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Inequalities on General L_p-Mixed Chord Integral Difference

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Abstract: In this article, we introduce the concept of general L_p-mixed chord integral difference of star bodies. Further, we establish the Brunn–Minkowski type, Aleksandrov–Fenchel type and cyclic inequalities for the L_p-mixed chord integral difference.

Keywords: general L_p-mixed chord integral difference; volume difference; L_p-radial Minkowski combination; L_p-radial Blaschke Minkowski homomorphism

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1. Introduction

The setting for this paper is n-dimensional Euclidean spaces $\mathbb{R}^n$ $(n \geq 1)$. Let K and L be two convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. V denotes the volume. If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, \infty)$, is defined by (see [1]):

$$\rho(K, u) = \max\{\lambda \geq 0, \lambda u \in K\}, \;\; u \in \mathcal{S}^{n-1}.$$

If ρ_K is positive and continuous, K is called a star body (about the origin), and S^n denotes the set of star bodies in $\mathbb{R}^n$. S_0^n is the subset of S^n containing the origin in their interiors. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathcal{S}^{n-1}$, and B denotes the standard unit ball in $\mathbb{R}^n$.

The classical Brunn–Minkowski inequality is (see [2])

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$

where $+$ denotes vector or the Minkowski sum of two sets, i.e., $A + B = \{a + b : a \in A, b \in B\}$.

In 2004, Leng (see [3]) presented a new generalization of the Brunn–Minkowski inequality for the volume difference of convex bodies.

Theorem 1. *Suppose that K, L and D are compact domains, and $D \subset K, D' \subset L$, D' is a homothetic copy of D. Then*

$$[V(K+L) - V(D+D')]^{\frac{1}{n}} \geq [V(K) - V(D)]^{\frac{1}{n}} + [V(L) - V(D')]^{\frac{1}{n}}.$$

The equality holds if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

Leng's result is a major extension of the classical Brunn–Minkowski inequality and attracts more and more attention (see [4–6]).

In 1977, Lutwak introduced the notion of a mixed width-integral of convex bodies (see [7]), and the dual notion, mixed chord-integrals of star bodies was defined by Lu (see [8]). Later, as a part of the asymmetric L_p Brunn–Minkowski theory, which has its origins in the work of Ludwig, Haberl and Schuster (see [9–13]), Feng and Wang generalized the mixed chord-integrals to general mixed chord-integrals of star bodies (see [14]). For $K_1, \cdots, K_n \in S_0^n$ and $\tau \in (-1, 1)$, the general mixed chord-integral $C^{(\tau)}(K_1, \cdots, K_n)$ is defined by

$$C^{(\tau)}(K_1, \cdots, K_n) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c^{(\tau)}(K_1, u) \cdots c^{(\tau)}(K_n, u) du,$$

here, $c^{(\tau)}(K, \cdot) = f_1(\tau)\rho(K, \cdot) + f_2(\tau)\rho(-K, \cdot)$, and the functions $f_1(\tau)$ and $f_2(\tau)$ are defined as follows

$$f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \quad f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)}.$$

In 2016, Li and Wang extended the general mixed chord-integral to the general L_p-mixed chord integral of star bodies (see [15]): For $K_1, \cdots, K_n \in S_0^n$, $p > 0$ and $\tau \in (-1, 1)$, the general L_p-mixed chord integral $C_p^{(\tau)}(K_1, \cdots, K_n)$ of $K_1, \cdots, K_n$ is defined by

$$C_p^{(\tau)}(K_1, \cdots, K_n) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K_1, u) \cdots c_p^{(\tau)}(K_n, u) du. \tag{1a}$$

Here, $c_p^{(\tau)}(K, \cdot)$ is defined by

$$c_p^{(\tau)}(K, u) = \left(f_1(\tau)\rho^p(K, u) + f_2(\tau)\rho^p(-K, u)\right)^{\frac{1}{p}},$$

for any $u \in \mathcal{S}^{n-1}$, and $f_1(\tau)$ and $f_2(\tau)$ are chosen as (see [16])

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \quad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}.$$

Obviously, $f_1(\tau)$ and $f_2(\tau)$ satisfy

$$f_1(\tau) + f_2(\tau) = 1,$$

$$f_1(-\tau) = f_2(\tau), \;\; f_2(-\tau) = f_1(\tau).$$

$C_{p,i}^{(\tau)}(K, L)$ denotes that K appears $n - i$ times, and L appears i times, which is

$$C_{p,i}^{(\tau)}(K, L) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K, u)^{n-i} c_p^{(\tau)}(L, u)^i du.$$

If constants $\lambda_1, \cdots, \lambda_n > 0$ exist such that $\lambda_1 c_p^{(\tau)}(K_1, u) = \cdots = \lambda_n c_p^{(\tau)}(K_n, u)$ for all $u \in \mathcal{S}^{n-1}$, star bodies $K_1, \cdots, K_n$ are said to have a similar general L_p-chord. For this general L_p-chord integral, Li and Wang gave the following inequalities (see [15]).

Theorem 2. *If $K, L \in S_o^n$ and $\tau \in (-1, 1)$, $p > 0$, then for $i \le n - p$,*

$$C_{p,i}^{(\tau)}(K \tilde{+}_p L)^{\frac{p}{n-i}} \le C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + C_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}, \tag{1b}$$

for $n - p < i < n$ or $i > n$,

$$C_{p,i}^{(\tau)}(K \tilde{+}_p L)^{\frac{p}{n-i}} \ge C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + C_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}, \tag{1c}$$

with equality in each inequality if and only if K and L have a similar general L_p-chord. Here and in the following Theorems, $K \tilde{+}_p L$ denotes the L_p-radial Minkowski combination of K and L.

Theorem 3. *If $K_1, \cdots, K_n \in S_o^n$ and $\tau \in (-1,1)$, $\;p > 0$, then for $1 < m \leq n$,*

$$C_p^{(\tau)}(K_1, \cdots, K_n)^m \leq \prod_{i=1}^{m} C_p^{(\tau)}(K_1, \cdots, K_{n-m}, K_{n-i+1}, K_{n-i+1}, \cdots, K_{n-i+1}), \tag{1d}$$

with equality if and only if $K_{n-m+1}, \cdots, K_n$ all have a similar general L_p-chord.

Theorem 4. *If $K, L \in S_o^n$ and $\tau \in (-1,1)$, $\;p > 0$, then for $i < j < k$,*

$$C_{p,j}^{(\tau)}(K,L)^{k-i} \leq C_{p,i}^{(\tau)}(K,L)^{k-j} C_{p,k}^{(\tau)}(K,L)^{j-i}, \tag{1e}$$

with equality if and only if K and L have a similar general L_p-chord.

2. Main Results

Inspired by Leng's idea, this article deals with the general L_p-chord integral of star bodies and gives some inequalities for the general L_p-chord integral difference.

Theorem 5. *Let $K, L, M, M' \in S_o^n$ and $\tau \in (-1,1)$, $\;p > 0$. If K and L have similar general L_p-chord and $M \subseteq K$, $M' \subseteq L$, then for $i \leq n - p$,*

$$[C_{p,i}^{(\tau)}(K \tilde{+}_p L) - C_{p,i}^{(\tau)}(M \tilde{+}_p M')]^{\frac{p}{n-i}} \geq [C_{p,i}^{(\tau)}(K) - C_{p,i}^{(\tau)}(M)]^{\frac{p}{n-i}} + [C_{p,i}^{(\tau)}(L) - C_{p,i}^{(\tau)}(M')]^{\frac{p}{n-i}}, \tag{1f}$$

and for $n - p < i < n$ or $i > n$,

$$[C_{p,i}^{(\tau)}(K \tilde{+}_p L) - C_{p,i}^{(\tau)}(M \tilde{+}_p M')]^{\frac{p}{n-i}} \leq [C_{p,i}^{(\tau)}(K) - C_{p,i}^{(\tau)}(M)]^{\frac{p}{n-i}} + [C_{p,i}^{(\tau)}(L) - C_{p,i}^{(\tau)}(M')]^{\frac{p}{n-i}}, \tag{1g}$$

with equality in each inequality if and only if M and M' have a similar general L_p-chord.

Theorem 6. *Let $K_1, \cdots, K_n$ and $M_1, \cdots, M_n \in S_o^n$, and $\tau \in (-1,1)$, $\;p > 0$. If $M_i \subseteq K_i$, $i = 1, 2, \cdots, n$, $K_1, \cdots K_n$ have similar general L_p-chord, then for $1 < m \leq n$,*

$$[C_p^{(\tau)}(K_1, \cdots, K_n) - C_p^{(\tau)}(M_1, \cdots, M_n)]^m \geq$$
$$\prod_{i=1}^{m} [C_p^{(\tau)}(K_1, \cdots, K_{n-m}, K_{n-i+1}, K_{n-i+1}, \cdots, K_{n-i+1}) - C_p^{(\tau)}(M_1, \cdots, M_{n-m}, K_{n-i+1}, M_{n-i+1}, \cdots, M_{n-i+1})], \tag{1h}$$

with equality if and only if $M_1, \cdots, M_n$ all have a similar general L_p-chord.

Theorem 7. *Let $K, L, M, M' \in S_o^n$ and $\tau \in (-1,1)$, $\;p > 0$. If K and L have similar general L_p-chord, then for $i < j < k$,*

$$[C_{p,j}^{(\tau)}(K,L) - C_{p,j}^{(\tau)}(M,M')]^{k-i} \geq [C_{p,i}^{(\tau)}(K,L) - C_{p,i}^{(\tau)}(M,M')]^{k-j} [C_{p,k}^{(\tau)}(K,L) - C_{p,k}^{(\tau)}(M,M')]^{j-i}, \tag{1i}$$

with equality if and only if K and L have a similar general L_p-chord.

3. Preliminaries

For $K, L \in S^n$, the radial Blaschke linear combination $K \check{+} L$ and the radial Minkowski linear combination are defined by Lutwak (see [17]), respectively:

$$\rho(K \check{+} L, u)^{n-1} = \rho(K,u)^{n-1} + \rho(L,u)^{n-1}, \tag{2a}$$

and

$$\rho(K \tilde{+} L, u) = \rho(K,u) + \rho(L,u). \tag{2b}$$

In 2007, Schuster introduced the notion of radial Blaschke–Minkowski homomorphism (see [18–22]) as follows.

Definition 1. *A map $\Psi : S^n \to S^n$ is called a radial Blaschke–Minkowski homomorphism if it satisfies the following conditions:*

(1) *Ψ is coninuous;*
(2) *Ψ is radial Blaschke Minkowski additive, i.e., $\Psi(K\check{+}L) = \Psi K\tilde{+}\Psi L$ for all $K, L \in S^n$;*
(3) *Ψ intertwines rotations, i.e., $\Psi(\phi K) = \phi\Psi K$, for all $\phi \in SO(n)$ and $K \in S^n$.*

Here, $\Psi K\tilde{+}\Psi L$ denotes the radial sum of ΨK and ΨL, and $K\check{+}L$ is the radial Blaschke sum of the star bodies K and L.

In 2011, Wang et al. (see [23]) extended the notion of radial Blaschke–Minkowski homomorphism to L_p-radial Minkowski homomorphism as follows.

Definition 2. *A map $\Psi_p : S^n \to S^n$ is called an L_p-radial Minkowski homomorphism if it satisfies the following conditions:*

(1) *Ψ_p is coninuous;*
(2) *Ψ_p is radial Minkowski additive, i.e., $\Psi_p(K\tilde{+}_{n-p}L) = \Psi_p K\tilde{+}_p\Psi_p L$ for all $K, L \in S^n$;*
(3) *Ψ_p intertwines rotations, i.e., $\Psi_p(\phi K) = \phi\Psi_p K$, for all $\phi_p \in SO(n)$ and $K \in S^n$.*

Here, $\Psi_p K\tilde{+}_{n-p}\Psi_p L$ denotes the L_{n-p} radial sum of $\Psi_p K$ and $\Psi_p L$, i.e., (see [9,24])

$$\rho(\Psi_p K\tilde{+}_{n-p}\Psi_p L, u)^{n-p} = \rho(\Psi_p K, u)^{n-p} + \rho(\Psi_p L, u)^{n-p}. \tag{2c}$$

For $0 < p < n$, the L_p-radial Blaschke linear combination $K\check{+}_p L$ was defined by Wang (see [25]):

$$\rho(K\check{+}_p L, u)^{n-p} = \rho(K, u)^{n-p} + \rho(L, u)^{n-p}. \tag{2d}$$

From Equations (2c) and (2d), we easily obtain

$$K\tilde{+}_{n-p}L = K\check{+}_p L. \tag{2e}$$

Here, we recall a special L_p-radial Minkowski homomorphism. In 2007, Yu, Wu and Leng (see [26]) introduced the quasi-L_p intersection body $I_p K$ of a star body. Let K be a star body in $\mathbb{R}^n$, then the quasi-L_p intersection body $I_p K$ of K is defined by:

$$\rho(I_p K, u)^p = \int_{S^{n-1}\bigcap u^{\perp}} \rho(K, u)^{n-p} du.$$

Further, Wang (see [23]) proved that the operator $I_p : S^n \to S^n$ has the following properties: (1) I_p is continuous with respect to radial metric; (2) $I_p(K\tilde{+}_{n-p}L) = I_p K\tilde{+}_p I_p L$ for all $K, L \in S^n$; (3) I_p intertwines rotations, i.e., $\Psi_p(\phi K) = \phi\Psi_p K$, for all $\phi_p \in SO(n)$ and $K \in S^n$, which means that the operator I_p is a special L_p-radial Minkowski homomorphism.

Now, we list three Lemmas useful in the proof of Theorems 5–7.

In 1997, Losonczi and Páles (see [27]) extended Bellman's inequality as follows:

Lemma 1. *Let $a = \{a_1, a_2, \cdots, a_n\}$ and $b = \{b_1, b_2, \cdots, b_n\}$ $(n \geq 1)$ be two sequences of positive real numbers and $p > 1$ such that $a_1^p - \Sigma_{i=2}^n a_i^p > 0$ and $b_1^p - \Sigma_{i=2}^n b_i^p > 0$. Then*

$$\left(a_1^p - \Sigma_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \Sigma_{i=2}^n b_i^p\right)^{\frac{1}{p}} \leq \left((a_1 + b_1)^p - \Sigma_{i=2}^n (a_i + b_i)^p\right)^{\frac{1}{p}}, \tag{2f}$$

If $p < 0$ or $0 < p < 1$, then

$$((a_1^p - \Sigma_{i=2}^n a_i^p)^{\frac{1}{p}} + (b_1^p - \Sigma_{i=2}^n b_i^p)^{\frac{1}{p}}))^p \geq (a_1 + b_1)^p - \Sigma_{i=2}^n (a_i + b_i)^p,$$

with equality if and only if $a = vb$, where v is a constant.

Lemma 2 ([28], p.26). *If $x_i > 0, y_i > 0, i = 1, 2, \cdots, n$, then*

$$\Big(\prod_{i=1}^{n}(x_i + y_i)\Big)^{\frac{1}{n}} \geq (\prod_{i=1}^{n} x_i)^{\frac{1}{n}} + (\prod_{i=1}^{n} y_i)^{\frac{1}{n}}, \tag{2g}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$.

Lemma 3 ([5]). *Suppose that f_i, g_i $(i = 1, 2)$ are non-negative continuous functions on $\mathcal{S}^{n-1}$ such that*

$$\int_{\mathcal{S}^{n-1}} f_1^s(u)du \geq \int_{\mathcal{S}^{n-1}} f_2^s(u)du,$$

$$\int_{\mathcal{S}^{n-1}} g_1^t(u)du \geq \int_{\mathcal{S}^{n-1}} g_2^t(u)du,$$

for $s > 1, \frac{1}{s} + \frac{1}{t} = 1$, and

$$f_1^s(u) = \lambda g_1^t(u), \forall u \in \mathcal{S}^{n-1},$$

where λ is a constant. Then

$$\Big(\int_{\mathcal{S}^{n-1}} (f_1^s - f_2^s)du\Big)^{\frac{1}{s}} \Big(\int_{\mathcal{S}^{n-1}} (g_1^t - g_2^t)du\Big)^{\frac{1}{t}} \leq \int_{\mathcal{S}^{n-1}} (f_1 g_1 - f_2 g_2)du, \tag{2h}$$

with equality if and only if $f_2^s(u) = \lambda g_2^t(u)$ for any $u \in \mathcal{S}^{n-1}$.

4. Proofs of Main Results

In this section, we prove Theorems 5–7.

Proof of Theorem 5. We only prove Equation (1f). The proof of Equation (1g) is similar to Equation (1f). Let $i \leq n - p$. Since K and L have similar general L_p-chord, by Equation (1b),

$$C_{p,i}^{(\tau)}(K\tilde{+}_p L)^{\frac{p}{n-i}} = C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + C_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}, \tag{3a}$$

for M and M',

$$C_{p,i}^{(\tau)}(M\tilde{+}_p M')^{\frac{p}{n-i}} \leq C_{p,i}^{(\tau)}(M)^{\frac{p}{n-i}} + C_{p,i}^{(\tau)}(M')^{\frac{p}{n-i}}. \tag{3b}$$

Let $a_1 = C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}, a_2 = C_{p,i}^{(\tau)}(M)^{\frac{p}{n-i}}$ and $b_1 = C_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}, b_2 = C_{p,i}^{(\tau)}(M')^{\frac{p}{n-i}}$, then from Equations (3a) and (3b) and Lemma 1, we have

$$\begin{aligned}
&\Big(C_{p,i}^{(\tau)}(K\tilde{+}_p L) - C_{p,i}^{(\tau)}(M\tilde{+}_p M')\Big)^{\frac{p}{n-i}} \\
&\geq \Big((C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + C_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}})^{\frac{n-i}{p}} + (C_{p,i}^{(\tau)}(M)^{\frac{p}{n-i}} + C_{p,i}^{(\tau)}(M')^{\frac{p}{n-i}})^{\frac{n-i}{p}}\Big)^{\frac{p}{n-i}} \\
&\geq \Big(C_{p,i}^{(\tau)}(K) - C_{p,i}^{(\tau)}(M)\Big)^{\frac{p}{n-i}} + \Big(C_{p,i}^{(\tau)}(L) - C_{p,i}^{(\tau)}(M')\Big)^{\frac{p}{n-i}}
\end{aligned}$$

This gives the desired inequality of Equation (1f) and according to the equality condition of Lemma 1, we obtain that equality holds if and only if M and M' have a similar general L_p-chord. □

Notice that from the notion of L_p-radial Minkowski homomorphism and Equation (2e), we have the following direct Corollary 1.

Corollary 1. *Let $K, L, M, M' \in S_o^n$ and $\tau \in (-1, 1)$, $p > 0$. Ψ_p is a radial Blaschke–Minkowski homomorphism. K and L have a similar general L_p-chord and $M \subseteq K$, $M' \subseteq L$, then for $i \leq n - p$,*

$$[C_{p,i}^{(\tau)}(\Psi_p(K\tilde{+}_p L)) - C_{p,i}^{(\tau)}(\Psi_p(M\check{+}_p M'))]^{\frac{p}{n-i}} \geq [C_{p,i}^{(\tau)}(\Psi_p K) - C_{p,i}^{(\tau)}(\Psi_p M)]^{\frac{p}{n-i}} + [C_{p,i}^{(\tau)}(\Psi_p L) - C_{p,i}^{(\tau)}(\Psi_p M')]^{\frac{p}{n-i}},$$

and for $n-p<i<n$ or $i>n$,

$$[C_{p,i}^{(\tau)}(\Psi_p(K\check{+}_pL))-C_{p,i}^{(\tau)}(\Psi_p(M\check{+}_pM'))]^{\frac{p}{n-i}}\leq[C_{p,i}^{(\tau)}(\Psi_pK)-C_{p,i}^{(\tau)}(\Psi_pM)]^{\frac{p}{n-i}}+[C_{p,i}^{(\tau)}(\Psi_pL)-C_{p,i}^{(\tau)}(\Psi_pM')]^{\frac{p}{n-i}},$$

with equality in each inequality if and only if M and M' have a similar general L_p-chord.

Further, since the L_p intersection map is a special L_p-radial Minkowski homomorphism, we have the following corollary

Corollary 2. *Let $K,L,M,M'\in S_o^n$ and $\tau\in(-1,1),\ \ p>0$. If K and L have a similar general L_p-chord and $M\subseteq K$, $M'\subseteq L$, then for $i\leq n-p$,*

$$[C_{p,i}^{(\tau)}(I_p(K\check{+}_pL))-C_{p,i}^{(\tau)}(I_p(M\check{+}_pM'))]^{\frac{p}{n-i}}\geq[C_{p,i}^{(\tau)}(I_pK)-C_{p,i}^{(\tau)}(I_pM)]^{\frac{p}{n-i}}+\mu[C_{p,i}^{(\tau)}(I_pL)-C_{p,i}^{(\tau)}(I_pM')]^{\frac{p}{n-i}},$$

and for $n-p<i<n$ or $i>n$,

$$[C_{p,i}^{(\tau)}(I_p(K\check{+}_pL))-C_{p,i}^{(\tau)}(I_p(M\check{+}_pM'))]^{\frac{p}{n-i}}\leq[C_{p,i}^{(\tau)}(I_pK)-C_{p,i}^{(\tau)}(I_pM)]^{\frac{p}{n-i}}+[C_{p,i}^{(\tau)}(I_pL)-C_{p,i}^{(\tau)}(I_pM')]^{\frac{p}{n-i}},$$

with equality in each inequality if and only if M and M' have a similar general L_p-chord.

Proof of Theorem 6. Since $K_1,\cdots,K_n$ have a similar general L_p-chord, from (1d) we have for $1<m\leq n$,

$$C_p^{(\tau)}(K_1,\cdots,K_n)^m=\prod_{i=1}^{m}C_p^{(\tau)}(K_1,\cdots,K_{n-m},K_{n-i+1},K_{n-i+1},\cdots,K_{n-i+1}). \tag{3c}$$

For $M_1,\cdots,M_n$,

$$C_p^{(\tau)}(M_1,\cdots,M_n)^m\leq\prod_{i=1}^{m}C_p^{(\tau)}(M_1,\cdots,M_{n-m},M_{n-i+1},M_{n-i+1},\cdots,M_{n-i+1}). \tag{3d}$$

The condition $M_i\subseteq K_i, i=1,2,\cdots,n$ means that $C_p^{(\tau)}(K_1,\cdots,K_n)^m\geq C_p^{(\tau)}(M_1,\cdots,M_n)^m$. From Equations (3c) and (3d) and Lemma 2, we obtain

$$\begin{aligned}&C_p^{(\tau)}(K_1,\cdots,K_n)-C_p^{(\tau)}(M_1,\cdots,M_n)\\&\geq\Big(\prod_{i=1}^{m}C_p^{(\tau)}(K_1,\cdots,K_{n-m},K_{n-i+1},K_{n-i+1},\cdots,K_{n-i+1})\Big)^{\frac{1}{m}}\\&\quad-\Big(\prod_{i=1}^{m}C_p^{(\tau)}(M_1,\cdots,M_{n-m},M_{n-i+1},M_{n-i+1},\cdots,M_{n-i+1})\Big)^{\frac{1}{m}}.\end{aligned}$$

Let $x_i+y_i=C_p^{(\tau)}(K_1,\cdots,K_{n-m},K_{n-i+1},K_{n-i+1},\cdots,K_{n-i+1})$ and $y_i=C_p^{(\tau)}(M_1,\cdots,M_{n-m},M_{n-i+1},M_{n-i+1},\cdots,M_{n-i+1})$ in Lemma 2. Then by Equation (2g)

$$\begin{aligned}&C_p^{(\tau)}(K_1,\cdots,K_n)-C_p^{(\tau)}(M_1,\cdots,M_n)\\&\geq\Big(\prod_{i=1}^{m}[C_p^{(\tau)}(K_1,\cdots,K_{n-m},K_{n-i+1},\cdots,K_{n-i+1})-C_p^{(\tau)}(M_1,\cdots,M_{n-m},M_{n-i+1},\cdots,M_{n-i+1})]\Big)^{\frac{1}{m}},\end{aligned}$$

which implies that Equation (1h) is proved. According to the equality condition of Lemma 2, we know that equality holds in Equation (1h) if and only if $M_1,\cdots M_n$ all have a similar general L_p-chord. □

Proof of Theorem 7. For $i<j<k$, let $s=\frac{k-i}{k-j},t=\frac{k-i}{j-i}$. Then, $s>1$ and $\frac{1}{s}+\frac{1}{t}=1$. Let

$$f_1^s=c_p^{(\tau)}(K,u)^{n-i}c_p^{(\tau)}(L,u)^i,\ \ f_2^s=c_p^{(\tau)}(M,u)^{n-i}c_p^{(\tau)}(M',u)^i$$

and
$$g_1^t = c_p^{(\tau)}(K,u)^{n-k} c_p^{(\tau)}(L,u)^k, \;\; g_2^t = c_p^{(\tau)}(M,u)^{n-k} c_p^{(\tau)}(M',u)^k.$$
After a simple calculation, we obtain
$$\begin{aligned}\int_{S^{n-1}} (f_1 g_1 - f_2 g_2) du &= \int_{S^{n-1}} (c_p^{(\tau)}(K,u)^{n-j} c_p^{(\tau)}(L,u)^j - c_p^{(\tau)}(M,u)^{n-j} c_p^{(\tau)}(M',u)^j) du \\ &= C_{p,j}^{(\tau)}(K,L) - C_{p,j}^{(\tau)}(M,M').\end{aligned}$$

The left-hand side of Equation (2h) leads to $[C_{p,i}^{(\tau)}(K,L) - C_{p,i}^{(\tau)}(M,M')]^{\frac{1}{s}} [C_{p,k}^{(\tau)}(K,L) - C_{p,k}^{(\tau)}(M,M')]^{\frac{1}{t}}$.

By Lemma 3, Equation (1i) immediately holds.

The equality condition of Equation (2h) means that $\frac{f_1^s}{g_1^t} = (\frac{c_p^{(\tau)}(K,u)}{c_p^{(\tau)}(L,u)})^{k-i}$ is a constant, that is, K and L have a similar general L_p-chord. This completes the proof. □

5. Conclusions

The asymmetric operators belong to a new and rapidly evolving asymmetric L_p-Brunn–Minkowski theory that has its origins in the work of Ludwig, Haberl and Schuster (see [9,11,12,16,18–20]). The general L_p-mixed chord integral difference of star bodies was motivated by the notion of mixed width-integrals of convex bodies. We hope that besides the inequalities mentioned in this article, we can deduce some other inequalities in the future.

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