \in [0,r)$ such that $t = s + nr + r_1$. Then

$$\begin{aligned} &\int_\Omega \|\Phi(t,s,\omega)x(\omega)\|dP(\omega) = \\ = &\int_\Omega \|\Phi(s+nr+r_1, s+nr, \varphi(s+nr,s,\omega)\Phi(s+nr,s,\omega)x(\omega)\|dP(\omega) \le \\ \le &Me^{\omega r_1}\int_\Omega \|\Phi(s+nr,s,\omega)x(\omega)\|dP(\omega) \le \\ \le &Me^{\omega r}\int_\Omega \|\Phi(s+nr, s+(n-1)r, \varphi(s+(n-1)r,s,\omega)\Phi(s+(n-1)r,s,\omega)x(\omega)\|dP(\omega) \le \\ \le &Mce^{\omega r}\int_\Omega \|\Phi(s+(n-1)r,s,\omega)x(\omega)\|dP(\omega) \le \\ \le &\ldots \le Mc^{n-1}e^{\omega r}\int_\Omega \|\Phi(s+r,s,\omega)x(\omega)\|dP(\omega) \le \\ \le &Me^{\omega r}c^n\int_\Omega \|x(\omega)\|dP(\omega) \end{aligned}$$

for all $(s,x) \in Y$. If we denote

$$\nu = -\frac{\ln c}{r}$$

then $\nu > 0$ and $c^n = e^{-\nu rn}$ and

$$\int_\Omega \|\Phi(t,s,\omega)x(\omega)\|dP(\omega) \le Me^{\omega r}e^{-\nu(t-s-r_1)}$$

$$\int_{\Omega} \|x(\omega)\| dP(\omega) \leq Me^{(\omega+\nu)r} e^{-\nu(t-s)} \int_{\Omega} \|x(\omega)\| dP(\omega)$$

for all $(t, s, x) \in Z$. □

Corollary 3. *If the stochastic skew-evolution semiflow* (Φ, φ) *has uniform polynomial growth in mean, then* (Φ, φ) *is uniformly polynomially stable in mean if and only if there exists* $r > e - 1$ *and* $c \in (0, 1)$ *with*

$$\int_{\Omega} \|\Phi(rs + r + s, s, \omega)x(\omega)\| dP(\omega) \leq c \int_{\Omega} \|x(\omega)\| dP(\omega)$$

for all $(s, x) \in Y$.

Proof. *Necessity.* If (Φ, φ) is uniformly polynomially stable in mean, then from Corollary 1, it yields that (Φ_1, φ_1) is uniformly exponentially stable in mean.

From Theorem 3, it follows that there are $u > 1$ and $c \in (0, 1)$ such that

$$\begin{aligned} \int_{\Omega} \|\Phi_1(u + v, v, \omega)x(\omega)\| dP(\omega) &= \int_{\Omega} \|\Phi(e^{u+v} - 1, e^{v} - 1, \omega)x(\omega)\| dP(\omega) \leq \\ &\leq c \int_{\Omega} \|x(\omega)\| dP(\omega) \end{aligned}$$

for all $(v, x) \in Y$.

Then for $v = ln(s + 1)$ where $s \geq 0$ and $r = e^{u} - 1$ we have that $r > e - 1$ and $e^{u+v} - 1 = (e^{u} - 1)(e^{v} - 1) + e^{u} - 1 + e^{v} - 1 = rs + r + s$. Finally, we have that

$$\begin{aligned} \int_{\Omega} \|\Phi(rs + r + s, s, \omega)x(\omega)\| dP(\omega) &= \int_{\Omega} \|\Phi(e^{u+v} - 1, e^{v} - 1, \omega)x(\omega)\| dP(\omega) \leq \\ &\leq c \int_{\Omega} \|x(\omega)\| dP(\omega) \end{aligned}$$

for all $(s, x) \in Y$.

Sufficiency. Let $u = \ln(1 + r)$ and $\nu \geq 0$. Then $u > 1$, $s = e^{v} - 1 \geq 0$ and $u + v = \ln(1 + r)(1 + s) = \ln(1 + r + s + rs)$

Thus $e^{u+v} - 1 = r + s + rs$

Then

$$\begin{aligned} \int_{\Omega} \|\Phi_1(u + v, v, \omega)x(\omega)\| dP(\omega) &= \int_{\Omega} \|\Phi(e^{u+v} - 1, e^{v} - 1, \omega)x(\omega)\| dP(\omega) = \\ &= \int_{\Omega} \|\Phi(rs + r + s, s, \omega)x(\omega)\| dP(\omega) \leq \\ &\leq c \int_{\Omega} \|x(\omega)\| dP(\omega) \end{aligned}$$

From Theorem 3, it follows that (Φ_1, φ_1) is uniformly exponentially stable in mean. Finally, according to Corollary 1, we see that (Φ, φ) is uniformly polynomially stable in mean. □

Corollary 4. *If the stochastic skew-evolution semiflow* (Φ, φ) *has uniform h-growth in mean, then* (Φ, φ) *is uniformly h-stable in mean if and only if there are* $r > e$ *and* $c \in (0, 1)$ *such that*

$$\int_{\Omega} \|\Phi(h^{-1}(rs), h^{-1}(s), \omega)x(\omega)\| dP(\omega) \leq c \int_{\Omega} \|x(\omega)\| dP(\omega)$$

for all $(s, x) \in Y$.

Proof. It follows from Theorems 1 and 3. □

Another characterization of polynomial uniform stability in mean is given by

Theorem 4. *If the stochastic skew-evolution semiflow (Φ, φ) has uniform polynomial growth in mean, then (Φ, φ) is uniformly polynomially stable in mean if and only if there exist $L > 1$ with*

$$\int_{\Omega} \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \ln \frac{t+1}{s+1} \le L \int_{\Omega} \|x(\omega)\| dP(\omega)$$

for all $(t,s,x) \in Z$.

Proof. *Necessity.* If (Φ, φ) is uniformly polynomially stable in mean, then there are the constants $N > 1$ and $\nu > 0$ such that

$$\int_{\Omega} \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \ln \frac{t+1}{s+1} \le \frac{N}{\nu} \left(\frac{t+1}{s+1}\right)^{-\nu} \ln \frac{t+1}{s+1} \int_{\Omega} \|x(\omega)\| dP(\omega)$$

for all $(t,s,x) \in Z$.

Using the inequality

$$\frac{\ln u}{u} \le \frac{1}{e}, \quad \text{for every} \quad u \ge 1$$

it results that

$$\int_{\Omega} \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \ln \frac{t+1}{s+1} \le \frac{N}{\nu e} < 1 + \frac{N}{\nu e} \stackrel{def}{=} L > 1$$

for all $(t,s) \in \Delta$ and all $x \in L(\Omega, X, P)$. □

Sufficiency. For $r \stackrel{def}{=} e^{4L} - 1 > e - 1$ and $c = \frac{1}{4} \in (0,1)$
We deduce that

$$\begin{aligned}
\int_{\Omega} \|\Phi(rs+r+s,s,\omega)x(\omega)\| dP(\omega) &\le \frac{L}{\frac{rs+r+s+1}{s+1}} \int_{\Omega} \|x(\omega)\| dP(\omega) = \\
&\le \frac{L}{\ln(r+1)} \int_{\Omega} \|x(\omega)\| dP(\omega) = \\
&= L \int_{\Omega} \|x(\omega)\| dP(\omega)
\end{aligned}$$

for all $(s,x) \in Y$.

From Corollary 3, it follows that (Φ, φ) is uniformly polynomially stable in mean.

Corollary 5. *If the stochastic skew-evolution semiflow (Φ, φ) has uniform h-growth in mean, then (Φ, φ) is uniformly h-stable in mean if and only if there exists $L > 1$ with*

$$\int_{\Omega} \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \ln \frac{h(t)}{h(s)} \le L \int_{\Omega} \|x(\omega)\| dP(\omega)$$

for all $(t,s,x) \in Z$.

Proof. *Necessity.* If (Φ, φ) is uniformly h-stable in mean, then from Theorem 2 it follows that

$$\Psi_h(u,v,\omega) \stackrel{def}{=} \Phi(h^{-1}(u+1), h^{-1}(v+1), \omega)$$

is uniformly polynomially stable in mean.

From Theorem 4, it results that there exists $L > 1$ with

$$\begin{aligned} &\int_\Omega \|\Phi(h^{-1}(u+1), h^{-1}(v+1), \omega)x(\omega)\| dP(\omega) \ln \frac{u+1}{v+1} = \\ = &\int_\Omega \|\Psi_h(u,v,\omega)x(\omega)\| dP(\omega) \ln \frac{u+1}{v+1} \leq L \int_\Omega \|x(\omega)\| dP(\omega) \end{aligned}$$

for all $(u, v, x) \in Z$.

Let $(t, s) \in \Delta$. Then for $u \stackrel{def}{=} h(t) - 1$ and $v \stackrel{def}{=} h(s) - 1$ we have $(u, v) \in \Delta$ and

$$\begin{aligned} &\int_\Omega \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \ln \frac{h(t)}{h(s)} = \\ = &\int_\Omega \|\Phi(h^{-1}(u+1), h^{-1}(v+1), \omega) dP(\omega) \ln \frac{u+1}{v+1} \leq L \int_\Omega \|x(\omega)\| dP(\omega) \end{aligned}$$

Sufficiency. Let $s \geq 1$ and $r = e^{4L}$. Then $r \geq e$ and from the hypothesis there exists $L > 1$ with

$$\int_\Omega \|\Phi(h^{-1}(rs), h^{-1}(s), \omega) dP(\omega) \leq \frac{L}{\ln r} \int_\Omega \|x(\omega)\| dP(\omega) = c \int_\Omega \|x(\omega)\| dP(\omega)$$

From Corollary 4, we have that (Φ, φ) is uniformly h-stable in mean. □

Corollary 6. *If the stochastic skew-evolution semiflow (Φ, φ) has uniform exponential growth in mean, then (Φ, φ) is uniformly exponentially stable in mean if and only if there exists $L > 1$ with*

$$(t-s) \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \leq L \int_\Omega \|x(\omega)\| dP(\omega)$$

for all $(t, s, x) \in Z$.

Proof. It follows from Corollary 5, taking $h(t) = e^t$. □

A majorization criterion for uniform exponential stability in mean is given in the next theorem.

Theorem 5. *If the stochastic skew-evolution semiflow (Φ, φ) has uniform exponential growth in mean, then (Φ, φ) is uniformly exponentially stable in mean if and only if there exist $M > 1$ and a nondecreasing application $g : [1, \infty) \to \mathbf{R}_+$ with $\lim_{t\to\infty} g(t) = \infty$ and*

$$g(t-s) \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \leq M \int_\Omega \|x(\omega)\| dP(\omega)$$

for all $(t, s, x) \in Z$

Proof. *Necessity.* From Corollary 5, for the case $h(t) = e^t$, it results that if Φ is uniformly exponentially stable in mean, then there exists $L > 1$ with

$$(t-s) \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \leq L \int_\Omega \|x(\omega)\| dP(\omega)$$

for all $(t, s) \in \Delta$, thus the condition of the theorem is satisfied for $g(t) = t$

Sufficiency. From $\lim_{t\to\infty} g(t) = \infty$, we have that there exists $\delta > 0$ with $M < g(\delta)$. Then, for all $(t, s) \in \Delta$, there are $n \in \mathbf{N}$ and $r \in [0, \delta)$ with $t = s + n\delta + r$.

Let $c = \frac{L}{g(\delta)}$. Then $c \in (0,1)$ and from uniform exponential growth of (Φ, φ) and by hypothesis we observe that there exist $M > 1, \omega > 0$ such that

$$\begin{aligned}
& \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) = \\
= & \int_\Omega \|\Phi(s+n\delta+r, s+n\delta, \varphi(s+n\delta,s,\omega))\Phi(s+n\delta,s,\omega)x(\omega)\| dP(\omega) \leq \\
\leq & \; Me^{\omega r} \int_\Omega \|\Phi(s+n\delta,s,\omega)x(\omega)\| dP(\omega) \leq \\
\leq & \; Me^{\omega\delta} \int_\Omega \|\Phi(s+n\delta, s+(n-1)\delta, \varphi(s+(n-1)\delta), s, \omega)\Phi(s+(n-1)\delta,s,\omega)x(\omega)\| dP(\omega) \leq \\
\leq & \; Mce^{\omega\delta} \int_\Omega \|\Phi(s+(n-1\delta),s,\omega)x(\omega)\| dP(\omega) \leq \\
\leq & \; ... \leq Mc^{n-1}e^{\omega\delta} \int_\Omega \|\Phi(s+\delta,s,\omega)x(\omega)\| dP(\omega) \leq \\
\leq & \; Mc^n e^{\omega\delta} \int_\Omega \|x(\omega)\| dP(\omega) = Me^{\omega\delta}e^{n\ln c} \int_\Omega \|x(\omega)\| dP(\omega) = \\
= & \; Me^{\omega\delta}e^{\frac{t-s-r}{\delta}\ln c} \int_\Omega \|x(\omega)\| dP(\omega) \leq \frac{Me^{\omega\delta}}{c} e^{-\nu(t-s)} \int_\Omega \|x(\omega)\| dP(\omega) \leq \\
\leq & \; Ne^{-\nu(t-s)} \int_\Omega \|x(\omega)
\end{aligned}$$

for all $(t,s,x) \in Z$ where $N = \frac{Me^{\omega\delta}}{c} > 1$ and $\nu = -\frac{\ln c}{\delta} > 0$

Finally, it results that (Φ, φ) is uniformly exponentially stable in mean. □

Corollary 7. *If the stochastic skew-evolution semiflow (Φ, φ) has uniform polynomial growth in mean, then (Φ, φ) is uniformly polynomially stable in mean if and only if there exists $M > 1$ and a nondecreasing application $g : [1, \infty) \to \mathbf{R}_+$ with $\lim_{t\to\infty} g(t) = \infty$ and*

$$g\left(\frac{t+1}{s+1}\right) \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \leq M \int_\Omega \|x(\omega)\| dP(\omega)$$

for all $(t,s,x) \in Z$.

Proof. *Necessity.* It follows from Theorem 4 for $g(t) = \ln t$.

Sufficiency. Let $(u,v) \in \Delta$ and $t = e^u - 1$, $s = e^v - 1$. Then from the inequality from the assumption, it follows that

$$\varphi_2(u-v) \int_\Omega \|\Phi_2(u,v,\omega)x(\omega)\| dP(\omega) \leq M \int_\Omega \|x(\omega)\| dP(\omega)$$

where $\varphi_2(t) = g(e^t)$ and $\Phi_2(u,v,\omega) = \Phi(\ln(u+1), \ln(v+1), \omega)$.

From Theorem 5, we obtain that (Φ_2, φ_2) is uniformly exponentially stable in mean and from Corollary 2 it results that (Φ, φ) is uniformly polynomially stable in mean. □

Corollary 8. *If the stochastic skew-evolution semiflow (Φ, φ) has uniform h-growth in mean, then (Φ, φ) is uniformly h-stable in mean if and only if there exist $M > 1$ and a nondecreasing application $g : [1, \infty) \to \mathbf{R}_+$ with $\lim_{t\to\infty} g(t) = \infty$ and*

$$g\left(\frac{h(t)}{h(s)}\right) \int_\Omega \|\Phi(t,s,\omega)x(\omega)\| dP(\omega) \leq M \int_\Omega \|x(\omega)\| dP(\omega)$$

for all $(t,s,x) \in Z$

Proof. *Necessity.* It follows from Corollary 5 for $g(t) = \ln t$.

Sufficiency. Let $(t,s,x) \in Z$ and $u = h^{-1}(t+1)$ and $v = h^{-1}(s+1)$. Then $(u,v) \in \Delta$ and

$$\begin{aligned} & g\left(\frac{t+1}{s+1}\right) \int_{\Omega} \|\Psi_h(t,s,\omega)x(\omega)\| dP(\omega) = \\ = \; & g\left(\frac{t+1}{s+1}\right) \int_{\Omega} \|\Phi(h^{-1}(t+1), h^{-1}(s+1), \omega)x(\omega)\| dP(\omega) \leq \\ \leq \; & M \int_{\Omega} \|x(\omega)\| dP(\omega) \end{aligned}$$

By assumption, it results that (Ψ_h, ψ_h) is uniformly exponentially stable in mean. From Theorem 2, we observe that (Φ, φ) is uniformly h-stable in mean. □

5. Conclusions

In this note, we have considered three concepts of uniform stability in mean for stochastic skew-evolution semiflows. These notions are natural generalizations from the deterministic context. The established relation between these concepts represents the first main goal. Thus, the relation between uniform h-stability in mean and uniform exponential stability in mean is established in Theorem 1, the relation between uniform polynomial stability in mean and uniform exponential stability in mean is given in Corollary 1, the relation between uniform h-stability in mean and uniform polynomial stability in mean is established in Theorem 2, and finally, the connection between uniform exponential stability in mean and uniform polynomial stability in mean is done in Corollary 2. Then, based on the present results from this work, characterizations of these concepts are exposed. The second main goal of the paper is to give three types of characterizations for these concepts of uniform stability in mean. These results are presented in Section 4. As open problems, the authors would like to investigate:

- A generalization of the previous results for the case of nonuniform h-stability in mean;
- Applications in control theory;
- Variants for h-dichotomy in mean and h-trichotomy in mean.

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