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Global Directed Dynamic Behaviors of a Lotka-Volterra Competition-Diffusion-Advection System

Lili Chen, Shilei Lin and Yanfeng Zhao *

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China; chenlili0819@foxmail.com (L.C.); linshilei2021@126.com (S.L.)
* Correspondence: zhaoyanfeng1101@126.com

Abstract: This paper investigates the problem of the global directed dynamic behaviors of a Lotka-Volterra competition-diffusion-advection system between two organisms in heterogeneous environments. The two organisms not only compete for different basic resources, but also the advection and diffusion strategies follow the dispersal towards a positive distribution. By virtue of the principal eigenvalue theory, the linear stability of the co-existing steady state is established. Furthermore, the classification of dynamical behaviors is shown by utilizing the monotone dynamical system theory. This work can be seen as a further development of a competition-diffusion system.

Keywords: competition-diffusion-advection; principal eigenvalue; dynamic behaviors; global asymptotic stability

MSC: 35B40; 35K57; 37C65; 92D25

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1. Introduction

In the past few decades, the dynamic behaviors of competition-diffusion systems (see [1]) in homogeneous or heterogeneous environments have been extensively studied. Until 2017, He and Ni [2,3] studied the dynamics of two organisms by changing their random diffusion coefficients, resource carrying capacity and competitiveness, and they also described the global dynamics of two organisms. Their research has made outstanding contributions to the competition-diffusion systems. For the competition model of two organisms, either both organisms survive or win with the extinction of the other organisms (see [4–6]). In 2019, Du et al. [7,8] studied a Lotka-Volterra competition system with periodic habitat advection. From a biological point of view, this pulsating travel front provided a way for two competing species to interact in heterogeneous habitats. Based on the assumption that the resource function in spatial variables is decreasing, Lou et al. [9] described the competition between two aquatic organisms with different diffusion strategies for the same resource in the Lotka-Volterra reaction-diffusion-advection system in 2019. Md. Kamrujjaman [10] studied the impact of diffusion strategies on the outcome of competition between two populations while the species are distributed according to their respective carrying capacities in competition-diffusion systems. However, in the competition-diffusion-advection systems, the study of different species with different distribution functions will be more complex. Tang and Chen [11] and Xu et al. [12] studied the population dynamics of competition between two organisms from the perspective of river ecology in 2020. One interesting feature of their system was that the boundary conditions at the upstream end and downstream end can represent the net loss of individuals. In some cases, both organisms leave the site of competition, neither coexisting nor becoming extinct. Such an environment is important enough to demonstrate how organisms change their density and survival time in competition (see [13]). In 2021, Ma and Guo [14] described the feature of the coincidence of bifurcating coexistence steady-state solution branches and the effect of advection on the stability of the bifurcating solution. However, it is worthwhile to point out that all the aforementioned works focus on the global dynamic behaviors of competition-diffusion systems (see [10,15,16]) or advection systems (see [17,18]), in which the diffusion rates and spatial carrying capacity are changed, or the periodic habitat of advection systems is studied, or the upstream and downstream boundary conditions are changed.

Motivated by the effort of the aforementioned studies, we investigate the problem of the global directed dynamic behaviors of a Lotka-Volterra advection system between two organisms in heterogeneous environments, where two organisms are competing for different fundamental resources, their advection and diffusion strategies follow the dispersal towards a positive distribution, and the functions of inter-specific competition ability are variable.

Hence, we discuss the following global dynamics of the advection system:

$$\begin{cases} U_t = \nabla\cdot [\kappa_1(x)\nabla(\frac{U}{Q(x)}) - \mu_1(x)\frac{U}{Q(x)}\nabla\omega_1(x)] + U[r_1(x) - U - \rho_2(x)V], & \\ & in\ \Omega\times\mathbb{R}^+, \\ V_t = \nabla\cdot [\kappa_2(x)\nabla(\frac{V}{Q(x)}) - \mu_2(x)\frac{V}{Q(x)}\nabla\omega_2(x)] + V[r_2(x) - \rho_1(x)U - V], & \\ & in\ \Omega\times\mathbb{R}^+, \\ \kappa_1(x)\frac{\partial}{\partial n}(\frac{U}{Q}) - \mu_1(x)\frac{U}{Q}\frac{\partial\omega_1(x)}{\partial n} = 0, & on\ \partial\Omega\times\mathbb{R}^+, \\ \kappa_2(x)\frac{\partial}{\partial n}(\frac{V}{Q}) - \mu_2(x)\frac{V}{Q}\frac{\partial\omega_2(x)}{\partial n} = 0, & on\ \partial\Omega\times\mathbb{R}^+, \\ U(x,0) = U_0(x) \geq, \not\equiv 0, & in\ \Omega, \\ V(x,0) = V_0(x) \geq, \not\equiv 0, & in\ \Omega, \end{cases} \tag{1}$$

where $U(x,t)$ and $V(x,t)$ are the population densities of biological organisms, location $x\ in\ \Omega$ and time $t > 0$, which are supposed to be nonnegative; $\kappa_1(x), \kappa_2(x) > 0$ correspond to the dispersal rates of two competing organisms U and V, respectively. ∇ is the gradient operator. $\mu_1(x), \mu_2(x) > 0$ correspond to the advection rates of two competing organisms U and V, and $\omega_1(x), \omega_2(x) \in C^2(\overline{\Omega})$ are the nonconstant functions and represent the advective direction. The intrinsic growth rates of the two competing organisms are bounded functions $r_1(x)$ and $r_2(x)$, respectively, two positive distributions are $Q(x)$. $\rho_1(x), \rho_2(x) > 0$ show the inter-specific competition ability. The habitat Ω is a bounded smooth domain in $\mathbb{R}^N$, $1 \leq N \in \mathbb{Z}$; n denotes the outward unit normal vector on the boundary $\partial\Omega$. For the sake of simplicity, we can suppose the initial data U_0 and V_0 not identically zero. The system (1) satisfies no-flux boundary conditions.

When $\kappa_1(x) = \kappa_1, \kappa_2(x) = \kappa_2, \mu_1(x) = \mu_1, \mu_2(x) = \mu_2, Q(x) \equiv 1, \rho_1(x) = \rho_1, \rho_2(x) = \rho_2, \omega_1(x) = \omega_2(x)$, the system (1) becomes the advection system studied by Zhou and Xiao [19]:

$$\begin{cases} U_t = \kappa_1\Delta U - \mu_1\nabla\cdot[U\nabla\omega(x)] + U[r_1(x) - U - \rho_2 V], & in\ \Omega\times\mathbb{R}^+, \\ V_t = \kappa_2\Delta V - \mu_2\nabla\cdot[V\nabla\omega(x)] + V[r_2(x) - \rho_1 U - V], & in\ \Omega\times\mathbb{R}^+, \\ \kappa_1\frac{\partial U}{\partial n} - \mu_1 U\frac{\partial\omega(x)}{\partial n} = 0, & on\ \partial\Omega\times\mathbb{R}^+, \\ \kappa_2\frac{\partial V}{\partial n} - \mu_2 V\frac{\partial\omega(x)}{\partial n} = 0, & on\ \partial\Omega\times\mathbb{R}^+, \\ U(x,0) = U_0(x) \geq, \not\equiv 0, & in\ \Omega, \\ V(x,0) = V_0(x) \geq, \not\equiv 0, & in\ \Omega, \end{cases} \tag{2}$$

where $\kappa_1, \kappa_2, \mu_1, \mu_2, \rho_1$ and ρ_2 are positive constants. $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator. If $\mu_1, \mu_2 > 0$, readers can take a look at the relevant literature [20] and for the case $\mu_1 > 0 = \mu_2$, please see the references [21–25].

If $\mu_1 = \mu_2 = 0$, the system (2) becomes a diffusion model (see [2,3,5,26]):

$$\begin{cases} U_t = \kappa_1 \Delta U + U[r_1(x) - U - V], & in\ \Omega \times \mathbb{R}^+, \\ V_t = \kappa_2 \Delta V + V[r_2(x) - U - V], & in\ \Omega \times \mathbb{R}^+, \\ \frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = 0, & on\ \partial\Omega \times \mathbb{R}^+, \\ U(x,0) = U_0(x) \geq, \not\equiv 0, V(x,0) = V_0(x) \geq, \not\equiv 0, & in\ \Omega. \end{cases} \tag{3}$$

According to the research of the above models, the purpose of our paper is to deal with a more broader model (1) in a high spatial dimensions. In this system, we consider that the diffusion and advection strategies follow the dispersal towards a positive distribution, growth rates and competitiveness of the two organisms are different. Thus, we have the following basic assumptions in this paper.

(A_1) $\frac{\mu_1(x)}{\kappa_1(x)}\omega_1(x) - \frac{\mu_2(x)}{\kappa_2(x)}\omega_2(x) := \zeta_1\omega_1(x) - \zeta_2\omega_2(x) \geq 0$, $\Lambda := \min_{x\in\overline{\Omega}} e^{\zeta_2\omega_2(x)-\zeta_1\omega_1(x)}$, ζ_1 and ζ_2 are positive constants;

(A_2) $(\rho_1(x), \rho_2(x)) \in \Pi_\Lambda := \{(\rho_1(x), \rho_2(x)) : \rho_1(x), \rho_2(x) > 0, \rho_1(x)\rho_2(x) \leq \Lambda\}$;

(A_3) $r_1(x) > 0, r_2(x) > 0$ in $L^\infty(\Omega)$;

(A_4) $Q(x) > 0$ is nonconstant, and $\frac{Q(x)}{r_1(x)}, \frac{Q(x)}{r_2(x)}$ are also nonconstant.

Conditions $(A_3) - (A_4)$ ensure that the distribution of resources is heterogeneous for two species and the positivity is imposed here to guarantee the existence of two semi-trivial steady states for later discussion convenience. Under the conditions of $(A_1) - (A_4)$, we show a complete classification of the global dynamics of the system (1). The rest of this paper is arranged as follows. In Section 2, we mainly do some preparatory work. Some related properties of the system (1) are deduced from the properties of a single organisms model (4). Besides, some lemmas are proved. In Section 3, we investigate our main results. By using principal eigenvalue theory, we obtain the linear stability of coexisting steady states (see Theorem 2). Then, the most important thing is that in virtue of the monotone dynamical system theory (see [4]), we show the classification of global dynamic behaviors (see Theorem 3). A discussion on the main results and problems that deserve future investigation is presented in Section 4.

2. Preliminaries

In order to describe our main results, we show a competition-diffusion-advection system for a single organisms as follows:

$$\begin{cases} U_t = \nabla \cdot [\kappa(x)\nabla(\frac{U}{Q(x)}) - \mu(x)\frac{U}{Q(x)}\nabla\omega(x)] + U[r(x) - U], & in\ \Omega \times \mathbb{R}^+, \\ \kappa(x)\frac{\partial}{\partial n}(\frac{U}{Q}) - \mu(x)\frac{U}{Q}\frac{\partial\omega(x)}{\partial n} = 0, & on\ \partial\Omega \times \mathbb{R}^+, \\ U(x,0) = U_0(x) \geq, \not\equiv 0, & in\ \Omega, \end{cases} \tag{4}$$

where $\kappa(x) > 0, \mu(x) > 0, Q(x) > 0$ and $r(x) > 0$, $r(x)$ is bounded. According to the relevant description in [27] and the case that $r(x) > 0$, there is a unique positive steady state $\theta_{d,Q,\mu,r}$ in the system (4). If we apply this result to the system (1) and the conditions $(A_3) - (A_4)$, there are two semi-trivial steady states $(\theta_{\kappa_1,Q,\mu_1,r_1}, 0)$ and $(0, \theta_{\kappa_2,Q,\mu_2,r_2})$, respectively.

Lemma 1. *Assume that $\kappa(x) > 0, \mu(x) > 0, Q(x) > 0$ and $r(x) > 0$, $r(x)$ is bounded. The elliptic boundary value Problem:*

$$\begin{cases} \nabla \cdot \left[\kappa(x)\nabla(\frac{\theta}{Q(x)}) - \mu(x)\frac{\theta}{Q(x)}\nabla\omega(x)\right] + \theta[r(x) - \theta] = 0, & in\ \Omega, \\ \kappa(x)\frac{\partial}{\partial n}(\frac{\theta}{Q}) - \mu(x)\frac{\theta}{Q}\frac{\partial\omega(x)}{\partial n} = 0, & on\ \partial\Omega, \end{cases} \tag{5}$$

has a unique positive solution denoted by θ.

Proof. It is known in [27] that the problem (5) admits a solution and the solution is positive, denoted by θ, owning to the positivity of $\kappa(x), \mu(x), Q(x), r(x)$. Next, assume that θ_1, θ_2 are any two positive solutions of (5) and $0 < \theta_1 \le \theta_2$. It is not difficult to see that

$$\kappa\nabla(\frac{\theta}{Q}) - \mu\frac{\theta}{Q}\nabla\omega = \kappa e^{\frac{\mu}{\kappa}\omega}[\nabla(e^{-\frac{\mu}{\kappa}\omega}\frac{\theta}{Q})].$$

Then

$$\begin{aligned}
&\int \nabla\cdot\{\kappa e^{\frac{\mu}{\kappa}\omega}[\nabla(e^{-\frac{\mu}{\kappa}\omega}\frac{\theta_1}{Q})]\}(e^{-\frac{\mu}{\kappa}\omega}\frac{\theta_2}{Q})\,\mathrm{d}x \\
=& -\int \kappa e^{\frac{\mu}{\kappa}\omega}[\nabla(e^{-\frac{\mu}{\kappa}\omega}\frac{\theta_1}{Q})][\nabla(e^{-\frac{\mu}{\kappa}\omega}\frac{\theta_2}{Q})]\,\mathrm{d}x \\
=& -\int [r-\theta_1]e^{-\frac{\mu}{\kappa}\omega}\frac{\theta_1\theta_2}{Q}\,\mathrm{d}x \\
=& -\int [r-\theta_2]e^{-\frac{\mu}{\kappa}\omega}\frac{\theta_1\theta_2}{Q}\,\mathrm{d}x.
\end{aligned} \tag{6}$$

We deduce

$$\int [\theta_1-\theta_2]e^{-\frac{\mu}{\kappa}\omega}\frac{\theta_1\theta_2}{Q}\,\mathrm{d}x = 0.$$

Therefore, $\theta_1 = \theta_2$. □

To give a complete classification of the global dynamic system (1), we define

$$(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Gamma := \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+.$$

Based on the approach in [2], we define

$$\begin{aligned}
\Sigma_U &:= \{(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Gamma : (\theta_{\kappa_1,Q,\mu_1,r_1}, 0) \textit{ is linearly stable}\}; \\
\Sigma_V &:= \{(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Gamma : (0, \theta_{\kappa_2,Q,\mu_2,r_2}) \textit{ is linearly stable}\}; \\
\Sigma_- &:= \{(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Gamma : (\theta_{\kappa_1,Q,\mu_1,r_1}, 0) \textit{ and} (0, \theta_{\kappa_2,Q,\mu_2,r_2}) \textit{ are linearly unstable}\}.
\end{aligned} \tag{7}$$

We first recall the well-known Krein-Rutman Theorem:

Theorem 1 (Krein-Rutman Theorem [28])**.** *Let X be a Banach space, $K \subset X$ a total cone and $T : X \to X$ a compact linear operator that is positive (i.e., $T(K) \subset K$) with positive spectral radius $r(T)$. Then $r(T)$ is an eigenvalue with an eigenvector $u \in K \setminus 0 : Tu = r(T)u$. Moreover, $r(T^*) = r(T)$ is an eigenvalue of T^* with an eigenvector $u^* \in K^*$.*

In order to better describe the linear stability of semi-trivial steady states, we give the definition of elliptic eigenvalue problem:

$$\begin{cases} \nabla\cdot[\kappa(x)\nabla(\frac{\phi}{Q}) - \mu(x)\frac{\phi}{Q}\nabla\omega(x)] + h(x)\phi + \sigma\phi = 0, & \textit{in } \Omega, \\ \kappa(x)\frac{\partial}{\partial n}(\frac{\phi}{Q}) - \mu(x)\frac{\phi}{Q}\frac{\partial\omega(x)}{\partial n} = 0, & \textit{on } \partial\Omega, \end{cases} \tag{8}$$

where $\kappa(x) > 0, \mu(x) > 0, Q(x) > 0$ and $h(x) \in L^\infty(\Omega)$. Let

$$A\phi = \nabla\cdot[\kappa(x)\nabla(\frac{\phi}{Q}) - \mu(x)\frac{\phi}{Q}\nabla\omega(x)] + h(x)\phi.$$

Since A is uniformly strongly elliptic operator, we declare that the operator A satisfies the conditions in Theorem 1. An eigenvalue σ_1 of the problem (8) is called a principal eigenvalue if $\sigma_1 \in \mathbb{R}$ and for any eigenvalue σ with $\sigma \neq \sigma_1$, we have $Re\ \sigma > \sigma_1$. Hence, the problem (8) has a principal eigenvalue, denoted by $\sigma_1(\kappa, Q, \mu, h)$, and its corresponding eigenfuntion $\phi(\kappa, Q, \mu, h) > 0$ in Ω. The principal eigenvalue is expressed as

$$\sigma_1(\kappa,Q,\mu,h)=\inf_{0\neq\phi\in H^1(\Omega)}\frac{\int\kappa e^{\frac{\mu}{\kappa}\omega}[\nabla(e^{-\frac{\mu}{\kappa}\omega}\frac{\phi}{Q})]^2\,\mathrm{d}x-\int h\cdot e^{-\frac{\mu}{\kappa}\omega}\frac{\phi^2}{Q}\,\mathrm{d}x}{\int e^{-\frac{\mu}{d}\omega}\frac{\phi^2}{Q}\,\mathrm{d}x}. \tag{9}$$

Next, we give a useful lemma related to eigenvalue comparison results, which is used for Lemma 3 and Theorem 3.

Lemma 2 ([5])**.** *If $h_1(x)\le h_2(x)$ within Ω, then $\sigma_1(\kappa,Q,\mu,h_1)\ge\sigma_1(\kappa,Q,\mu,h_2)$ and the equality holds if and only if $h_1(x)\equiv h_2(x)$ in Ω.*

According to the description of theory of monotone semi-flow in the literature [6], let X denote the standard Banach space consisting of all continuous functions from $\overline{\Omega}$ to $\mathbb{R}$, i.e., $X:=C(\overline{\Omega})$, and X^+ be the set of all non-negative continuous functions from $\overline{\Omega}$ to $\mathbb{R}^+\cup 0$. Define $K:=X^+\times(-X^+)$ as the usual cone for the study of competitive systems with nonempty interior. Then we define the notion of linear stability of a given steady state (U,V). Linearizing the steady state problem of (1) at (U,V), we obtain

$$\begin{cases}\nabla\cdot[\kappa_1(x)\nabla(\frac{\varphi}{Q(x)})-\mu_1(x)\frac{\varphi}{Q(x)}\nabla\omega_1(x)]+[r_1(x)-U-\rho_2(x)V]\varphi & \\ -U[\varphi+\rho_2(x)\psi]+\lambda\varphi=0, & in\ \Omega,\\ \nabla\cdot[\kappa_2(x)\nabla(\frac{\psi}{Q(x)})-\mu_2(x)\frac{\psi}{Q(x)}\nabla\omega_2(x)]+[r_2(x)-\rho_1(x)U-V]\psi & \\ -V[\rho_1(x)\varphi+\psi]+\lambda\psi=0, & in\ \Omega,\\ \kappa_1(x)\frac{\partial}{\partial n}(\frac{\varphi}{Q})-\mu_1(x)\frac{\varphi}{Q}\frac{\partial\omega_1(x)}{\partial n}=0, & on\ \partial\Omega,\\ \kappa_2(x)\frac{\partial}{\partial n}(\frac{\psi}{Q})-\mu_2(x)\frac{\psi}{Q}\frac{\partial\omega_2(x)}{\partial n}=0, & on\ \partial\Omega.\end{cases} \tag{10}$$

Similar to the scalar problem (8), we can define the principal eigenvalue for the system (10), that is, an eigenvalue λ_1 of the problem (10) is called a principal eigenvalue if $\lambda_1\in\mathbb{R}$ and for any eigenvalue λ with $\lambda\neq\lambda_1$, we have $Re\,\lambda>\lambda_1$. Based on the approach in [6], by using Theorem 1, the problem (10) has a principal eigenvalue $\lambda_1\in\mathbb{R}$. In fact, we can select the corresponding eigenfunction (φ_1,ψ_1), which satisfies $\varphi_1>0>\psi_1$ in $\overline{\Omega}$. Here, for the convenience of readers to better understand the problem (10), we provide a simple illustration. Let us do this simple transformation

$$\Phi=e^{-\frac{\mu_1}{\kappa_1}\omega_1}\varphi\ and\ \Psi=-e^{-\frac{\mu_2}{\kappa_2}\omega_2}\psi,$$

then the problem (10) can be changed to

$$\begin{cases}\nabla[\kappa_1(x)\nabla(\frac{\Phi}{Q(x)})]+\mu_1(x)\nabla\omega_1(x)\cdot\nabla[\frac{\Phi}{Q(x)}]+[r_1(x)-2U-\rho_2(x)V]\Phi & \\ +\rho_2(x)Ue^{(\zeta_2\omega_2(x)-\zeta_1\omega_1(x))}\Psi+\lambda\Phi=0, & in\ \Omega,\\ \nabla[\kappa_2(x)\nabla(\frac{\Psi}{Q(x)})]+\mu_2(x)\nabla\omega_2(x)\cdot\nabla[\frac{\Psi}{Q(x)}]+\rho_1(x)Ve^{(\zeta_1\omega_1(x)-\zeta_2\omega_2(x))}\Phi & \\ +[r_2(x)-\rho_1(x)U-2V]\Psi+\lambda\Psi=0, & in\ \Omega,\\ \frac{\partial}{\partial n}(\frac{\Phi}{Q})=\frac{\partial}{\partial n}(\frac{\Psi}{Q})=0, & on\ \partial\Omega,\end{cases} \tag{11}$$

which is a linear cooperative elliptic system. Suppose now L is the elliptic operator, let

$$\begin{aligned}L\Phi&=\nabla[\kappa_1(x)\nabla(\frac{\Phi}{Q(x)})]+\mu_1(x)\nabla\omega_1(x)\cdot\nabla[\frac{\Phi}{Q(x)}]+[r_1(x)-2U-\rho_2(x)V]\Phi,\\ L\Psi&=\nabla[\kappa_2(x)\nabla(\frac{\Psi}{Q(x)})]+\mu_2(x)\nabla\omega_2(x)\cdot\nabla[\frac{\Psi}{Q(x)}]+[r_2(x)-\rho_1(x)U-2V]\Psi.\end{aligned}$$

According to [28,29], the problem (11) has $C^\alpha(\overline{\Omega})$ coefficients and is strictly uniformly elliptic in the bounded domain Ω which has $C^{2,\alpha}$ boundary. Let K be the positive cone in $X:=C_0^{1,\alpha}(\overline{\Omega})$ consisting of nonnegative functions. For any $\Phi_1,\Psi_1\in X$, then we can deduce that $T:X\to X$ defined by $T(\Phi_1,\Psi_1)=(\Phi,\Psi)$ is a positive compact linear operator. By applying Theorem 1 for positive compact linear operators and the Neumann type boundary condition, the problem (11) admits a principal eigenvalue $\lambda_1 \in \mathbb{R}$, and the corresponding eigenfunction (Φ, Ψ) can be chosen to satisfy $\Phi \geq 0$ and $\Psi \geq 0$ in $\overline{\Omega}$. Notice that (Φ, Ψ) is the solution of the problem (11). Moreover, since the off-diagonal elements $\rho_2(x)Ue^{(\zeta_2\omega_2(x)-\zeta_1\omega_1(x))}$ and $\rho_1(x)Ve^{(\zeta_1\omega_1(x)-\zeta_2\omega_2(x))}$ are strictly positive in Ω, it can be further concluded that λ_1 is simple and it is the unique eigenvalue corresponding to a pair of strictly positive eigenfunctions, i.e., $\Phi > 0$ and $\Psi > 0$ in Ω. In fact, we have $\Phi > 0$ and $\Psi > 0$ in $\overline{\Omega}$ due to Hopf boundary lemma, which in turn allows us to choose $\varphi > 0 > \psi$ in $\overline{\Omega}$. See [30] using semi-group theory and [31] using maximum principle, [1,6] for detailed explanation. For the principal eigenvalue theory of general linear cooperative elliptic systems, we refer the interested readers to [29]. If $\lambda \neq \lambda_1$ is an eigenvalue of (10) and the boundary condition is Neumann type, then $Re\,\lambda > \lambda_1$ in the coexistence case.

Based on [26], (Corollary 2.10), the following lemma is about the linear stability of $(\theta_{\kappa_1,Q,\mu_1,r_1}, 0)$ and $(0, \theta_{\kappa_2,Q,\mu_2,r_2})$.

Lemma 3. *The linear stability of $(\theta_{\kappa_1,Q,\mu_1,r_1}, 0)$, $(0, \theta_{\kappa_2,Q,\mu_2,r_2})$ and $(0,0)$ in the system (1) are determined by the sign of $min\{\sigma_1(\kappa_1, Q, \mu_1, r_1), \sigma_1(\kappa_2, Q, \mu_2, r_2)\}$, $\sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$ and $\sigma_1(\kappa_1, Q, \mu_1, r_1 - \rho_2\theta_{\kappa_2,Q,\mu_2,r_2})$.*

Proof. For the linear stability of $(\theta_{\kappa_1,Q,\mu_1,r_1}, 0)$, when $(U, V) = (\theta_{\kappa_1,Q,\mu_1,r_1}, 0)$ in (10), we have

$$\begin{cases} \nabla \cdot [\kappa_1(x)\nabla(\frac{\varphi}{Q(x)}) - \mu_1(x)\frac{\varphi}{Q(x)}\nabla\omega_1(x)] + [r_1(x) - 2\theta_{\kappa_1,Q,\mu_1,r_1}]\varphi & \\ +\lambda\varphi = \theta_{\kappa_1,Q,\mu_1,r_1}\rho_2(x)\psi, & in\ \Omega, \\ \nabla \cdot [\kappa_2(x)\nabla(\frac{\psi}{Q(x)}) - \mu_2(x)\frac{\psi}{Q(x)}\nabla\omega_2(x)] + [r_2(x) - \rho_1(x)\theta_{\kappa_1,Q,\mu_1,r_1}]\psi & \\ +\lambda\psi = 0, & in\ \Omega, \\ \kappa_1(x)\frac{\partial}{\partial n}(\frac{\varphi}{Q}) - \mu_1(x)\frac{\varphi}{Q}\frac{\partial\omega_1(x)}{\partial n} = 0, & on\ \partial\Omega, \\ \kappa_2(x)\frac{\partial}{\partial n}(\frac{\psi}{Q}) - \mu_2(x)\frac{\psi}{Q}\frac{\partial\omega_2(x)}{\partial n} = 0, & on\ \partial\Omega. \end{cases} \tag{12}$$

Let λ be an principal eigenvalue of (12) with the eigenfunction (φ, ψ). We get

$$\lambda = min\{\sigma_1(\kappa_1, Q, \mu_1, r_1 - 2\theta_{\kappa_1,Q,\mu_1,r_1}), \sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})\}. \tag{13}$$

If $\psi \neq 0$, then λ belonging to an eigenvalue of the second equation in (12), is real and the inequality $\lambda \geq \sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$ holds. Perhaps, if $\psi = 0$, then $\varphi \neq 0$ and λ is an eigenvalue of the first equation, we get

$$\begin{cases} \nabla \cdot [\kappa_1(x)\nabla(\frac{\varphi}{Q(x)}) - \mu_1(x)\frac{\varphi}{Q(x)}\nabla\omega_1(x)] + [r_1(x) - 2\theta_{\kappa_1,Q,\mu_1,r_1}]\varphi + \lambda\varphi = 0, & in\ \Omega, \\ \kappa_1(x)\frac{\partial}{\partial n}(\frac{\varphi}{Q}) - \mu_1(x)\frac{\varphi}{Q}\frac{\partial\omega_1(x)}{\partial n} = 0, & on\ \partial\Omega. \end{cases} \tag{14}$$

Due to the fact that λ is real and satisfies $\lambda \geq \sigma_1(\kappa_1, Q, \mu_1, r_1 - 2\theta_{\kappa_1,Q,\mu_1,r_1})$. It follows

$$\lambda \geq min\{\sigma_1(\kappa_1, Q, \mu_1, r_1 - 2\theta_{\kappa_1,Q,\mu_1,r_1}), \sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})\}.$$

If now $\sigma_1(\kappa_1, Q, \mu_1, r_1 - 2\theta_{\kappa_1,Q,\mu_1,r_1}) \leq \sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$, letting φ be the first eigenfunction corresponding to $\sigma_1(\kappa_1, Q, \mu_1, r_1 - 2\theta_{\kappa_1,Q,\mu_1,r_1})$, then $\sigma_1(\kappa_1, Q, \mu_1, r_1 - 2\theta_{\kappa_1,Q,\mu_1,r_1})$ is an eigenvalue of (12) with the eigenfunction $(\varphi_1, 0)$, which deduces $\lambda = \sigma_1(\kappa_1, Q, \mu_1, r_1 - 2\theta_{\kappa_1,Q,\mu_1,r_1})$.

Suppose that $\sigma_1(\kappa_1, Q, \mu_1, r_1 - 2\theta_{\kappa_1,Q,\mu_1,r_1}) > \sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$. Let ψ be the first eigenfunction corresponding to $\sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$, then $\sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$ is an eigenvalue of (12) with the eigenfunction $(\varphi, \psi) = (\varphi^*, \psi)$, that means $\lambda = \sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$. Here φ^* satisfies

$$\begin{cases} \nabla\cdot[\kappa_1(x)\nabla(\frac{\varphi^*}{Q(x)})-\mu_1(x)\frac{\varphi^*}{Q(x)}\nabla\omega_1(x)]+[r_1(x)-2\theta_{\kappa_1,Q,\mu_1,r_1}]\varphi^* & \\ +\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1})\varphi^*=\theta_{\kappa_1,Q,\mu_1,r_1}\rho_2(x)\psi, & in\ \Omega, \\ \kappa_1(x)\frac{\partial}{\partial n}(\frac{\varphi^*}{Q})-\mu_1(x)\frac{\varphi^*}{Q}\frac{\partial\omega_1(x)}{\partial n}=0, & on\ \partial\Omega. \end{cases} \tag{15}$$

The existence of φ^* is inferred from

$$\begin{aligned} &\sigma_1(\kappa_1,Q,\mu_1,r_1-2\theta_{\kappa_1,Q,\mu_1,r_1}+\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1})) \\ =&\sigma_1(\kappa_1,Q,\mu_1,r_1-2\theta_{\kappa_1,Q,\mu_1,r_1})-\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1})>0. \end{aligned}$$

So our claim is right. Owing to (6) and (9), it is inferred that $\sigma_1(\kappa_1,Q,\mu_1,r_1-\theta_{\kappa_1,Q,\mu_1,r_1})=0$. Hence, according to Lemma 2, we gain

$$\sigma_1(\kappa_1,Q,\mu_1,r_1-2\theta_{\kappa_1,Q,\mu_1,r_1})>\sigma_1(\kappa_1,Q,\mu_1,r_1-\theta_{\kappa_1,Q,\mu_1,r_1})=0,$$

then λ has the same sign as the first eigenvalue $\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$. Applying the definition of λ and linear stability, we deduce that the linear stability of $(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ is determined by the sign of $\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$.

Through completely similar arguments, we demonstrate that the stability of $(0,0)$ and $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$, is determined by $min\{\sigma_1(\kappa_1,Q,\mu_1,r_1),\sigma_1(\kappa_2,Q,\mu_2,r_2)\},\sigma_1(\kappa_1,Q,\mu_1,r_1-\rho_2\theta_{\kappa_2,Q,\mu_2,r_2})$ respectively. □

Remark 1. *From the variational characteristics of the first eigenvalue, we can see that* $(0,0)$ *is linearly unstable for any* $\kappa_1(x),\kappa_2(x),\mu_1(x),\mu_2(x),\rho_2(x),\rho_1(x)>0$.

Therefore, we give equivalent descriptions of (7) below:

$$\begin{aligned} \Sigma_U :=&\{(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Gamma:\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1})>0\}; \\ \Sigma_V :=&\{(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Gamma:\sigma_1(\kappa_1,Q,\mu_1,r_1-\rho_2\theta_{\kappa_2,Q,\mu_2,r_2})>0\}; \\ \Sigma_- :=&\{(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Gamma:\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1})<0 \ and \\ &\sigma_1(\kappa_1,Q,\mu_1,r_1-\rho_2\theta_{\kappa_2,Q,\mu_2,r_2})<0\}. \end{aligned}$$

The neutrally stable case is defined as follows

$$\begin{aligned} \Sigma_{U,0} :=&\{(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Gamma:\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1})=0\}; \\ \Sigma_{V,0} :=&\{(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Gamma:\sigma_1(\kappa_1,Q,\mu_1,r_1-\rho_2\theta_{\kappa_2,Q,\mu_2,r_2})=0\}; \\ \Sigma_{0,0} :=&\{(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Gamma:\sigma_1(\kappa_2,Q,\mu_2,r_2-\rho_1\theta_{\kappa_1,Q,\mu_1,r_1}) \\ &=\sigma_1(\kappa_1,Q,\mu_1,r_1-\rho_2\theta_{\kappa_2,Q,\mu_2,r_2})=0\}. \end{aligned}$$

By the definition, it is easy to see $\Sigma_{0,0}=\Sigma_{U,0}\cap\Sigma_{V,0}$.

In the following, "*g.a.s*" is used to mean that the steady state is globally asymptotically stable among all non-negative and not identically zero initial conditions.

Lemma 4 ([5])**.** *For any* $\kappa_1,\kappa_2,\mu_1,\mu_2>0$, *assume that* $(A_1)-(A_4)$ *hold and every coexistence steady state of the system (1), if it exists, is asymptotically stable. Then one of the following alternatives holds:*

(i) There exists a unique coexistence steady state of (1) that is g.a.s.

(ii) The system (1) has no coexistence steady state and either one of $(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ *or* $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$ *is g.a.s, while the other is unstable.*

3. Main Results

In this section, we present the results which are related to the co-existence steady state and the classification of global dynamic behaviors of the system (1).

Theorem 2. *Suppose that* $(A_1) - (A_4)$ *hold. For any* $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Gamma \setminus \Sigma_{0,0}$*, then every co-existence steady state of the system* (1)*, if exists, is linearly stable, i.e.,* $\lambda_1 > 0$.

Theorem 3. *Suppose that* $(A_1) - (A_4)$ *hold. Then we have the mutually disjoint decomposition of* Γ*:*

$$\Gamma = (\Sigma_U \cup \Sigma_{U,0} \setminus \Sigma_{0,0}) \cup (\Sigma_V \cup \Sigma_{V,0} \setminus \Sigma_{0,0}) \cup \Sigma_- \cup \Sigma_{0,0}. \tag{16}$$

Moreover, the following statements hold for the system (1)*:*

(i) *For all* $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in (\Sigma_U \cup \Sigma_{U,0} \setminus \Sigma_{0,0})$, $(\theta_{\kappa_1,Q,\mu_1,r_1}, 0)$ *is g.a.s;*

(ii) *For all* $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in (\Sigma_V \cup \Sigma_{V,0} \setminus \Sigma_{0,0})$, $(0, \theta_{\kappa_2,Q,\mu_2,r_2})$ *is g.a.s;*

(iii) *For all* $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Sigma_-$*, the system* (1) *has a unique coexistence steady state that is g.a.s;*

(iv) *For all* $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Sigma_{0,0}$, $\theta_{\kappa_1,Q,\mu_1,r_1} \equiv \rho_2(x)\theta_{\kappa_2,Q,\mu_2,r_2}$ *in* $\overline{\Omega}$ *and the system* (1) *has a compact global attractor consisting of a continuum of steady states*

$$\{(\eta(x)\theta_{\kappa_1,Q,\mu_1,r_1}, (1-\eta(x))\frac{\theta_{\kappa_1,Q,\mu_1,r_1}}{\rho_2(x)}) : \eta(x) \in [0,1]\}$$

connecting with two semi-trivial steady states.

3.1. Co-Existence Steady State

In order to prove Theorem 2, we assume that (U, V) is the co-existence steady state of the following system (1):

$$\begin{cases} \nabla \cdot [\kappa_1(x)\nabla(\frac{U}{Q(x)}) - \mu_1(x)\frac{U}{Q(x)}\nabla\omega_1(x)] + U[r_1(x) - U - \rho_2(x)V] = 0, & \text{in } \Omega, \\ \nabla \cdot [\kappa_2(x)\nabla(\frac{V}{Q(x)}) - \mu_2(x)\frac{V}{Q(x)}\nabla\omega_2(x)] + V[r_2(x) - \rho_1(x)U - V] = 0, & \text{in } \Omega, \\ \kappa_1(x)\frac{\partial}{\partial n}(\frac{U}{Q}) - \mu_1(x)\frac{U}{Q}\frac{\partial\omega_1(x)}{\partial n} = 0, & \text{on } \partial\Omega, \\ \kappa_2(x)\frac{\partial}{\partial n}(\frac{V}{Q}) - \mu_2(x)\frac{V}{Q}\frac{\partial\omega_2(x)}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Similar to the problem (10), then we get the linear eigenvalue model by linearize system (1) at (U, V),

$$\begin{cases} \nabla \cdot [\kappa_1(x)\nabla(\frac{\varphi}{Q(x)}) - \mu_1(x)\frac{\varphi}{Q(x)}\nabla\omega_1(x)] + [r_1(x) - U - \rho_2(x)V]\varphi & \\ -U[\varphi + \rho_2(x)\psi] + \lambda\varphi = 0, & \text{in } \Omega, \\ \nabla \cdot [\kappa_2(x)\nabla(\frac{\psi}{Q(x)}) - \mu_2(x)\frac{\psi}{Q(x)}\nabla\omega_2(x)] + [r_2(x) - \rho_1(x)U - V]\psi & \\ -V[\rho_1(x)\varphi + \psi] + \lambda\psi = 0, & \text{in } \Omega, \\ \kappa_1(x)\frac{\partial}{\partial n}(\frac{\varphi}{Q}) - \mu_1(x)\frac{\varphi}{Q}\frac{\partial\omega_1(x)}{\partial n} = 0, & \text{on } \partial\Omega, \\ \kappa_2(x)\frac{\partial}{\partial n}(\frac{\psi}{Q}) - \mu_2(x)\frac{\psi}{Q}\frac{\partial\omega_2(x)}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{18}$$

According to the problem (8) and using Theorem 1, we can deduce that the problem (18) has a principal eigenvalue λ_1. Moreover, we can choose the corresponding eigenfunction (φ, ψ), it satisfies $\varphi > 0 > \psi$ in $\overline{\Omega}$.

Now, we are ready to discuss Theorem 2.

Proof of Theorem 2. Obviously, as long as we can obtain $\lambda_1 > 0$ when $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Gamma \setminus \Sigma_{0,0}$. Multiplying the first equation in (18) and (17) by $\frac{U}{Q(x)}$ and $\frac{\varphi}{Q(x)}$, respectively, and subtracting the obtained equations, we obtain

$$\begin{aligned}&[\nabla(\kappa_1\nabla(\frac{\varphi}{Q}) - \mu_1\tfrac{\varphi}{Q}\nabla\omega_1)]\tfrac{U}{Q} - [\nabla(\kappa_1\nabla(\frac{U}{Q}) - \mu_1\tfrac{U}{Q}\nabla\omega_1)]\tfrac{\varphi}{Q}\\ &\qquad -\tfrac{U^2}{Q}[\varphi + \rho_2(x)\psi] = -\tfrac{\lambda_1 U\varphi}{Q}.\end{aligned} \tag{19}$$

In the similar way, it can be derived from the second equation in (17) and (18) that

$$\begin{aligned}&[\nabla(\kappa_2\nabla(\frac{\psi}{Q}) - \mu_2\tfrac{\psi}{Q}\nabla\omega_2)]\tfrac{V}{Q} - [\nabla(\kappa_2\nabla(\frac{V}{Q}) - \mu_2\tfrac{V}{Q}\nabla\omega_2)]\tfrac{\psi}{Q}\\ &\qquad -\tfrac{V^2}{Q}[\rho_1(x)\varphi + \psi] = -\tfrac{\lambda_1 V\psi}{Q}.\end{aligned} \tag{20}$$

Furthermore, multiplying (19) by $e^{-\frac{\mu_1}{\kappa_1}\omega_1} \cdot \frac{\varphi^2}{U^2}$, then we integrate over Ω and deduce (for simplicity, we replace $\int_\Omega$ with $\int$)

$$\begin{aligned}&\lambda_1 \int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ}\,\mathrm{d}x\\ =&-\int [\nabla(\kappa_1\nabla(\frac{\varphi}{Q}) - \mu_1\frac{\varphi}{Q}\nabla\omega_1)]e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^2}{UQ}\,\mathrm{d}x\\ &+\int [\nabla(\kappa_1\nabla(\frac{U}{Q}) - \mu_1\frac{U}{Q}\nabla\omega_1)]e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{U^2Q}\,\mathrm{d}x\\ &+\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^2}{Q}[\varphi + \rho_2(x)\psi]\,\mathrm{d}x\\ =&\int [\kappa_1\nabla(\frac{\varphi}{Q}) - \mu_1\frac{\varphi}{Q}\nabla\omega_1][\nabla\cdot(e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^2}{UQ})]\,\mathrm{d}x\\ &-\int [\kappa_1\nabla(\frac{U}{Q}) - \mu_1\frac{U}{Q}\nabla\omega_1][\nabla\cdot(e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{U^2Q})]\,\mathrm{d}x\\ &+\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^2}{Q}[\varphi + \rho_2(x)\psi]\,\mathrm{d}x\\ :=&I_1 - I_2 + I_3.\end{aligned} \tag{21}$$

By using the similarly method for the Equation (20), we get

$$\begin{aligned}&\lambda_1 \int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\psi^3}{VQ}\,\mathrm{d}x\\ =&-\int [\nabla(\kappa_2\nabla(\frac{\psi}{Q}) - \mu_2\frac{\psi}{Q}\nabla\omega_2)]e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\psi^2}{VQ}\,\mathrm{d}x\\ &+\int [\nabla(\kappa_2\nabla(\frac{V}{Q}) - \mu_2\frac{V}{Q}\nabla\omega_2)]e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\psi^3}{V^2Q}\,\mathrm{d}x\\ &+\int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\psi^2}{Q}[\rho_1(x)\varphi + \psi]\,\mathrm{d}x\\ =&\int [\kappa_2\nabla(\frac{\psi}{Q}) - \mu_2\frac{\psi}{Q}\nabla\omega_2][\nabla\cdot(e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\psi^2}{VQ})]\,\mathrm{d}x\\ &-\int [\kappa_2\nabla(\frac{V}{Q}) - \mu_2\frac{V}{Q}\nabla\omega_2][\nabla\cdot(e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\psi^3}{V^2Q})]\,\mathrm{d}x\\ &+\int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\psi^2}{Q}[\rho_1(x)\varphi + \psi]\,\mathrm{d}x\\ :=&J_1 - J_2 + J_3.\end{aligned} \tag{22}$$

We now simplify the formulas I_1, I_2, J_1 and J_2. Then we find

$$
\begin{aligned}
I_1 :=& \int \left[\kappa_1 \nabla(\frac{\varphi}{Q}) - \mu_1 \frac{\varphi}{Q} \nabla\omega_1\right] \left[\nabla \cdot (e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^2}{UQ})\right] \mathrm{d}x \\
=& \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{\varphi}{Q}) - \mu_1 \frac{\varphi}{Q} \nabla\omega_1][\frac{2\varphi U \nabla\varphi - \varphi^2 \nabla U}{U^2 Q}] \,\mathrm{d}x \\
&- \int \frac{\mu_1}{\kappa_1} \frac{\varphi^2}{UQ} e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{\varphi}{Q}) - \mu_1 \frac{\varphi}{Q} \nabla\omega_1] \cdot [\nabla\omega_1] \,\mathrm{d}x \\
&- \int \frac{\varphi^2}{UQ^2} e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{\varphi}{Q}) - \mu_1 \frac{\varphi}{Q} \nabla\omega_1][\nabla Q] \,\mathrm{d}x \\
=& \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^3}{UQ^2} [\kappa_1 \nabla(\frac{\varphi}{Q}) \frac{Q}{\varphi} - \mu_1 \nabla\omega_1][2\frac{\nabla\varphi}{\varphi} - \frac{\nabla U}{U}] \,\mathrm{d}x \\
&- \int \frac{\mu_1}{\kappa_1} \frac{\varphi^3}{UQ^2} e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{\varphi}{Q}) \frac{Q}{\varphi} - \mu_1 \nabla\omega_1] \cdot [\nabla\omega_1] \,\mathrm{d}x \\
&- \int \frac{\varphi^3}{UQ^3} e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{\varphi}{Q}) \frac{Q}{\varphi} - \mu_1 \nabla\omega_1][\nabla Q] \,\mathrm{d}x \\
:=& \Delta_1 - \Delta_2 - \Delta_3,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 :=& \int \left[\kappa_1 \nabla(\frac{U}{Q}) - \mu_1 \frac{U}{Q} \nabla\omega_1\right] \left[\nabla \cdot (e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^3}{U^2 Q})\right] \mathrm{d}x \\
=& \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{U}{Q}) - \mu_1 \frac{U}{Q} \nabla\omega_1][\frac{3\varphi^2 U^2 \nabla\varphi - \varphi^3 2U \nabla U}{U^4 Q}] \,\mathrm{d}x \\
&- \int \frac{\mu_1}{\kappa_1} \frac{\varphi^3}{U^2 Q} e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{U}{Q}) - \mu_1 \frac{U}{Q} \nabla\omega_1] \cdot [\nabla\omega_1] \,\mathrm{d}x \\
&- \int \frac{\varphi^3}{U^2 Q^2} e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{U}{Q}) - \mu_1 \frac{U}{Q} \nabla\omega_1][\nabla Q] \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
=& \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^3}{UQ^2} [\kappa_1 \nabla(\frac{U}{Q}) \frac{Q}{U} - \mu_1 \nabla\omega_1][3\frac{\nabla\varphi}{\varphi} - 2\frac{\nabla U}{U}] \,\mathrm{d}x \\
&- \int \frac{\mu_1}{\kappa_1} \frac{\varphi^3}{UQ^2} e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{U}{Q}) \frac{Q}{U} - \mu_1 \nabla\omega_1] \cdot [\nabla\omega_1] \,\mathrm{d}x \\
&- \int \frac{\varphi^3}{UQ^3} e^{-\frac{\mu_1}{\kappa_1}\omega_1} [\kappa_1 \nabla(\frac{U}{Q}) \frac{Q}{U} - \mu_1 \nabla\omega_1][\nabla Q] \,\mathrm{d}x \\
:=& \Delta_4 - \Delta_5 - \Delta_6.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&\Delta_1-\Delta_4\\
=&\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}[\kappa_1\nabla(\frac{\varphi}{Q})\frac{Q}{\varphi}-\mu_1\nabla\omega_1][2\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}]\,\mathrm{d}x\\
&-\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}[\kappa_1\nabla(\frac{U}{Q})\frac{Q}{U}-\mu_1\nabla\omega_1][3\frac{\nabla\varphi}{\varphi}-2\frac{\nabla U}{U}]\,\mathrm{d}x\\
=&\int \kappa_1 e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}\{[\nabla(\frac{\varphi}{Q})\frac{Q}{\varphi}][2\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}]-[\nabla(\frac{U}{Q})\frac{Q}{U}][3\frac{\nabla\varphi}{\varphi}-2\frac{\nabla U}{U}]\}\,\mathrm{d}x\\
&+\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}[\mu_1\nabla\omega_1][\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}]\,\mathrm{d}x\\
=&\int \kappa_1 e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}\{[\frac{\nabla\varphi}{Q}-\frac{\varphi\nabla\varphi}{Q^2}]\frac{Q}{\varphi}[2\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}]-[\frac{\nabla U}{Q}-\frac{U\nabla Q}{Q^2}]\frac{Q}{U}[3\frac{\nabla\varphi}{\varphi}\\
&-2\frac{\nabla U}{U}]\}\,\mathrm{d}x+\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}[\mu_1\nabla\omega_1][\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}]\,\mathrm{d}x\\
=&\int 2\kappa_1 e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}(\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U})^2\,\mathrm{d}x\\
&+\int \kappa_1 e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^3}[\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}][\nabla Q]\,\mathrm{d}x\\
&+\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}[\mu_1\nabla\omega_1][\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}]\,\mathrm{d}x.
\end{aligned}
\tag{23}
$$

By the similar method, we deduce that

$$
\Delta_2-\Delta_5=\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}[\mu_1\nabla\omega_1][\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}]\,\mathrm{d}x,
\tag{24}
$$

and

$$
\Delta_3-\Delta_6=\int \kappa_1 e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^3}[\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U}][\nabla Q]\,\mathrm{d}x.
\tag{25}
$$

Thus

$$
\begin{aligned}
I_1-I_2=&[\Delta_1-\Delta_4]-[\Delta_2-\Delta_5]-[\Delta_3-\Delta_6]\\
=&\int 2\kappa_1 e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\varphi^3}{UQ^2}(\frac{\nabla\varphi}{\varphi}-\frac{\nabla U}{U})^2\,\mathrm{d}x.
\end{aligned}
\tag{26}
$$

By a similar method, one obtains

$$
J_1-J_2=\int 2\kappa_2 e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\psi^3}{VQ^2}(\frac{\nabla\psi}{\psi}-\frac{\nabla V}{V})^2\,\mathrm{d}x.
\tag{27}
$$

Replace (21) and (22) with (26) and (27), respectively. Multiplying (22) by $\rho_2(x)^3$ and subtracting it from (21), we can obtain

$$
\begin{aligned}
&\lambda_1 \int [e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^3}{UQ} - e^{-\frac{\mu_2}{\kappa_2}\omega_2} \frac{\rho_2(x)^3\psi^3}{VQ}] \,\mathrm{d}x \\
&= \int 2\kappa_1 e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^3}{UQ^2} (\frac{\nabla\varphi}{\varphi} - \frac{\nabla U}{U})^2 \,\mathrm{d}x + \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^2}{Q}[\varphi + \rho_2(x)\psi] \,\mathrm{d}x \\
&\quad - \int 2\kappa_2 e^{-\frac{\mu_2}{\kappa_2}\omega_2} \frac{\rho_2(x)^3\psi^3}{VQ^2} (\frac{\nabla\psi}{\psi} - \frac{\nabla V}{V})^2 \mathrm{d}x \\
&\quad - \int e^{-\frac{\mu_2}{\kappa_2}\omega_2} \frac{\rho_2(x)^2\psi^2}{Q}[\rho_1(x)\rho_2(x)\varphi + \rho_2(x)\psi] \,\mathrm{d}x \\
&\geq \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^2}{Q}[\varphi + \rho_2(x)\psi] \,\mathrm{d}x - \int e^{-\frac{\mu_2}{\kappa_2}\omega_2} \frac{\rho_2(x)^3\psi^3}{Q} \,\mathrm{d}x \\
&\quad - \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\rho_2(x)^2\psi^2\varphi}{Q}[\rho_1(x)\rho_2(x) e^{(\frac{\mu_1}{\kappa_1}\omega_1 - \frac{\mu_2}{\kappa_2}\omega_2)}] \,\mathrm{d}x \\
&\geq \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\varphi^2}{Q}[\varphi + \rho_2(x)\psi] \,\mathrm{d}x - \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\rho_2(x)^3\psi^3}{Q} \,\mathrm{d}x \\
&\quad - \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{\rho_2(x)^2\psi^2\varphi}{Q} \,\mathrm{d}x \\
&= \int e^{-\frac{\mu_1}{\kappa_1}\omega_1} \frac{1}{Q}[\varphi + \rho_2(x)\psi]^2[\varphi - \rho_2(x)\psi] \,\mathrm{d}x \geq 0,
\end{aligned} \tag{28}
$$

where we use the facts $(A_1), (A_2)$ and $\varphi > 0 > \psi$ in $\overline{\Omega}$. So, $\lambda_1 \geqslant 0$.

Next, we will prove that $\lambda_1 = 0$ can not happen. According to (28), we infer that $\lambda_1 = 0$ if and only if

$$
\rho_1(x)\rho_2(x) = 1, \ \frac{\mu_1}{\kappa_1}\omega_1 = \frac{\mu_2}{\kappa_2}\omega_2, \ \frac{\nabla\varphi}{\varphi} = \frac{\nabla U}{U}, \ \frac{\nabla\psi}{\psi} = \frac{\nabla V}{V}, \ \varphi = -\rho_2(x)\psi, \tag{29}
$$

which means that

$$
\frac{\nabla U}{U} = \frac{\nabla V}{V},
$$

i.e.,

$$
\nabla[lnU] = \nabla[lnV].
$$

Then, one obtains

$$
U = dV \quad \text{for some constant} \quad d > 0. \tag{30}
$$

In addition, by applying (30) to (17), and the uniqueness of the positive steady state of the system (4), it can be concluded that

$$
(1 + \frac{\rho_2(x)}{d})U = \theta_{\kappa_1,Q,\mu_1,r_1} \quad \text{and} \quad (\rho_1(x)d + 1)V = \theta_{\kappa_2,Q,\mu_2,r_2}.
$$

Noting that $\rho_1(x)\rho_2(x) = 1$, we deduce

$$
\frac{\theta_{\kappa_1,Q,\mu_1,r_1}}{\theta_{\kappa_2,Q,\mu_2,r_2}} = \frac{(1 + \frac{\rho_2(x)}{d})U}{(\rho_1(x)d + 1)V} = \frac{U + \rho_2(x)V}{\rho_1(x)U + V} = \rho_2(x) = \frac{1}{\rho_1(x)}. \tag{31}
$$

Based on (31), one can easily check

$$
\sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1}) = \sigma_1(\kappa_2, Q, \mu_2, r_2 - \theta_{\kappa_2,Q,\mu_2,r_2}) = 0,
$$

and

$$\sigma_1(\kappa_1, Q, \mu_1, r_1 - \theta_{\kappa_1,Q,\mu_1,r_1}) = \sigma_1(\kappa_1, Q, \mu_1, r_1 - \rho_2\theta_{\kappa_2,Q,\mu_2,r_2}) = 0.$$

According to the assumption $(\kappa_1, \kappa_2, \mu_1, \mu_2) \notin \Sigma_{0,0}$, we get $\lambda_1 \neq 0$. Therefore, $\lambda_1 > 0$. □

3.2. Classification of the Global Dynamics

In this subsection, we are ready to discuss the complete global dynamic behaviors of the system (1).

Proof of Theorem 3. According to the proof procedure (see [2,5]), this proof can be divided into two steps.

Step 1. On the proof of the disjoint decomposition in (16).

Obviously, we can get the decomposition in (16). According to the relevant conclusion and definitions, we only need to prove

$$(\Sigma_U \cup \Sigma_{U,0} \setminus \Sigma_{0,0}) \cap (\Sigma_V \cup \Sigma_{V,0} \setminus \Sigma_{0,0}) = \emptyset. \tag{32}$$

By Lemma 3, the linear stability of $(\theta_{\kappa_1,Q,\mu_1,r_1}, 0)$, $(0, \theta_{\kappa_2,Q,\mu_2,r_2})$ can be determined by the sign of $\sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1})$, $\sigma_1(\kappa_1, Q, \mu_1, r_1 - \rho_2\theta_{\kappa_2,Q,\mu_2,r_2})$ respectively. For the sake of convenience of in writing, let

$$\theta_{\kappa_1,Q,\mu_1,r_1} := \theta_1,\ \sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1}) := \sigma_1(\theta_1^*),$$
$$\theta_{\kappa_2,Q,\mu_2,r_2} := \theta_2,\ \sigma_1(\kappa_1, Q, \mu_1, r_1 - \rho_2\theta_{\kappa_2,Q,\mu_2,r_2}) := \sigma_1(\theta_2^*).$$

According to the properties of the variational characterization and (6), we obtain

$$\begin{aligned}
\sigma_1(\theta_2^*) &= \inf_{0\neq\phi\in H^1(\Omega)} \frac{\int \kappa_1 e^{\frac{\mu_1}{\kappa_1}\omega_1}[\nabla(e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\phi}{Q})]^2\,\mathrm{d}x - \int[r_1 - \rho_2(x)\theta_2]e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\phi^2}{Q}\,\mathrm{d}x}{\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\phi^2}{Q}\,\mathrm{d}x} \\
&\leq \frac{\int \kappa_1 e^{\frac{\mu_1}{\kappa_1}\omega_1}[\nabla(e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\theta_1}{Q})]^2\,\mathrm{d}x - \int[r_1 - \rho_2(x)\theta_2]e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\theta_1^2}{Q}\,\mathrm{d}x}{\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\theta_1{}^2}{Q}\,\mathrm{d}x} \\
&= \frac{\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\theta_1^2}{Q}[\rho_2(x)\theta_2 - \theta_1]\,\mathrm{d}x}{\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\theta_1^2}{Q}\,\mathrm{d}x},
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\sigma_1(\theta_1^*) &= \inf_{0\neq\phi\in H^1(\Omega)} \frac{\int \kappa_2 e^{\frac{\mu_2}{\kappa_2}\omega_2}[\nabla(e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\phi}{Q})]^2\,\mathrm{d}x - \int[r_2 - \rho_1(x)\theta_1]e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\phi^2}{Q}\,\mathrm{d}x}{\int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\phi^2}{Q}\,\mathrm{d}x} \\
&\leq \frac{\int \kappa_2 e^{\frac{\mu_2}{\kappa_2}\omega_2}[\nabla(e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\theta_2}{Q})]^2\,\mathrm{d}x - \int[r_2 - \rho_1(x)\theta_1]e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\theta_2^2}{Q}\mathrm{d}x}{\int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\theta_2{}^2}{Q}\,\mathrm{d}x} \\
&= \frac{\int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\theta_2^2}{Q}[\rho_1(x)\theta_1 - \theta_2]\,\mathrm{d}x}{\int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\theta_2^2}{Q}\,\mathrm{d}x}.
\end{aligned} \tag{34}$$

Since $0 < \rho_1(x)\rho_2(x) \leq 1$, combining with (33) and (34) together, we have

$$
\begin{aligned}
&\sigma_1(\theta_2^*)\cdot\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\theta_1^2}{Q}\,\mathrm{d}x+\sigma_1(\theta_1^*)\cdot\int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\rho_2(x)^3\theta_2^2}{Q}\,\mathrm{d}x\\
\le&\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\theta_1^2}{Q}[\rho_2(x)\theta_2-\theta_1]\,\mathrm{d}x-\int e^{-\frac{\mu_2}{\kappa_2}\omega_2}\frac{\rho_2(x)^3\theta_2^3}{Q}\,\mathrm{d}x\\
&+\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\rho_2(x)^2\theta_2^2\theta_1}{Q}[\rho_1(x)\rho_2(x)e^{(\frac{\mu_1}{\kappa_1}\omega_1-\frac{\mu_2}{\kappa_2}\omega_2)}]\,\mathrm{d}x\\
\le&\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\theta_1^2}{Q}[\rho_2(x)\theta_2-\theta_1]\,\mathrm{d}x-\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\rho_2(x)^3\theta_2^3}{Q}\,\mathrm{d}x\\
&+\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{\rho_2(x)^2\theta_2^2\theta_1}{Q}\,\mathrm{d}x\\
=&-\int e^{-\frac{\mu_1}{\kappa_1}\omega_1}\frac{1}{Q}[\rho_2(x)\theta_2-\theta_1]^2[\theta_1+\rho_2(x)\theta_2]\,\mathrm{d}x\\
\le&0,
\end{aligned}
\tag{35}
$$

where all the inequalities become equalities if and only if

$$\rho_1(x)\rho_2(x)=1,\ \frac{\mu_1}{\kappa_1}\omega_1=\frac{\mu_2}{\kappa_2}\omega_2\ \textit{and}\ \theta_{\kappa_1,Q,\mu_1,r_1}\equiv\rho_2(x)\theta_{\kappa_2,Q,\mu_2,r_2}\ \textit{in}\ \overline{\Omega}.$$

It follows from (35) that the conclusion (32) holds.

Step 2. On the proof of the statements $(i)-(iv)$.

Firstly, we will prove the statements $(i)-(iii)$ hold. In consideration of (16) in Theorem 3 and (35) in step 1, we see that for any $(\kappa_1,\kappa_2,\mu_1,\mu_2)\in(\Gamma\setminus\Sigma_{0,0})$, there are five possibilities as follows:

(b_1) $(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Sigma_U$, i.e., $(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ is linearly stable, $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$ is linearly unstable;

(b_2) $(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Sigma_V$, i.e., $(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ is linearly unstable, $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$ is linearly stable;

(b_3) $(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Sigma_-$, i.e., both $(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ and $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$ are linearly unstable;

(b_4) $(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Sigma_{U,0}\setminus\Sigma_{0,0}$, i.e., $(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ is neutrally stable, $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$ is linearly unstable;

(b_5) $(\kappa_1,\kappa_2,\mu_1,\mu_2)\in\Sigma_{V,0}\setminus\Sigma_{0,0}$, i.e., $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$ is neutrally stable, $(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ is linearly unstable.

By Lemma 4, we immediately deduce the following conclusion:

$(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ and $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$ are $g.a.s$ based on the assumptions (b_1) and (b_2), respectively, and there is a unique co-existence steady state under the condition (b_3).

We now claim that there is no coexistence steady state under the condition (b_4) or the condition (b_5). Then we can infer that $(\theta_{\kappa_1,Q,\mu_1,r_1},0)$ and $(0,\theta_{\kappa_2,Q,\mu_2,r_2})$ are also $g.a.s$ based on the assumptions (b_4) and (b_5), respectively, from Lemma 4.

We only need to verify the above statement for the case (b_4). Indeed, if the system (1) has a co-existence steady state $(\widetilde{U},\widetilde{V})$ for some $(\kappa_1,\kappa_2,\mu_1,\mu_2)=(\widetilde{\kappa_1},\widetilde{\kappa_2},\widetilde{\mu_1},\widetilde{\mu_2})\in\Sigma_{U,0}\setminus\Sigma_{0,0}$ and $(\widetilde{U},\widetilde{V})$ satisfies

$$
\begin{cases}
\nabla\cdot[\widetilde{\kappa_1}(x)\nabla(\frac{\widetilde{U}}{\widetilde{Q}(x)})-\widetilde{\mu_1}(x)\frac{\widetilde{U}}{\widetilde{Q}(x)}\nabla\omega_1(x)]+\widetilde{U}[r_1(x)-\widetilde{U}-\rho_2(x)\widetilde{V}]=0, & in\ \Omega,\\
\nabla\cdot[\widetilde{\kappa_2}(x)\nabla(\frac{\widetilde{V}}{\widetilde{Q}(x)})-\widetilde{\mu_2}(x)\frac{\widetilde{V}}{\widetilde{Q}(x)}\nabla\omega_2(x)]+\widetilde{V}[r_2(x)-\rho_1(x)\widetilde{U}-\widetilde{V}]=0, & in\ \Omega,\\
[\widetilde{\kappa_1}(x)\frac{\partial}{\partial n}(\frac{\widetilde{U}}{\widetilde{Q}})-\widetilde{\mu_1}(x)\frac{\widetilde{U}}{\widetilde{Q}}\frac{\partial\omega_1(x)}{\partial n}]\mid_{\partial\Omega}=[\widetilde{\kappa_2}(x)\frac{\partial}{\partial n}(\frac{\widetilde{V}}{\widetilde{Q}})-\widetilde{\mu_2}(x)\frac{\widetilde{V}}{\widetilde{Q}}\frac{\partial\omega_2(x)}{\partial n}]\mid_{\partial\Omega}=0.
\end{cases}
\tag{36}
$$

We have

$$\sigma_1(\widetilde{\kappa_2},\widetilde{Q},\widetilde{\mu_2},r_2-\rho_1\theta_{\widetilde{\kappa_1},\widetilde{Q},\widetilde{\mu_1},r_1})=0\ \textit{and}\ \sigma_1(\widetilde{\kappa_1},\widetilde{Q},\widetilde{\mu_1},r_1-\rho_2\theta_{\widetilde{\kappa_2},\widetilde{Q},\widetilde{\mu_2},r_2})<0. \tag{37}$$

Define the operator $G : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{M}_1 \times \mathbb{M}_1 \to \mathbb{M}_2 \times \mathbb{M}_2$

$$G : (\varrho_1, \varrho_2, w_1, w_2) \longmapsto \begin{pmatrix} \widetilde{\kappa_1}\Delta(\frac{w_1}{Q}) - \widetilde{\mu_1}\nabla \cdot [\frac{w_1}{Q}\nabla\omega_1] + w_1[r_1 - w_1 - \varrho_2 w_2] \\ \widetilde{\kappa_2}\Delta(\frac{w_2}{Q}) - \widetilde{\mu_2}\nabla \cdot [\frac{w_2}{Q}\nabla\omega_2] + w_2[r_2 - \varrho_1 w_1 - w_2] \end{pmatrix},$$

with

$$\mathbb{M}_1 := \{\varphi \in W^{2,p}(\Omega) : [\widetilde{\kappa_1}\frac{\partial}{\partial n}(\frac{\varphi}{Q}) - \widetilde{\mu_1}(\frac{\varphi}{Q})\frac{\partial\omega_1}{Q}] \mid_{\partial\Omega} = 0\},\ \mathbb{M}_2 := L^p(\Omega), p > N.$$

From (36), $G(\rho_1(x), \rho_2(x), \widetilde{U}, \widetilde{V}) = 0$ and Theorem 2, it yields that

$$\frac{\partial G(\varrho_1, \varrho_2, w_1, w_2)}{\partial(w_1, w_2)} \mid_{(\varrho_1,\varrho_2,w_1,w_2) = (\rho_1(x),\rho_2(x),\widetilde{U},\widetilde{V})} \ \textit{is invertible}.$$

Applying implicit function theorem, one gets (ϱ_1, ϱ_2) is closed to $(\rho_1(x), \rho_2(x))$. We have a positive solution $(\widetilde{U}, \widetilde{V})$ to the equation $G(\varrho_1, \varrho_2, w_1, w_2) = 0$. Let us choose $(\overline{\varrho_1}, \overline{\varrho_2})$, which implies (the solution corresponding to $(\overline{\varrho_1}, \overline{\varrho_2})$ is denoted by $(\overline{U}, \overline{V})$)

$$\overline{\varrho_1} > \rho_1(x),\ 0 < \overline{\varrho_2} < \rho_2(x) \quad \textit{and} \quad \overline{\varrho_1} \cdot \overline{\varrho_2} \le \Lambda. \tag{38}$$

Let us see the following auxiliary problem

$$\begin{cases} U_t = \nabla \cdot [\widetilde{\kappa_1}(x)\nabla(\frac{U}{Q}) - \widetilde{\mu_1}(x)\frac{U}{Q}\nabla\omega_1(x)] + U[r_1(x) - U - \overline{\varrho_2}V] = 0, & \textit{in } \Omega \times \mathbb{R}^+, \\ V_t = \nabla \cdot [\widetilde{\kappa_2}(x)\nabla(\frac{V}{Q}) - \widetilde{\mu_2}(x)\frac{V}{Q}\nabla\omega_2(x)] + V[r_2(x) - \overline{\varrho_1}U - V] = 0, & \textit{in } \Omega \times \mathbb{R}^+, \\ [\widetilde{\kappa_1}(x)\frac{\partial}{\partial n}(\frac{U}{Q}) - \widetilde{\mu_1}(x)\frac{U}{Q}\frac{\partial\omega_1(x)}{\partial n}] \mid_{\partial\Omega} = 0, \\ [\widetilde{\kappa_2}(x)\frac{\partial}{\partial n}(\frac{V}{Q}) - \widetilde{\mu_2}(x)\frac{V}{Q}\frac{\partial\omega_2(x)}{\partial n}] \mid_{\partial\Omega} = 0, \end{cases} \tag{39}$$

which has the same semi-trivial steady states $(\theta_{\widetilde{\kappa_1},\widetilde{Q},\widetilde{\mu_1},r_1}, 0)$ and $(0, \theta_{\widetilde{\kappa_2},\widetilde{Q},\widetilde{\mu_2},r_2})$. From (37), (38) and Lemma 2, it then follows that

$$\sigma_1(\widetilde{\kappa_2}, \widetilde{Q}, \widetilde{\mu_2}, r_2 - \overline{\varrho_1}\theta_{\widetilde{\kappa_1},\widetilde{Q},\widetilde{\mu_1},r_1}) > 0 \textit{ and } \sigma_1(\widetilde{\kappa_1}, \widetilde{Q}, \widetilde{\mu_1}, r_1 - \overline{\varrho_2}\theta_{\widetilde{\kappa_2},\widetilde{Q},\widetilde{\mu_2},r_2}) < 0. \tag{40}$$

According to the case (b_1), $(\theta_{\kappa_1,Q,\mu_1,r_1}, 0)$ is also $g.a.s$ in the system (39) which contradicts with the existence of $(\overline{U}, \overline{V})$. Therefore, there is no coexistence steady state under the condition (b_4). Similarly, we can get the conclusion that there is also no coexistence steady state under the condition (b_5). The above descriptions of the cases $(b_1) - (b_5)$ represent the expected results described in the statements $(i) - (iii)$.

Secondly, we prove the statement (iv). We will show

$$\Sigma_{0,0} = \Sigma^{\sim} := \{(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Gamma : \rho_1(x)\rho_2(x) = 1, \theta_{\kappa_1,Q,\mu_1,r_1} \equiv \rho_2(x)\theta_{\kappa_2,Q,\mu_2,r_2} \textit{ in } \overline{\Omega}\}. \tag{41}$$

It makes the same description of $\Sigma_{0,0}$, which means the expected result in the statement (iv).

Let $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Sigma^{\sim}$, then

$$\rho_1(x)\rho_2(x) = 1 \textit{ and } \theta_{\kappa_1,Q,\mu_1,r_1} \equiv \rho_2(x)\theta_{\kappa_2,Q,\mu_2,r_2} \textit{ in } \overline{\Omega}.$$

Based on the proof of Theorem 2, we get

$$\sigma_1(\kappa_2, Q, \mu_2, r_2 - \rho_1\theta_{\kappa_1,Q,\mu_1,r_1}) = \sigma_1(\kappa_1, Q, \mu_1, r_1 - \rho_2\theta_{\kappa_2,Q,\mu_2,r_2}) = 0, \tag{42}$$

which implies $\Sigma^{\sim} \subset \Sigma_{0,0}$. When (42) holds, the last three inequalities in (35) become equalities, we have

$$\rho_1(x)\rho_2(x) = 1,\ \ \theta_{\kappa_1,Q,\mu_1,r_1} \equiv \rho_2(x)\theta_{\kappa_2,Q,\mu_2,r_2}\ in\ \overline{\Omega},$$

which shows $\Sigma_{0,0} \subset \Sigma^{\sim}$. Hence, the equality (41) is confirmed.

Let $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Sigma_{0,0}$ and (U, V) be the corresponding coexistence steady state of (1). We claim that $\frac{U}{V} \equiv$ constant. Let λ_1 be a principal eigenvalue for (U, V). Moreover, we choose the corresponding eigenfunction (φ, ψ), which satisfies $\varphi > 0 > \psi$ in $\overline{\Omega}$ and $\|\varphi\|_2^2 + \|\psi\|_2^2 = 1$. In order to prove it, it is enough to show that (29) holds. Suppose that (29) is not true. Then (28) means $\lambda_1 > 0$. Similar to the proof of the case (b_4), we get (29) holds, i.e., $\frac{U}{V} \equiv$ constant. This yields that

$$(U, V) = (\eta(x)\theta_{\kappa_1,Q,\mu_1,r_1}, (1 - \eta(x))\frac{\theta_{\kappa_1,Q,\mu_1,r_1}}{\rho_2(x)}) : \eta(x) \in [0, 1].$$

Therefore, we conclude that for any $(\kappa_1, \kappa_2, \mu_1, \mu_2) \in \Sigma_{0,0}$, the set of equilibria of (1) is

$$\{(0, 0)\} \cup \{(\eta(x)\theta_{\kappa_1,Q,\mu_1,r_1}, (1 - \eta(x))\frac{\theta_{\kappa_1,Q,\mu_1,r_1}}{\rho_2(x)}) : \eta(x) \in [0, 1]\},$$

where $(0, 0)$ is a repeller by Remark 1. Then each solution of (1) converges to a single equilibrium $\{(\eta(x)\theta_{\kappa_1,Q,\mu_1,r_1}, (1 - \eta(x))\frac{\theta_{\kappa_1,Q,\mu_1,r_1}}{\rho_2(x)}) : \eta(x) \in [0, 1]\}$. □

4. Discussion

In this paper, by using principal eigenvalue theory and monotone dynamical system theory, we mainly analyzed the global directed dynamic behaviors of a Lotka-Volterra competition-diffusion-advection system between two organisms in heterogeneous environments. The two organisms compete for different fundamental resources, their advection and diffusion strategies follow a positive diffusion distribution, the functions of interspecific competition ability are variable. Our work can be seen as a further development of Wang [5] for the competition-diffusion system, where we bring new ingredients in the arguments to overcome the difficulty caused by the involvement of advection.

In the future, exploring the global directed dynamic behaviors under the condition of cross-diffusion may be an interesting research point. We leave this challenge to future investigations.

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