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θ^*-Weak Contractions and Discontinuity at the Fixed Point with Applications to Matrix and Integral Equations

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Abstract: In this paper, the notion of θ^*-weak contraction is introduced, which is utilized to prove some fixed point results. These results are helpful to give a positive response to certain open question raised by Kannan and Rhoades on the existence of contractive definition which does not force the mapping to be continuous at the fixed point. Some illustrative examples are also given to support our results. As applications of our result, we investigate the existence and uniqueness of a solution of non-linear matrix equations and integral equations of Volterra type as well.

Keywords: θ*-weak contraction; fixed point; discontinuity at the fixed point; property P; matrix equation; integral equation

MSC: 47H10; 54H25

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1. Introduction and Preliminaries

In order to study the existence of fixed point for discontinuous mappings, Kannan [1] introduced a weaker contraction condition and proved the following theorem:

Every self-mapping S defined on a complete metric space (M, d) satisfying the condition

$$d(Sz, Sw) \leq \beta[d(z, Sz) + d(w, Sw)], \text{where } \beta \in \left[0, \frac{1}{2}\right), \tag{1}$$

$\forall z, w \in M$, has a unique fixed point. We refer such a mappings as Kannan type mappings. Reader can find a lot of literature in this conntext. One such type of result can be seen in [2].

In his paper, [3], Rhoades presented 250 contractive definitions (including (1)) and compared them. He found that though most of them do not force the mapping to be continuous in the entire domain but under these definitions, all the mapping are continuous at the fixed point. Rhoades [4] constructed a very fascinating open problem:

Open Question 1. *Does there exist a contractive definition which is strong enough to ensure the existence and uniqueness of a fixed point but does not force the mapping to be continuous at the underlying fixed point?*

After more than a decade, Pant [5] was the first to give an answer to this Open Question 1.

In other direction, Jleli and Samet [6] introduced another class of mappings and by using it, they defined θ-contractions.

Definition 1 ([6–8]). *Let $\theta : (0, \infty) \to (1, \infty)$ be a mapping satisfying the following conditions:*
$\Theta 1$: *θ is non-decreasing;*
$\Theta 2$: *for each sequence $\{\beta_k\} \subset (0, \infty)$, $\lim_{k\to\infty} \theta(\beta_k) = 1 \iff \lim_{k\to\infty} \beta_k = 0$;*

$\Theta3$: $\exists\, r \in (0,1)$ *and* $l \in (0,\infty]$ *such that*

$$\lim_{\beta\to 0^+} \frac{\theta(\beta)-1}{\beta^r} = l;$$

$\Theta4$: θ *is continuous.*

We consider the following class of mappings:
$\Theta_{1,2,3}$: the class of mappings satisfying $\Theta1$–$\Theta3$.
$\Theta_{1,2,4}$: the class of mappings satisfying $\Theta1$, $\Theta2$ and $\Theta4$.
$\Theta_{2,3}$: the class of mappings satisfying $\Theta2$ and $\Theta3$.
$\Theta_{2,4}$: the class of mappings satisfying $\Theta2$ and $\Theta4$.

Jleli and Samet [6] used the class of functions $\Theta_{1,2,3}$ and proved the following result.

Theorem 1 ([6])**.** *Let (M,d) be a complete generalized metric space and $S : M \to M$ a given mapping. Suppose that there exist $\theta \in \Theta_{1,2,3}$ and $k \in (0,1)$ such that*

$$d(Sz, Sw) > 0 \implies \theta(d(Sz,Sw)) \le [\theta(d(z,w))]^k, \tag{2}$$

$\forall z, w \in M$. Then S has a unique fixed point.

Later on, this contraction condition was modified by many authors. In this direction, Ahmad et al. [9] proved the same result by using class of functions $\Theta_{1,2,4}$. It was already remarked that the monotonicity of θ implies just the continuity of the mapping S, but continuity of S can also be obtained by $\Theta2$, without using $\Theta1$.

Let S be a self-mapping defined on a metric space (M,d) satisfying condition $\Theta2$. If $z, w \in M$ such that $d(z,w)$ tends to 0, then condition $\Theta2$ implies that $\theta(d(z,w))$ tends to 1 and (2) yields that $\theta(d(Sz,Sw))$ tends to 1. Again, condition $\Theta2$ implies that $d(Sz,Sw)$ tends to 0. Hence, S is continuous mapping. Imdad et al. [10] observed that continuity of S still holds even if $\Theta1$ is removed. So they used $\theta \in \Theta_{2,3}$ (or $\theta \in \Theta_{2,4}$) and established that Theorem 1 still held true by considering these class of mappings, i.e., Theorem 1 can survive without $\Theta1$.

In the sequel, it is substantial to state the following lemma.

Lemma 1 ([11])**.** *Let $\{z_n\}$ be a sequence in a metric space (M,d). If $\{z_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two subsequences $\{z_{n(k)}\}$ and $\{z_{m(k)}\}$ of $\{z_n\}$ such that*

$$k \le m(k) < n(k), \quad d(z_{m(k)}, z_{n(k)}) \ge \epsilon \text{ and } d(z_{m(k)}, z_{n(k)-1}) < \epsilon, \quad \forall\, k \in \mathbb{N}.$$

Furthermore, $\lim_{k\to\infty} d(z_{m(k)}, z_{n(k)}) = \epsilon$, provided $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$.

The aim of this paper is five-fold narrated as follows:

1. To introduce the concept of θ^*-weak contractions.
2. To prove some new fixed point results.
3. To provide a new answer to the *Open Question* 1 using θ^*-weak contractions.
4. To investigate the existence and uniqueness of a solution of non-linear matrix equation.
5. To investigate the existence and uniqueness of a solution of integral equation of Volterra type.

In the sequel, $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers, respectively, and $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$. The set of all fixed points of a self-mapping S is denoted by $Fix(S)$.

2. Main Results

Let Θ' be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following condition:
$\Theta2'$: for every sequence $\{\beta_k\} \subset (0,\infty)$, $\lim_{k\to\infty} \theta(\beta_k) = 1 \Rightarrow \lim_{k\to\infty} \beta_k = 0$.

Obviously, $\Theta_{1,2,3} \subset \Theta'$. However, the converse inclusion is not true in general as substantiated by the following examples:

Example 1 ([10]). *Let $\theta : (0, \infty) \to (1, \infty)$ be given by: $\theta(\alpha) = e^{\frac{\alpha}{2} + \sin\alpha}$. It is clear that θ satisfies $\Theta2$ and $\Theta4$. However, it dose not satisfy $\Theta1$.*

Example 2. *Let $\theta : (0, \infty) \to (1, \infty)$ be given by: $\theta(\alpha) = e^{e^{\cos\alpha - \frac{1}{\alpha}}}$. It is clear that θ satisfies $\Theta2$ and $\Theta4$. However, it dose not satisfy $\Theta1$ and $\Theta3$.*

Example 3. *Let $\theta : (0, \infty) \to (1, \infty)$ be given by: $\theta(\alpha) = e^{\cos\alpha - (1+\alpha)}$. It is clear that θ satisfies $\Theta2$ and $\Theta4$. However, it dose not satisfy $\Theta1$ and $\Theta3$.*

Example 4. *Let $\theta : (0, \infty) \to (1, \infty)$ be given by: $\theta(\alpha) = e^{\arctan\alpha - \sin\alpha}$. It is clear that θ satisfies $\Theta2$ and $\Theta4$. However, it dose not satisfy $\Theta1$ and $\Theta3$.*

Now, we are ready to define the notion of θ^*-weak contractions as follows.

Definition 2. *Let (M, d) be a metric space. A self-mapping S on M is said to be a* θ^*-weak contraction *if there exist $k \in (0, 1)$ and $\theta \in \Theta'$ such that*

$$d(Sz, Sw) > 0 \Rightarrow \theta(d(Sz, Sw)) \leq [\theta(m(z, w))]^k, \tag{3}$$

where $m(z, w) = \max\{d(z, w), d(z, Sz), d(w, Sw)\}$.

Now, we state and prove our main results as follows:

Theorem 2. *Let (M, d) be a complete metric space and $S : M \to M$ a θ^*-weak contraction. If θ is continuous, then*

(a) *S has a unique fixed point (say $z^* \in M$),*
(b) $\lim_{n\to\infty} S^n z = z^*$, $\forall z \in M$.

Moreover, S is continuous at z^* if and only if $\lim_{z\to z^*} m(z, z^*) = 0$.

Proof. Let $z_0 \in M$ be an arbitrary point. Define a Picard sequence $\{z_n\} \subseteq M$ based at z_0, i.e., $z_{n+1} = Sz_n$, $\forall n \in \mathbb{N}_0$. If there exists some $n_0 \in \mathbb{N}_0$ such that $z_{n_0} = Sz_{n_0}$, then we are done. Assume that $z_{n+1} \neq z_n$, $\forall n \in \mathbb{N}_0$. On using (3), we have ($\forall n \in \mathbb{N}_0$)

$$\theta(d(z_{n+1}, z_n)) \leq [\theta(m(z_n, z_{n-1}))]^k,$$

where

$$m(z_n, z_{n-1}) \quad = \quad \max\{d(z_n, z_{n-1}), d(z_n, z_{n+1}), d(z_n, z_{n-1})\}.$$

Now, $m(z_n, z_{n-1}) \neq d(z_n, z_{n+1})$, otherwise $\theta(d(z_n, z_{n+1})) \leq [\theta(d(z_n, z_{n+1}))]^k$ a contradiction. Hence, $m(z_n, z_{n-1}) = d(z_n, z_{n-1})$. Thus, we have

$$\theta(d(z_{n+1}, z_n)) \leq [\theta(d(z_n, z_{n-1}))]^k \leq [\theta(d(z_{n-1}, z_{n-2}))]^{k^2} \leq ... \leq [\theta(d(z_1, z_0))]^{k^n}.$$

On letting $n \to \infty$, we obtain

$$1 \leq \lim_{n\to\infty} \theta(d(z_{n+1}, z_n)) \leq \lim_{n\to\infty} [\theta(d(z_1, z_0))]^{k^n} = 1,$$

i.e., $\lim_{n\to\infty} \theta(d(z_{n+1}, z_n)) = 1$ which by $\Theta2'$ yields that

$$\lim_{n\to\infty} d(z_{n+1}, z_n) = 0. \tag{4}$$

Now, we show that $\{z_n\}$ is a Cauchy sequence employing a contradiction. Suppose on contrary that it is not so, then (in view of Lemma 1) there exist $\epsilon_0 > 0$ and two subsequences $\{z_{n(k)}\}$ and $\{z_{m(k)}\}$ of $\{z_n\}$ such that

$$k \leq m(k) < n(k), \;\; d(z_{n(k)-1}, z_{m(k)}) < \epsilon_0 \leq d(z_{n(k)}, z_{m(k)}), \;\; \forall k \in \mathbb{N}_0.$$

We observe that

$$\epsilon_0 \leq d(z_{n(k)}, z_{m(k)}) \leq d(z_{n(k)}, z_{n(k)-1}) + d(z_{n(k)-1}, z_{m(k)}) < d(z_{n(k)}, z_{n(k)-1}) + \epsilon_0$$

so that

$$\lim_{k\to\infty} d(z_{n(k)}, z_{m(k)}) = \lim_{k\to\infty} d(z_{n(k)-1}, z_{m(k)}) = \epsilon_0. \tag{5}$$

Furthermore, we have

$$\begin{aligned} \epsilon_0 &\leq d(z_{n(k)}, z_{m(k)}) \\ &\leq d(z_{n(k)}, z_{m(k)-1}) + d(z_{m(k)-1}, z_{m(k)}) \\ &\leq d(z_{n(k)}, z_{n(k)-1}) + d(z_{n(k)-1}, z_{m(k)-1}) + d(z_{m(k)-1}, z_{m(k)}) \\ &\leq d(z_{n(k)}, z_{n(k)-1}) + d(z_{n(k)-1}, z_{m(k)}) + 2d(z_{m(k)-1}, z_{m(k)}) \end{aligned}$$

so that

$$\lim_{k\to\infty} d(z_{n(k)}, z_{m(k)-1}) = \lim_{k\to\infty} d(z_{n(k)-1}, z_{m(k)-1}) = \epsilon_0. \tag{6}$$

Thus, there exists $N \in \mathbb{N}_0$ such that $d(z_{n(k)-1}, z_{m(k)-1}) > 0$, $\forall k \geq N$ so that on applying (3) with $z = z_{n(k)-1}$ and $w = z_{m(k)-1}$, we have

$$\Theta(d(z_{n(k)}, z_{m(k)})) \leq [\theta(m(z_{n(k)-1}, z_{m(k)-1}))]^k, \tag{7}$$

where

$$m(z_{n(k)-1}, z_{m(k)-1}) = \max\left\{d(z_{n(k)-1}, z_{m(k)-1}), d(z_{n(k)-1}, z_{n(k)}), d(z_{m(k)-1}, z_{m(k)})\right\}.$$

On taking $k \to \infty$ in the above relation and making use of (4) and (6), we get

$$\lim_{k\to\infty} m(z_{n(k)-1}, z_{m(k)-1}) = \epsilon_0. \tag{8}$$

Next, on taking $k \to \infty$ in (7) and using (5) and (8), using the continuity of θ, we obtain $\theta(\epsilon_0) \leq [\theta(\epsilon_0)]^k$, which is a contradiction. Hence, $\{z_n\}$ is a Cauchy sequence in (M, d). As M is complete, so there exists $z^* \in M$ such that $\{z_n\} \to z^*$.

The next step is to prove the point z^* to be a fixed point of S. For this, we consider a set, say $Q = \{n \in \mathbb{N}_0 : z_n = Sz^*\}$. Then, two cases come into existence depending on Q. The first one is, if Q is an infinite set. Then there exists a subsequence $\{z_{n(k)}\} \subseteq \{z_n\}$ which converges to Sz^*. By the property of uniqueness of limit, we arrive at the conclusion that $Sz^* = z^*$. The second is, if Q is a finite set. Then $d(z_n, Sz^*) > 0$ for infinitely many $n \in \mathbb{N}_0$. Hence, there exists a subsequence $\{z_{n(k)}\} \subseteq \{z_n\}$ such that $d(z_{n(k)}, Sz^*) > 0$, $\forall k \in \mathbb{N}_0$. Making use of (3), we obtain ($\forall k \in \mathbb{N}_0$)

$$\theta(d(z_{n(k)}, Sz^*)) \leq [\theta(m(z_{n(k)-1}, z^*))]^k, \tag{9}$$

where

$$m(z_{n(k)-1}, z^*) = \max\left\{d(z_{n(k)-1}, z^*), d(z_{n(k)-1}, z_{n(k)}), d(z^*, Sz^*)\right\}.$$

Now, if $d(Sz^*, z^*) > 0$, then $\lim_{k\to\infty} m(z_{n(k)-1}, z^*) = d(Sz^*, z^*)$. Taking $k \to \infty$ in (9), it yields $\theta(d(z^*, Sz^*)) \leq [\theta(d(z^*, Sz^*))]^k$, which is a contradiction. Therefore, our supposition is wrong and $d(Sz^*, z^*) = 0$ and hence $Sz^* = z^*$.

Now we come to prove the uniqueness of this fixed point. For the same, assume that z^{**} is one more fixed point of S. Then, (3) yields $\theta(d(z^*, z^{**})) \leq [\theta(d(z^*, z^{**}))]^k$, a contradiction. Hence, the fixed point of S is unique.

Now we come to prove the last part of this theorem. Assume that S is continuous at its fixed point z^* and a sequence $\{w_n\} \to z^*$. Then, we obtain $\{Sw_n\} \to Sz^* = z^*$ and $\lim_{n\to\infty} d(w_n, Sw_n) = 0$. Thus, we have $\lim_{n\to\infty} m(w_n, z^*) = 0$.

To establish the converse part, let $\{w_n\} \to z^*$. If we assume that $\lim_{n\to\infty} m(w_n, z^*) = 0$, then $\lim_{n\to\infty} d(w_n, Sw_n) = 0$. This implies that $\lim_{n\to\infty} Sw_n = \lim_{n\to\infty} w_n = z^* = Sz^*$ so that S is continuous at z^*. □

Next, we deduce the following results, which are new for the existing literature by combining Theorem 2 with Examples 1–4:

Corollary 1. *Let (M, d) be a complete metric space and $S : M \to M$. If there exists $k \in (0, 1)$ such that*

$$d(Sz, Sw) > 0 \Rightarrow e^{2\sin(d(Sz,Sw))+d(Sz,Sw)} \leq e^{k(2\sin(m(z,w))+m(z,w))},$$

for all $z, w \in M$, then S has a unique fixed point (say $z^* \in M$) and $\lim_{n\to\infty} S^n z = z^*$, $\forall z \in M$. Moreover, S is continuous at z^* if and only if $\lim_{z\to z^*} m(z, z^*) = 0$.

Corollary 2. *Let (M, d) be a complete metric space and $S : M \to M$. If there exists $k \in (0, 1)$ such that*

$$d(Sz, Sw) > 0 \Rightarrow e^{e^{\cos(d(Sz,Sw))-\frac{1}{d(Sz,Sw)}}} \leq e^{ke^{\left(\cos(m(z,w))-\frac{1}{m(z,w)}\right)}},$$

for all $z, w \in M$, then S has a unique fixed point (say $z^* \in M$) and $\lim_{n\to\infty} S^n z = z^*$, $\forall z \in M$. Moreover, S is continuous at z^* if and only if $\lim_{z\to z^*} m(z, z^*) = 0$.

Corollary 3. *Let (M, d) be a complete metric space and $S : M \to M$. If there exists $k \in (0, 1)$ such that*

$$d(Sz, Sw) > 0 \Rightarrow e^{\cos(d(Sz,Sw))-d(Sz,Sw)} \leq e^{k\left(\cos(m(z,w))-m(z,w)-1+\frac{1}{k}\right)},$$

for all $z, w \in M$, then S has a unique fixed point (say $z^* \in M$) and $\lim_{n\to\infty} S^n z = z^*$, $\forall z \in M$. Moreover, S is continuous at z if and only if $\lim_{z\to z^*} m(z, z^*) = 0$.

Corollary 4. *Let (M, d) be a complete metric space and $S : M \to M$. If there exists $k \in (0, 1)$ such that*

$$d(Sz, Sw) > 0 \Rightarrow e^{\arctan(d(Sz,Sw))-\sin(d(Sz,Sw))} \leq e^{k(\arctan(m(Sz,Sw))-\sin(m(Sz,Sw)))},$$

for all $z, w \in M$, then S has a unique fixed point (say $z^* \in M$) and $\lim_{n\to\infty} S^n z = z^*$, $\forall z \in M$. Moreover, S is continuous at z^* if and only if $\lim_{z\to z^*} m(z, z^*) = 0$.

The following example demonstrates Theorem 2.

Example 5. *Let $M = [0, 1]$ endowed with the usual metric. Define $S : M \to M$ by*

$$Sz = \begin{cases} \frac{1}{3}, & \text{for } z \in [0, 1), \\ \frac{1}{6}, & \text{for } z = 1. \end{cases}$$

Now,

$$d(Sz, Sw) > 0 \iff z \in [0, 1) \text{ and } w = 1 \text{ (and vice versa)},$$

so we get $d(Sz, S1) = d(\frac{1}{3}, \frac{1}{6}) = \frac{1}{6}$ and $m(z, 1) \geq d(1, S1) = \frac{5}{6}$. Consider θ as given in Example 2 and $k = \frac{3}{10}$, then we easily show that (3) holds for such θ and k. Thus, S is θ^*-weak *contraction. Hence, Theorem 2 (Corollary 2) shows that S has a unique fixed point (namely $z^* = \frac{1}{3}$). Furthermore, $\lim_{z \to \frac{1}{3}} m(z, \frac{1}{3}) = 0$ and S is continuous at $\frac{1}{3}$, though it is discontinuous on $[0, 1]$.*

Now, we deduce an integral-type result via Theorem 2.

Theorem 3. *Let (M, d) be a complete metric space and $S : M \to M$ a self mapping satisfying the following: $\forall z, w \in M$, there exists $k \in \left(0, \frac{1}{2}\right)$ and $\theta \in \Theta'$ such that*

$$\int_0^{d(Sz,Sw)} \phi(t)dt > 0 \implies \theta\Big(\int_0^{d(Sz,Sw)} \phi(t)dt\Big) \leq \Big[\theta\Big(\int_0^{m(z,w)} \phi(t)dt\Big)\Big]^k,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping satisfying $\int_0^\epsilon \phi(t)dt > 0$, $\forall \epsilon > 0$. Then S has a unique fixed point.

In next lines, we prove a result analogous to Theorem 2 avoiding the continuity of θ.

Theorem 4. *Let (M, d) be a complete metric space and $S : M \to M$ a θ^*-weak contraction. Assume that S^2 is continuous and there exists $z_0 \in M$ such that $\{S^n z_0\}$ is bounded, then*

(a) S has a unique fixed point (say $z^* \in M$),
(b) $\lim_{n \to \infty} S^n z = z^*$, $\forall z \in M$, provided S is bounded.

Moreover, S is continuous at z^* if and only if $\lim_{z \to z^*} m(z, z^*) = 0$.

Proof. Assume that $z_0 \in M$ such that the sequence $\{S^n z_0\}$ is bounded. On the same steps of proof of Theorem 2, we arrive at the following:

$$\theta(d(z_{n+1}, z_n)) \leq [\theta(d(z_n, z_{n-1}))]^k, \ \forall n \in \mathbb{N}_0$$

so that, $\forall n, l \geq 1$, we have

$$\begin{aligned} \theta(d(z_{n+l}, z_n)) &\leq [\theta(d(z_{n+l-1}, z_{n-1}))]^k \leq [\theta(d(z_{n+l-2}, z_{n-2}))]^{k^2} \leq ... \\ &\leq [\theta(d(z_l, z_0))]^{k^n} \leq [\theta(C)]^{k^n} \to 1, \text{ as } n \to \infty, \end{aligned}$$

where $C = \sup_{l \geq 1} d(z_0, S^l z_0)$. Now, making use of $\Theta 2'$, we obtain

$$\lim_{n \to \infty} d(z_{n+l}, z_n) = 0.$$

Hence, $\{z_n\}$ is a Cauchy sequence. As M is a complete metric space, this fact implies that there exists $z^* \in M$ such that $\{z_n\}$ converges to z^*. As S^2 is continuous, so $\{S^2 z_n = z_{n+2}\} \to S^2 z^*$. Owing to the uniqueness of the limit, we have $S^2 z^* = z^*$. Next, we claim that $Sz^* = z^*$. Assume on contrary that $Sz^* \neq z^*$. Then, we have $m(z^*, Sz^*) = d(z^*, Sz^*)$ and hence,

$$\theta(d(z^*, Sz^*)) = \theta(d(S^2 z^*, Sz^*)) \leq [\theta(m(z^*, Sz^*))]^k = [\theta(d(z^*, Sz^*))]^k,$$

a contradiction. Thus, $Sz^* = z^*$. Observe that if S is bounded, then z_0 chosen in the beginning may be any arbitrary point of M and hence, part (b) is established. The rest of the proof is followed on the lines of the proof of Theorem 2. □

In the following example, we furnish a mapping which is discontinuous at its fixed point exhibiting the utility of Theorem 4.

Example 6. *Let $M = [0, 1]$ endowed with the usual metric. Define $S : M \to M$ by*

$$Sz = \begin{cases} \frac{1}{2}, & \text{for } z \in [0, \frac{1}{2}], \\ 0, & \text{for } z \in (\frac{1}{2}, 1]. \end{cases}$$

We see that that S is bounded and S^2 is continuous as well. Next, define $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(\alpha) = \begin{cases} e^{e^{cos\alpha - \frac{1}{\alpha}}}, & \text{for } \alpha \in (0, \frac{1}{2}], \\ e^{\frac{\alpha}{2} + sin\alpha}, & \text{for } \alpha \in (\frac{1}{2}, \infty). \end{cases}$$

Clearly, $\theta \in \Theta'$. Now,

$$d(Sz, Sw) > 0 \iff z \in \left[0, \frac{1}{2}\right] \text{ and } w \in \left(\frac{1}{2}, 1\right] \text{ (or vice versa)},$$

so that $d(Sz, Sw) = \frac{1}{2}$ and $m(z, w) > \frac{1}{2}$. It is very easy to show that (3) holds for θ and $k = \frac{9}{20}$ by routine calculation. Thus, all the hypotheses of Theorem 4 are satisfied and hence, S has a unique fixed point (namely $z^* = \frac{1}{2}$). Notice that $\lim_{z \to \frac{1}{2}} m(z, \frac{1}{2})$ does not exist and S is discontinuous at $z^* = \frac{1}{2}$.

Remark 1. *Notice that in Example 6, Theorem 1 as well as Theorem 2.1 of [8] is not applicable as neither θ nor S is continuous.*

Remark 2. *θ^*-weak contraction is sufficiently providing an answer to the Open Question 1.*

Next, we consider Θ'', the class of mappings $\theta : (0, \infty) \to (1, \infty)$ satisfying $\Theta 2'$ and $\Theta 3$. We recall the following notion before presenting our next result.

Definition 3. ***Property P:*** *A self-mapping S has property P if*

$$Fix(S^n) = Fix(S), \text{ for every } n \in \mathbb{N}.$$

Theorem 5. *Let (M, d) be a complete metric space and $S : M \to M$ a continuous mapping. If there exist $k \in (0, 1)$ and $\theta \in \Theta''$ such that*

$$d(Sz, S^2z) > 0 \implies \theta(d(Sz, S^2z)) \le [\theta(m(z, Sz))]^k, \quad (10)$$

$\forall z \in M$, then S has the property P.

Proof. Let $z_0 \in M$ be an arbitrary point. Define a Picard sequence $\{z_n\} \subseteq M$ based at z_0, i.e., $z_{n+1} = Sz_n, \forall n \in \mathbb{N}_0$. If there exists some $n_0 \in \mathbb{N}_0$ such that $z_{n_0} = z_{n_0+1}$, then we are done. Henceforth, assume that $z_n \neq z_{n+1}, \forall n \in \mathbb{N}_0$, i.e., $d(Sz_{n-1}, S^2z_{n-1}) > 0, \forall n \in \mathbb{N}$. Thus, (10) implies that

$$\theta(d(Sz_{n-1}, S^2z_{n-1})) \le [\theta(m(z_{n-1}, Sz_{n-1}))]^k$$

or

$$\theta(d(z_n, z_{n+1})) \le [\theta(m(z_{n-1}, z_n))]^k, \quad (11)$$

where

$$\begin{aligned} m(z_{n-1}, z_n) &= \max\{d(z_{n-1}, z_n), d(z_{n-1}, Sz_{n-1}), d(z_n, Sz_n)\} \\ &= \max\{d(z_{n-1}, z_n), d(z_n, z_{n+1})\}. \end{aligned}$$

If $m(z_{n-1}, z_n) = d(z_n, z_{n+1})$, then (11) yields that $\theta(d(z_n, z_{n+1})) \le [\theta(d(z_n, z_{n+1}))]^k$, $k \in (0, 1)$, which is a contradiction. Therefore, $m(z_{n-1}, z_n) = d(z_{n-1}, z_n)$. Now, in view of (11), we have

$$\theta(d(z_n, z_{n+1})) \le [\theta(d(z_{n-1}, z_n))]^k, \ \forall n \in \mathbb{N} \text{ and } k \in (0, 1).$$

Hence, we get

$$1 < \theta(d(z_n, z_{n+1})) \leq [\theta(d(z_{n-1}, z_n))]^k \leq [\theta(d(z_{n-2}, z_{n-1}))]^{k^2} \leq ... \leq [\theta(d(z_0, z_1))]^{k^n}. \quad (12)$$

Taking limit $n \to \infty$ in (12), we obtain

$$\lim_{n\to\infty} \theta(d(z_n, z_{n+1})) = 1,$$

which by $\Theta 2'$ gives

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0.$$

Now, $\Theta 3$ implies that there exist $r \in (0,1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(z_n, z_{n+1})) - 1}{[d(z_n, z_{n+1})]^r} = l.$$

Firstly, assume that $l < \infty$. Let $C = \frac{l}{2}$. Then, by the definition of the limit, there exists $N_1 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d(z_n, z_{n+1})) - 1}{[d(z_n, z_{n+1})]^r} - l \right| \leq C, \ \forall n \geq N_1$$

implying that

$$\frac{\theta(d(z_n, z_{n+1})) - 1}{[d(z_n, z_{n+1})]^r} \geq l - C = C, \ \forall n \geq N_1.$$

So, we have

$$n[d(z_n, z_{n+1})]^r \leq nD[\theta(d(z_n, z_{n+1})) - 1], \ \forall n \geq N_1 \text{ and } D = \frac{1}{C}.$$

Secondly, suppose that $l = \infty$. Let $C > 0$ be a given real number. Then from the definition of the limit, there exists $N_2 \in \mathbb{N}$ such that

$$\frac{\theta(d(z_n, z_{n+1})) - 1}{[d(z_n, z_{n+1})]^r} \geq C, \ \forall n \geq N_2$$

implying that

$$n[d(z_n, z_{n+1})]^r \leq nD[\theta(d(z_n, z_{n+1})) - 1], \ \forall n \geq N_2 \text{ and } D = \frac{1}{C}.$$

Thus, in all, there exist $D > 0$ and $N = \max\{N_1, N_2\}$ such that

$$n[d(z_n, z_{n+1})]^r \leq nD[\theta(d(z_n, z_{n+1})) - 1], \ \forall n \geq N.$$

From (12), we have

$$n[d(z_n, z_{n+1})]^r \leq nD[[\theta(d(z_0, z_1))]^{k^n} - 1], \ \forall n \geq N_2 \text{ and } D = \frac{1}{C}.$$

Taking limit $n \to \infty$, we obtain

$$n[d(z_n, z_{n+1})]^r = 0. \quad (13)$$

Now, (13) ensures the existence of N' such that

$$n[d(z_n, z_{n+1})]^r \leq 1, \ \forall n \geq N',$$

which implies that

$$d(z_n, z_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}}, \ \forall n \geq N'.$$

Now, for $m > n \geq N'$, we have

$$d(z_n, z_m) \leq \sum_{i=n}^{m-1} d(z_i, z_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{r}}}.$$

As $0 < r < 1$, so $\left\{ \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}} \right\}$ converges and hence,

$$\lim_{m,n\to\infty} d(z_n, z_m) = 0,$$

i.e., $\{z_n\}$ is a Cauchy sequence. Now, by the completeness of M, we get the ensurance of the existence of $z^* \in M$ such that $z_n \to z^*$, as $n \to \infty$.

By the continuity of S, we have

$$z_{n+1} = Sz_n \to Sz^*, \text{ as } n \to \infty.$$

By the uniqueness of the limit, we have $z^* = Sz^*$, i.e., S has a fixed point.

Now, we will show that

$$Fix(S^n) = Fix(S), \ \forall n \in \mathbb{N}.$$

Suppose on contrary that there exists some $z' \in Fix(S^n)$ such that $z' \notin Fix(S)$. Then, we have

$$d(z', Sz') > 0.$$

Now, we have

$$\begin{aligned} 1 < \theta(d(z', Sz')) =& \theta(d(S(S^{n-1}z'), S^2(S^{n-1}z')) \leq [\theta(m(S^{n-1}z', S^n z'))]^k \\ =& [\theta(d(S^{n-1}z', S^n z'))]^k \leq [\theta(d(S^{n-2}z', S^{n-1}z'))]^{k^2} \\ \leq& ... \leq [\theta(d(z', Sz'))]^{k^n}. \end{aligned}$$

Taking limit $n \to \infty$, we get $\theta(d(z', Sz')) = 1$, *i.e.*, $d(z', Sz') = 0$, a contradiction. This completes the proof. □

3. Application to Nonlinear Matrix Equations

Throughout this section, we use the following notations:

$\mathcal{M}(n)$ = the set of all $n \times n$ complex matrices.

$\mathcal{H}(n)$ = the set of all Hermitian matrices in $\mathcal{M}(n)$.

$\mathcal{P}(n)$ = the set of all positive definite matrices in $\mathcal{M}(n)$.

$\mathcal{H}^+(n)$ = the set of all positive semidefinite matrices in $\mathcal{M}(n)$.

For $Z \in \mathcal{P}(n)$ (resp. $Z \in \mathcal{H}^+(n)$), we write $Z \succ 0$ (resp. $Z \succeq 0$). The symbol $\|.\|$ symbolizes the spectral norm of a matrix A defined by $\|A\| = \sqrt{\lambda^+(A^*A)}$, where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A, where A^* is the conjugate transpose of A. Furthermore, $\|A\|_{tr} = \sum_{k=1}^{n} s_k(A) = tr((A^*A)^{\frac{1}{2}})$, where $s_k(A)$ $(1 \leq k \leq n)$ are the singular values of $A \in \mathcal{M}(n)$. In case if A is a Hermition matrix, this definition reduces to: $\|A\|_{tr} = tr(A)$. Here, $(\mathcal{H}(n), \|.\|_{tr})$ is complete metric space (for more details see [12–15]).

In this section, we apply our result (viz. Theorem 2) to prove the existence and uniqueness of a solution of the nonlinear matrix equation

$$Z = P + \sum_{k=1}^{m} A_k^* \mathcal{F}(Z) A_k, \tag{14}$$

where P is a Hermitian positive definite matrix and $\mathcal{F}$ is a continuous mapping from $\mathcal{H}(n)$ into $\mathcal{P}(n)$ such that $\mathcal{Q}(0) = 0$, A_k are arbitrary $n \times n$ matrices and A_k^* their conjugates.

In the sequel, we need the following lemmas:

Lemma 2 ([12])**.** *If $A \succeq 0$ and $B \succeq 0$ are $n \times n$ matrices, then $0 \leq tr(AB) \leq \|A\| tr(B)$.*

Lemma 3 ([16])**.** *If $A \in \mathcal{H}(n)$ such that $A \prec I_n$, then $\|A\| < 1$.*

Theorem 6. *Consider the matrix Equation (14). Assume that there exist two positive real numbers R and $M \geq 1$ such that:*

(i) $\sum_{k=1}^{m} A_k A_k^* \preceq R I_n$ *and*
(ii) *for every $Z, W \in \mathcal{H}(n)$ with $\sum_{k=1}^{n} A_k^* \mathcal{F}(Z) A_k \neq \sum_{k=1}^{n} A_k^* \mathcal{F}(W) A_k$, we have*

$$d(\mathcal{F}(Z), \mathcal{F}(W)) \leq \frac{e^{-M}}{R} m(Z, W),$$

where $M \geq 1$ and $m(Z, W)$ is as defined in Definition 2.

Then the matrix Equation (14) has a unique solution. Moreover, the iteration $Z_n = P + \sum_{k=1}^{n} A_k^* \mathcal{F}(Z_{n-1}) A_k$ converges in the sense of trace norm $\|.\|_{tr}$ to the solution of the matrix Equation (14), where $Z_0 \in \mathcal{H}(n)$ such that $Z_0 \preceq \sum_{k=1}^{m} A_k^* \mathcal{F}(Z_0) A_k$.

Proof. Define a mapping $S : \mathcal{H}(n) \to \mathcal{H}(n)$ by:

$$S(Z) = P + \sum_{k=1}^{n} A_k^* \mathcal{F}(Z) A_k, \quad \forall Z \in \mathcal{H}(n). \tag{15}$$

Observe that S is well defined and X is a fixed point of S if and only if it is a solution of the matrix Equation (14). To accomplish this, we need to show that S is θ^*-weak contraction wherein the mapping $\theta : (0, \infty) \to (1, \infty)$ is given by: $\theta(\alpha) = e^{\sqrt{\alpha}}$, $\forall \alpha \in (0, \infty)$, which is continuous and belongs to Θ'.

Let $Z, W \in \mathcal{H}(n)$ be such that $S(Z) \neq S(W)$. Consider

$$\begin{aligned}
\|S(Z) - S(W)\|_{tr} &= tr\big(S(Z) - S(W)\big) \\
&= tr\Big(\sum_{k=1}^{m} A_k^* (\mathcal{F}(Z) - \mathcal{F}(W)) A_k\Big) \\
&= \sum_{k=1}^{m} tr\big(A_k^* (\mathcal{F}(Z) - \mathcal{F}(W)) A_k\big) \\
&= \sum_{k=1}^{m} tr\big(A_k^* A_k (\mathcal{F}(Z) - \mathcal{F}(W))\big) \\
&= tr\Big(\Big(\sum_{k=1}^{m} A_k^* A_k\Big)(\mathcal{F}(Z) - \mathcal{F}(W))\Big) \\
&\leq \Big\|\sum_{k=1}^{m} A_k^* A_k\Big\| \|\mathcal{F}(Z) - \mathcal{F}(W)\|_{tr} \\
&\leq \frac{e^{-M}}{R} \Big\|\sum_{k=1}^{m} A_k^* A_k\Big\| m(Z, W) \\
&\leq e^{-M} m(Z, W),
\end{aligned}$$

so that

$$d(SZ, SW) \leq e^{-M} m(Z, W),$$

which implies that

$$e^{\sqrt{d(SZ,SW)}} \leq e^{\sqrt{e^{-M} m(Z,W)}}, \; M \geq 1.$$

This yields that
$$e^{\sqrt{d(SZ,SW)}} \leq \left[e^{\sqrt{m(Z,W)}}\right]^k,$$
where $k = \sqrt{e^{-M}}$. The supposition that $M \geq 1$ implies that $k \in (0,1)$ which shows that S is a θ^*-weak contraction. Thus, all the hypotheses of Theorem 2 are satisfied. Hence, there exists a unique $Z \in \mathcal{H}(n)$ such that $SZ = Z$, i.e., the matrix Equation (14) has a unique solution in $\mathcal{H}(n)$. This completes the proof. □

4. Application to Integral Equations

In this section, we investigate the existence and uniqueness of a solution of a Volterra type integral equation with the help of Theorem 2. Suppose the integral equation is given as follows:
$$z(t) = \int_0^t K(t,s,z(s))ds + h(t), \ \ t \in [0,T], \tag{16}$$
where $T > 0$, $K : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$, $h : [0,T] \to \mathbb{R}$.

Consider the space $C([0,T],\mathbb{R})$ of all continuous functions $z : [0,T] \to \mathbb{R}$ equipped with the Bielecki's norm
$$\|z\| = \sup_{t\in[0,T]} e^{-\alpha t}|z(t)|, \ \ \alpha \geq 1.$$

Then, the space $(C([0,T],\mathbb{R}),d)$ is a complete metric space with
$$d(z,w) = \|z - w\|, \ \ \forall z,w \in C([0,T],\mathbb{R}).$$

One can go through [17–19] for more literature.

Now, by utilizing Theorem 2, we state and prove the following result.

Theorem 7. *Assume that there exists $\alpha \geq 1$ such that*
$$|K(t,s,z) - K(t,s,w)| \leq \alpha e^{-\alpha} m^*(z,w),$$
$\forall t,s \in [0,T]$ and $z,w \in C([0,T],\mathbb{R})$, where
$$m^*(z,w) = \max\{|z-w|, |z-Sz|, |w-Sw|\}.$$

Then the integral Equation (16) has a unique solution in $C([0,T],\mathbb{R})$.

Proof. Define the mapping $S : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ by:
$$Sz(t) = \int_0^t K(t,s,z(s))ds + h(t), \ \ z \in C([0,T],\mathbb{R}).$$

Observe that S is well defined and z is a fixed point of S if and only if it is a solution of the integral Equation (14). Now, define $\theta : (0,\infty) \to (1,\infty)$ by $\theta(\alpha) = e^{\alpha}$, $\alpha \in (0,\infty)$. Then θ is continuous and belongs to Θ'. Now, consider

$$\begin{aligned}|Sz(t) - Sw(t)| =& \left| \int_0^t (K(t,s,z(s)) - K(t,s,w(s)))ds \right| \\ \leq& \int_0^t |(K(t,s,z(s)) - K(t,s,w(s)))|ds \\ \leq& \int_0^t \alpha e^{-\alpha} m^*(z,w)ds \\ =& \alpha e^{-\alpha} \int_0^t e^{s\alpha} \max\{|z(s) - w(s)|e^{-s\alpha}, |z(s) - Sz(s)|e^{-s\alpha}, \\ & |w(s) - Sw(s)|e^{-s\alpha}\}ds \\ \leq& \alpha e^{-\alpha} \int_0^t e^{s\alpha} \max\{d(z,w), d(z,Sz), d(w,Sw)\}ds \\ =& \alpha e^{-\alpha} m(z,w) \int_0^t e^{s\alpha} ds \\ \leq& m(z,w)e^{-\alpha(1-t)}.\end{aligned}$$

This implies that

$$|Sz(t) - Sw(t)|e^{-\alpha t} \leq e^{-\alpha} m(z,w).$$

Taking supremum over t of both sides, we get

$$d(Sz, Sw) \leq e^{-\alpha} m(z,w),$$

which implies that

$$e^{d(Sz,Sw)} \leq e^{e^{-\alpha} m(z,w)} = [e^{m(z,w)}]^k, \ \forall z,w \in C([0,T], \mathbb{R}),$$

where $k = e^{-\alpha}$. Since $\alpha \geq 1$, so $k \in (0,1)$. Therefore, S is θ^*-weak contraction. By Theorem 2, S has a unique solution of integral Equation (16). This ends the proof. □

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