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Normed Spaces Which Are Not Mackey Groups

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Abstract: It is well known that every normed (even quasibarrelled) space is a Mackey space. However, in the more general realm of locally quasi-convex abelian groups an analogous result does not hold. We give the first examples of normed spaces which are not Mackey groups.

Keywords: normed space; Mackey group; locally quasi-convex; compatible group topology

MSC: 46A8; 46E10; 54H11

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1. Introduction

Let (E, τ) be a locally convex space (lcs for short). A locally convex vector topology ν on E is called *compatible with* τ if the spaces (E, τ) and (E, ν) have the same topological dual space. The famous Mackey–Arens Theorem states that there is a finest locally convex vector space topology μ on E compatible with τ. The topology μ is called the *Mackey topology* on E associated with τ, and if $\mu = \tau$, the space E is called a *Mackey space*. The most important class of Mackey spaces is the class of quasibarrelled spaces. This class is sufficiently rich and contains all metrizable locally convex spaces. In particular, every normed space is a Mackey space.

For an abelian topological group (G, τ) we denote by $\widehat{G}$ the group of all continuous characters of (G, τ). Two topologies μ and ν on an abelian group G are said to be *compatible* if $\widehat{(G, \mu)} = \widehat{(G, \nu)}$. Being motivated by the concept of Mackey spaces, the following notion was implicitly introduced and studied in [1], and explicitly defined in [2] (for all relevant definitions see the next section): A locally quasi-convex abelian group (G, μ) is called a *Mackey group* if for every locally quasi-convex group topology ν on G compatible with μ it follows that $\nu \leq \mu$.

Every lcs considered as an abelian topological group is locally quasi-convex. So, it is natural to ask whether every Mackey space is also a Mackey group. Surprisingly, the answer to this question is negative. Indeed, answering a question posed in [2], we show in [3] that there is even a metrizable lcs which is not a Mackey group. Recall that for every Tychonoff space X, the space $C_p(X)$ of all continuous functions on X endowed with the pointwise topology is quasibarrelled, and hence it is a Mackey space. However, in [4] we proved that the space $C_p(X)$ is a Mackey group if and only if it is barrelled. In particular, the metrizable space $C_p(\mathbb{Q})$ is *not* a Mackey group. These results motivate the following question. For $1 \leq p \leq \infty$, denote with $\mathfrak{T}_{\ell_p}$ the topology on the direct sum $\mathbb{R}^{(\mathbb{N})} := \bigcup_{n\in\mathbb{N}} \mathbb{R}^n$ induced from ℓ_p.

Problem 1 ([3]). *Does there exist a* normed *space E which is not a Mackey group? What about $(\mathbb{R}^{(\mathbb{N})}, \mathfrak{T}_{\ell_p})$?*

The main goal of this note is to answer Problem 1 in the affirmative. More precisely, we show that the normed spaces $c_{00} := (\mathbb{R}^{(\mathbb{N})}, \mathfrak{T}_{\ell_\infty})$ and $\ell_{00}^1 := (\mathbb{R}^{(\mathbb{N})}, \mathfrak{T}_{\ell_1})$ are not Mackey groups.

2. Main Result

Set $\mathbb{N} := \{1, 2, \dots\}$. Denote by $\mathbb{S}$ the unit circle group and set $\mathbb{S}_+ := \{z \in \mathbb{S} : \operatorname{Re}(z) \geq 0\}$.

Let G be an abelian topological group. A character $\chi \in \widehat{G}$ is a continuous homomorphism from G into $\mathbb{S}$. A subset A of G is called *quasi-convex* if for every $g \in G \setminus A$ there exists $\chi \in \widehat{G}$ such that $\chi(g) \notin \mathbb{S}_+$ and $\chi(A) \subseteq \mathbb{S}_+$. An abelian topological group is called *locally quasi-convex* if it admits a neighborhood base at the neutral element 0 consisting of quasi-convex sets. It is well known that the class of locally quasi-convex abelian groups is closed under taking products and subgroups.

The following group plays an essential role in the proof of our main results, Theorems 1 and 2. Set

$$c_0(\mathbb{S}) := \{(z_n) \in \mathbb{S}^{\mathbb{N}} : z_n \to 1\},$$

and denote by $\mathfrak{F}_0(\mathbb{S})$ the group $c_0(\mathbb{S})$ endowed with the metric $d\big((z_n^1), (z_n^2)\big) = \sup\{|z_n^1 - z_n^2|, n \in \mathbb{N}\}$. Then $\mathfrak{F}_0(\mathbb{S})$ is a Polish group, and the sets of the form $V^{\mathbb{N}} \cap c_0(\mathbb{S})$, where V is a neighborhood at the unit $1 \in \mathbb{S}$, form a base at the identity $1 = (1_n) \in \mathfrak{F}_0(\mathbb{S})$. In [5] (Theorem 1), we proved that the group $\mathfrak{F}_0(\mathbb{S})$ is reflexive and hence locally quasi-convex.

A proof of the next important result can be found in [6] [Proposition 2.3].

Fact 1. *Let E be a real lcs. Then the map $\psi : E' \to \widehat{E}, \psi(\chi) := e^{2\pi i \chi}$, is an algebraic isomorphism.*

We use the next standard notations. Let $\{e_n\}_{n \in \mathbb{N}}$ be the standard basis of the Banach space $(c_0, \|\cdot\|_\infty)$, and let $\{e_n^*\}_{n \in \mathbb{N}}$ be the canonical basis in the dual Banach space $(c_0)' = \ell_1$, i.e.,

$$e_n = (0, \dots, 0, 1, 0, \dots) \quad \text{and} \quad e_n^* = (0, \dots, 0, 1, 0, \dots),$$

where 1 is placed in position n. Then $c_{00} = (\mathbb{R}^{(\mathbb{N})}, \mathfrak{T}_{\ell_\infty})$ is a dense subspace of c_0 consisting of all vectors with finite support.

Theorem 1. *The normed space c_{00} is not a Mackey group.*

Proof. For simplicity and clearness of notations we set $E := c_{00}$ and $\tau := \mathfrak{T}_{\ell_\infty}$. For every $n \in \mathbb{N}$, set $\chi_n := n e_n^*$. It is clear that $\chi_n \to 0$ in the weak* topology on E' and hence in $\sigma(\widehat{E}, E)$. Therefore we can define the linear injective operator $F : E \to E \times c_0$ and the monomorphism $p : E \to E \times \mathfrak{F}_0(\mathbb{S})$ setting (for all $x = (x_n) \in E$)

$$F(x) := \big(x, R(x)\big), \text{ where } R(x) := \big(\chi_n(x)\big) = (n x_n) \in c_0,$$
$$p(x) := \big(x, R_0(x)\big), \text{ where } R_0(x) := Q \circ R(x) = \big(\exp\{2\pi i \chi_n(x)\}\big) = \big(\exp\{2\pi i n x_n\}\big) \in \mathfrak{F}_0(\mathbb{S}).$$

Denote with $\mathfrak{T}$ and $\mathfrak{T}_0$ the topologies on E induced from $E \times c_0$ and $E \times \mathfrak{F}_0(\mathbb{S})$, respectively. So $\mathfrak{T}$ is a locally convex vector topology on E and $\mathfrak{T}_0$ is a locally quasi-convex group topology on E. By construction, $\tau \leq \mathfrak{T}_0 \leq \mathfrak{T}$, so taking into account Fact 1 and the Hahn–Banach extension theorem, we obtain

$$\psi(E') = \psi(\ell_1) \subseteq \widehat{(E, \mathfrak{T}_0)} \subseteq \psi\big((E, \mathfrak{T})'\big) \subseteq \psi\big(\ell_1 \times \ell_1\big). \tag{1}$$

Step 1: The topologies τ and $\mathfrak{T}_0$ are compatible. By (1), it is sufficient to show that each continuous character of $(E, \mathfrak{T}_0)$ belongs to $\psi(\ell_1)$. Fix $\chi \in \widehat{(E, \mathfrak{T}_0)}$. Then (1) implies that $\chi = \psi(\eta) = \exp\{2\pi i \eta\}$ for some

$$\eta = \big(\nu, (c_n)\big) \in \ell_1 \times \ell_1, \text{ where } \nu \in \ell_1 \text{ and } (c_n) \in \ell_1,$$

and

$$\eta(x) = \nu(x) + \sum_{n \in \mathbb{N}} c_n \chi_n(x) = \nu(x) + \sum_{n \in \mathbb{N}} c_n \cdot n x_n \quad \big(x = (x_n) \in E\big).$$

To prove that $\chi \in \psi(\ell_1)$ it is sufficient (and also necessary) to show that $(c_n n)_n \in \ell_1$. Replacing, if needed, η by $\eta - \nu$, we assume that $\nu = 0$.

Suppose for a contradiction that $\sum_n |c_n|n = \infty$. Since χ is continuous, Fact 1 shows that, for every $\varepsilon < 0.01$, there is a $\delta < \varepsilon$ such that

$$\eta(x) = \sum_{n\in\mathbb{N}} nc_n x_n \in (-\varepsilon,\varepsilon) + \mathbb{Z}, \text{ for every } x \in U_\delta, \tag{2}$$

where U_δ is a canonical $\mathfrak{T}_0$-neighborhood of zero

$$U_\delta := \{x = (x_n) \in E : |x_n| \le \delta \text{ and } nx_n \in [-\delta,\delta] + \mathbb{Z} \text{ for every } n \in \mathbb{N}\}. \tag{3}$$

In what follows ε and δ are fixed as above. We distinguish between three cases.

Case 1: There is a subsequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $|c_{n_k}|n_k \to \infty$ as $k \to \infty$. As $|c_{n_k}|n_k \to \infty$ and $c_n \to 0$, there is $k \in \mathbb{N}$ such that

$$\frac{1}{8|c_{n_k}|} > 1 \text{ and } \frac{3}{8|c_{n_k}|n_k} < \delta. \tag{4}$$

The first inequality in (4) implies that there is

$$m_k \in \left(\frac{1}{8|c_{n_k}|}, \frac{3}{8|c_{n_k}|}\right) \cap \mathbb{N}. \tag{5}$$

Set $x = (x_n) := (0,\dots,0,\operatorname{sign}(c_{n_k})\frac{m_k}{n_k},0,\dots)$, where the nonzero element is placed in position n_k. Then $nx_n \in \mathbb{Z}$ for every $n \in \mathbb{N}$, and the second inequality of (4) and (5) imply

$$\|x\|_\infty = |x_{n_k}| = \frac{m_k}{n_k} < \frac{3}{8|c_{n_k}|n_k} < \delta.$$

Therefore $x \in U_\delta$. On the other hand, (5) implies

$$\frac{1}{8} < \eta(x) = \sum_{n\in\mathbb{N}} c_n n x_n = |c_{n_k}|n_k\frac{m_k}{n_k} = |c_{n_k}|m_k < \frac{3}{8}.$$

Hence $\eta(x) \notin (-\varepsilon,\varepsilon) + \mathbb{Z}$ since $\varepsilon < 0.01$. However, this contradicts (2).

Case 2: There is a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and a number $a > 0$ such that $|c_{n_k}|n_k \to a$ as $k \to \infty$. Choose $N \in \mathbb{N}$ such that

$$\frac{a}{2} < |c_{n_k}|n_k < \frac{3a}{2} \text{ for all } k \ge N. \tag{6}$$

Choose a finite subset F of $\{n_k\}_{k\ge N}$ and, for every $n \in F$, a natural number i_n such that the following two conditions are satisfied:

$$i_n \in \{1,2,\dots,\lfloor \delta n\rfloor\} \text{ for every } n \in F, \tag{7}$$

and

$$\frac{10}{72a} < \sum_{n\in F}\frac{i_n}{n} < \frac{30}{72a} \tag{8}$$

(this is possible because $\frac{1}{n} \le \frac{i_n}{n} \le \frac{\lfloor \delta n\rfloor}{n} \approx \delta$ and $n \to \infty$: so, if $\frac{10}{72a} < \delta$ the set F can be chosen to have only one element, and if $\delta \le \frac{10}{72a}$, the set F also can be easily chosen to be finite). Now we define $x = (x_n) \in E$ by

$$x_n = \operatorname{sign}(c_n)\cdot\frac{i_n}{n} \text{ if } n \in F, \text{ and } x_n = 0 \text{ if } n \in \mathbb{N}\backslash F.$$

Then $nx_n \in \mathbb{Z}$ for every $n \in \mathbb{Z}$, and, by (7), $\|x\|_\infty = \max\left\{\frac{i_n}{n} : n \in F\right\} \le \delta$. Therefore $x \in U_\delta$. On the other hand, (6) and (8) imply

$$\frac{5}{24} < \frac{a}{2}\sum_{n\in F}\frac{i_n}{n} < \eta(x) = \sum_{n\in\mathbb{N}} c_n n x_n = \sum_{n\in F}|c_n|n\cdot\frac{i_n}{n} < \frac{3a}{2}\sum_{n\in F}\frac{i_n}{n} < \frac{5}{8}.$$

Hence $\eta(x) \notin (-\varepsilon, \varepsilon) + \mathbb{Z}$ which contradicts (2).

Case 3: $\lim_n |c_n| n = 0$. Choose $N \in \mathbb{N}$ such that (recall that $(c_n) \in \ell_1$)

$$\sum_{n \geq N} |c_n| < \tfrac{\delta}{100} \quad \text{and} \quad \sup\{|c_n| n : n \geq N\} < \tfrac{\delta}{100}. \tag{9}$$

Since $\sum_n |c_n| n = \infty$, choose a finite subset $F \subseteq \{N, N+1, \dots\}$ such that

$$\sum_{n \in F} |c_n| n > \tfrac{2}{\delta}. \tag{10}$$

Define $x = (x_n) \in E$ by

$$x_n := \Delta_n \cdot \operatorname{sign}(c_n) \cdot \tfrac{\lfloor \delta n \rfloor}{n} \ \text{ if } \ n \in F, \text{ and } \ x_n := 0 \ \text{ if } \ n \in \mathbb{N} \setminus F,$$

where $\Delta_n \in \{0, 1\}$ will be chosen afterwards. Then, for all $n \in \mathbb{N}$ and arbitrary Δ_ns, we have $x_n \cdot n \in \mathbb{Z}$ and $|x_n| \leq \delta$. Therefore $x \in U_\delta$. On the other hand, we have

$$0 < \eta(x) = \sum_{n \in \mathbb{N}} c_n n x_n = \sum_{n \in F} |c_n| n \Delta_n \cdot \tfrac{\lfloor \delta n \rfloor}{n} \leq \sum_{n \in F} |c_n| n\delta + \sum_{n \in F} |c_n| n \tfrac{\lfloor \delta n \rfloor - \delta n}{n}, \tag{11}$$

(to obtain the last inequality we put $\Delta_n = 1$ for all $n \in F$) and (9) and (10) imply

$$\sum_{n \in F} |c_n| n\delta + \sum_{n \in F} |c_n| n \tfrac{\lfloor \delta n \rfloor - \delta n}{n} > 2 - \tfrac{\delta}{100} > 1. \tag{12}$$

From the second inequality in (9), we have

$$c_n n x_n = |c_n| n \Delta_n \leq |c_n| n < \tfrac{\delta}{100} < \tfrac{1}{100} \quad \text{for every} \quad n \in F.$$

Using this inequality and (11) and (12), one can easily find a family $\{\Delta_n : n \in F\}$ such that

$$\tfrac{1}{4} < \eta(x) = \sum_{n \in \mathbb{N}} c_n n x_n < \tfrac{3}{4},$$

and hence $\eta(x) \notin (-\varepsilon, \varepsilon) + \mathbb{Z}$ which contradicts (2).

Cases 1–3 show that the assumption $\sum_n |c_n| n = \infty$ is wrong. Thus the topologies τ and $\mathfrak{T}_0$ are compatible.

Step 2. The topology $\mathfrak{T}_0$ is strictly finer than the original topology τ. Thus, E is not a Mackey group. Indeed, it is clear that $\frac{1}{2k} e_k \to 0$ in the norm topology τ on E. On the other hand, since

$$R_0\left(\tfrac{1}{2k} e_k\right) = \left(\exp\left\{2\pi i \cdot \chi_n\left(\tfrac{1}{2k} e_k\right)\right\}\right)_{n \in \mathbb{N}} = (1, \dots, 1, -1, 1, \dots,) \quad \text{for every} \quad k \in \mathbb{N},$$

where -1 is placed in position k, we obtain that $\frac{1}{2k} e_k \not\to 0$ in the topology $\mathfrak{T}_0$. Since, by construction, $\tau \leq \mathfrak{T}_0$ we obtain $\tau \subsetneq \mathfrak{T}_0$ as desired. □

Analogously we prove that the normed space $\ell^1_{00} = (\mathbb{R}^{(\mathbb{N})}, \mathfrak{T}_{\ell_1})$ is not a Mackey group. To this end, let $\{e_n\}_{n \in \mathbb{N}}$ be the standard basis of the Banach space $(\ell_1, \|\cdot\|_1)$, and let $\{e_n^*\}_{n \in \mathbb{N}}$ be the canonical dual sequence in the dual Banach space $(\ell_1)' = \ell_\infty$, i.e.,

$$e_n = (0, \dots, 0, 1, 0, \dots) \quad \text{and} \quad e_n^* = (0, \dots, 0, 1, 0, \dots),$$

where 1 is placed in position n. Then ℓ^1_{00} is a dense subspace of ℓ_1 consisting of all vectors with finite support.

Theorem 2. *The normed space ℓ^1_{00} is not a Mackey group.*

Proof. For simplicity and clearness of notations we set $E := \ell_{00}^1$ and $\tau := \mathfrak{T}_{\ell_1}$. For every $n \in \mathbb{N}$, set $\chi_n := ne_n^*$. It is clear that $\chi_n \to 0$ in the weak* topology on E' and hence in $\sigma(\widehat{E}, E)$. Therefore we can define the linear injective operator $F : E \to E \times c_0$ and the monomorphism $p : E \to E \times \mathfrak{F}_0(\mathbb{S})$ setting (for all $x = (x_n) \in E$)

$$F(x) := \big(x, R(x)\big), \text{ where } R(x) := \big(\chi_n(x)\big) = (nx_n) \in c_0,$$
$$p(x) := \big(x, R_0(x)\big), \text{ where } R_0(x) := Q \circ R(x) = \big(\exp\{2\pi i \chi_n(x)\}\big) = \big(\exp\{2\pi i n x_n\}\big) \in \mathfrak{F}_0(\mathbb{S}).$$

Denote with $\mathfrak{T}$ and $\mathfrak{T}_0$ the topologies on E induced from $E \times c_0$ and $E \times \mathfrak{F}_0(\mathbb{S})$, respectively. So $\mathfrak{T}$ is a locally convex vector topology on E and $\mathfrak{T}_0$ is a locally quasi-convex group topology on E. By construction, $\tau \le \mathfrak{T}_0 \le \mathfrak{T}$, so taking into account Fact 1 and the Hahn–Banach extension theorem we obtain

$$\psi(E') = \psi(\ell_\infty) \subseteq \widehat{(E, \mathfrak{T}_0)} \subseteq \psi\big((E, \mathfrak{T})'\big) \subseteq \psi(\ell_\infty \times \ell_1). \tag{13}$$

Step 1: The topologies τ and $\mathfrak{T}_0$ are compatible. By (13), it is sufficient to show that each continuous character of $(E, \mathfrak{T}_0)$ belongs to $\psi(\ell_\infty)$. Fix $\chi \in \widehat{(E, \mathfrak{T}_0)}$. Then (1) implies that $\chi = \psi(\eta) = \exp\{2\pi i \eta\}$ for some

$$\eta = \big(\nu, (c_n)\big) \in \ell_\infty \times \ell_1, \ \text{ where } \nu \in \ell_\infty \ \text{ and } \ (c_n) \in \ell_1,$$

and

$$\eta(x) = \nu(x) + \sum_{n \in \mathbb{N}} c_n \chi_n(x) = \nu(x) + \sum_{n \in \mathbb{N}} c_n \cdot n x_n \quad \big(x = (x_n) \in E\big).$$

To prove that $\chi \in \psi(\ell_\infty)$ it is sufficient (and also necessary) to show that $(c_n n)_n \in \ell_\infty$. Replacing if needed η by $\eta - \nu$, we assume that $\nu = 0$.

Suppose for a contradiction that $(|c_n|n)_n$ is unbounded. Then there is a subsequence $\{n_k\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $|c_{n_k}|n_k \to \infty$ as $k \to \infty$. Since χ is continuous, Fact 1 shows that, for every $\varepsilon < 0.01$, there is a $\delta < \varepsilon$ such that

$$\eta(x) = \sum_{n \in \mathbb{N}} n c_n x_n \in (-\varepsilon, \varepsilon) + \mathbb{Z}, \ \text{ for every } \ x \in U_\delta, \tag{14}$$

where U_δ is a canonical $\mathfrak{T}_0$-neighborhood of zero

$$U_\delta := \big\{x = (x_n) \in E : \|x\|_1 \le \delta \ \text{ and } \ n x_n \in [-\delta, \delta] + \mathbb{Z} \ \text{ for every } \ n \in \mathbb{N}\big\}. \tag{15}$$

As $|c_{n_k}|n_k \to \infty$ and $c_n \to 0$, there is $k \in \mathbb{N}$ such that

$$\tfrac{1}{8|c_{n_k}|} > 1 \ \text{ and } \ \tfrac{3}{8|c_{n_k}|n_k} < \delta. \tag{16}$$

The first inequality in (16) implies that there is

$$m_k \in \left(\tfrac{1}{8|c_{n_k}|}, \tfrac{3}{8|c_{n_k}|}\right) \cap \mathbb{N}. \tag{17}$$

Set $x = (x_n) := (0, \dots, 0, \operatorname{sign}(c_{n_k})\frac{m_k}{n_k}, 0, \dots)$, where the nonzero element is placed in position n_k. Then $n x_n \in \mathbb{Z}$ for every $n \in \mathbb{N}$, and the second inequality of (16) and (17) imply

$$\|x\|_1 = |x_{n_k}| = \tfrac{m_k}{n_k} < \tfrac{3}{8|c_{n_k}|n_k} < \delta.$$

Therefore $x \in U_\delta$. On the other hand, (17) implies

$$\tfrac{1}{8} < \eta(x) = \sum_{n \in \mathbb{N}} c_n n x_n = |c_{n_k}|n_k \tfrac{m_k}{n_k} = |c_{n_k}|m_k < \tfrac{3}{8}.$$

Hence $\eta(x) \notin (-\varepsilon, \varepsilon) + \mathbb{Z}$ since $\varepsilon < 0.01$. However, this contradicts (14).

Step 2. The topology $\mathfrak{T}_0$ is strictly finer than the original topology τ. Thus E is not a Mackey group. Indeed, it is clear that $\frac{1}{2k}e_k \to 0$ in the norm topology τ on E. On the other hand, since

$$R_0\left(\tfrac{1}{2k}e_k\right) = \left(\exp\left\{2\pi i \cdot \chi_n\left(\tfrac{1}{2k}e_k\right)\right\}\right)_{n\in\mathbb{N}} = (1,\dots,1,-1,1,\dots,) \quad \text{for every} \quad k \in \mathbb{N},$$

where -1 is placed in position k, we obtain that $\frac{1}{2k}e_k \not\to 0$ in the topology $\mathfrak{T}_0$. Since, by construction, $\tau \le \mathfrak{T}_0$ we obtain $\tau \subsetneq \mathfrak{T}_0$ as desired. □

We finish this note with the following problem.

Problem 2. *Let E be a real normed (metrizable, bornological or quasibarrelled) locally convex space. Is it true that E is a Mackey group if and only if it is barrelled?*

Note that every barrelled lcs is a Mackey group, see [1].

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