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Strictly Convex Banach Algebras

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Abstract: We discuss two facets of the interaction between geometry and algebra in Banach algebras. In the class of unital Banach algebras, there is essentially one known example which is also strictly convex as a Banach space. We recall this example, which is finite-dimensional, and consider the open question of generalising it to infinite dimensions. In C^*-algebras, we exhibit one striking example of the tighter relationship that exists between algebra and geometry there.

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Banach algebras are a class of objects with three structures: a vector space structure, a metrizable topology defined by a norm, and a multiplication. The associated scalar field is almost always the complex numbers. Well-known algebraic axioms ensure sensible interplay between the multiplication and the vector space structure. The Banach space axioms regulate the interplay between the norm and the vector space structure; in particular, the norm is positively homogeneous, and the metric it gives rise to is complete. The relationship between the norm and the multiplication is given by submultiplicativity, i.e., by assuming that $\|xy\| \leq \|x\|\|y\|$ for all x, y. In most cases, this is a strict inequality, as the next paragraph shows.

Helpfully, submultiplicativity does ensure that multiplication is jointly continuous. However, by itself, this is not a particularly strong assumption. Indeed, any Banach space can be made into a Banach algebra simply by defining all products to be zero. Thus, being a Banach algebra does not impose any geometric restrictions on the underlying Banach space. In such an example, the equality $\|xy\| = \|x\|\|y\|$ holds if, and only if, either x or y is 0.

A Banach algebra is called unital if it has an identity element for multiplication. In this case, it is conventional to assume that the identity element has norm one. For unital Banach algebras, the situation is more interesting. Clearly the submultiplicativity relation becomes an equality in a case where one of the elements is the identity, but in general, the inequality is strict. To see this, suppose that $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$. Then, the assumptions $\|y_n\| = 1$ and $xy_n \to 0$ imply that $x = 0$; i.e., A has no topological divisors of zero. Thanks to the Gelfand–Mazur–Kaplansky theorem [1], this tells us that A is 1-dimensional, i.e., isomorphic to the complex field.

The *unit sphere* of a Banach space is the collection of all elements with norm one. An element x in the unit sphere of a Banach space X is said to be a *smooth* point if (and only if) there is only one norm one functional $f \in X^*$ with $f(x) = 1$, i.e., the unit ball has a unique supporting hyperplane at x. Then X itself is said to be smooth if (and only if) X is smooth at each point x in the unit sphere. These definitions were introduced by Klee [2], although the concept was investigated earlier.

At the other extreme, a point x in the unit sphere is called a *vertex* of X if it has so many hyperplanes of support that the intersection of all of them consists of the point x alone. Standard duality arguments show that this is equivalent to the linear span of the set of support functionals of x being dense in the weak* topology in X^*. This concept was introduced by Bohnenblust and Karlin [3], who were the first to investigate the geometry of the unit ball in the more specific context of Banach *algebras*.

Theorem 1. *([3], Theorem 2). The unit element of a Banach algebra is a vertex of the unit sphere.*

In particular, the unit element of a Banach algebra is never a smooth point, and a unital Banach algebra is never a smooth Banach space. A stronger result was established independently in [4,5], namely that just the algebraic linear span of the set of support functionals of the unit of a Banach algebra A is in fact equal to A^*. This theorem can also be generalised to a class of Banach spaces called numerical range spaces, which include unital Banach algebras; see [6] and the references therein for some recent developments.

An *extreme point* of the unit ball of a Banach space X is any point which is **not** the midpoint of any line segment I which lies entirely in the unit ball of X. Clearly, extreme points can only be found on the unit sphere. If a point x in the unit sphere is the midpoint of such a line segment I, then I must lie entirely in the unit sphere, and any supporting hyperplane for the unit ball at x must contain all of I. It follows that a vertex of the unit ball cannot be the midpoint of such a line segment, i.e., every vertex is an extreme point.

One says that X is *strictly convex* if every point on the unit sphere is an extreme point of the unit ball [7] (p. 397). This is equivalent to the requirement that the unit sphere contains no proper line segment.

Theorem 2. *[2] ((A1.1), p. 37) There is a natural duality between smoothness and strict convexity.*

(i) *If a norm one point x is not an extreme point of the unit ball of X, and $f \in X^*$ is any support functional for x, then X^* is not smooth at f. Consequently, if X^* is smooth, then X is strictly convex.*

(ii) *If a norm one point x has two distinct support functionals f and g, then $\frac{1}{2}(f+g)$ has norm one, but is not an extreme point of the unit ball of X^*. Consequently, if X^* is strictly convex, then X is smooth.*

Proof. (i) The hypothesis means that x is the midpoint of some interval $[y, z]$ contained in the unit sphere of X. Then y and z, considered as elements of the bidual X^{**}, will be distinct support functionals for f. Obviously, this implies the second assertion.

(ii) Clearly, $\frac{1}{2}(f+g)$ is also a support functional for x, and so lies in the unit sphere. Again, the second assertion follows easily. □

This result shows that smoothness and strict convexity are in perfect duality in reflexive Banach spaces. However, non-reflexive examples show that none of the implications can be reversed, in general.

The following reformulation of strict convexity is sometimes useful.

Proposition 1. *[8] A Banach space X is strictly convex if, and only if, $\|x+y\| < \|x\| + \|y\|$ whenever x and y are linearly independent.*

Having seen that a unital Banach algebra cannot be smooth, it is natural to ask about strict convexity. None of the common examples of Banach algebras have this combination of properties, although a nondegenerate nonunital example is a Hilbert sequence space equipped with pointwise multiplication. The following example [9] (Example 2) shows that a unital Banach algebra can be strictly convex. Since this is our main concern now, we repeat the proof of submultiplicativity given there.

Example 1. *There are strictly convex unital Banach algebras.*

Proof. For a first example, take $A = \mathbb{C}^2$, with the usual pointwise operations, but the norm

$$\|(x,y)\| = \sqrt{\frac{1}{2}(|x|^2+|y|^2)} + |x-y|.$$

Proposition 1 makes it routine to verify that A is a strictly convex Banach space; clearly, $\|(1,1)\| = 1$. In order to establish that $\|(ac,bd)\| \le \|(a,b)\|\|(c,d)\|$, we begin by proving two more straightforward inequalities. First,

$$\begin{aligned} |ac-bd| &= \left|\frac{1}{2}(a+b)(c-d) + \frac{1}{2}(c+d)(a-b)\right| \\ &\le \frac{1}{2}(|a|+|b|)|c-d| + \frac{1}{2}(|c|+|d|)|a-b| \\ &\le \sqrt{\frac{1}{2}(|a|^2+|b|^2)}|c-d| + \sqrt{\frac{1}{2}(|c|^2+|d|^2)}|a-b|, \end{aligned}$$

where the last step used the Cauchy–Schwarz inequality.

Second, beginning with a difference of squares and dividing by one factor, we have

$$\begin{aligned} \sqrt{\frac{1}{2}(|ac|^2+|bd|^2)} - \sqrt{\frac{1}{2}(|a|^2+|b|^2)}\sqrt{\frac{1}{2}(|c|^2+|d|^2)} &= \frac{\frac{1}{4}(|a|^2-|b|^2)(|c|^2-|d|^2)}{\sqrt{\frac{1}{2}(|ac|^2+|bd|^2)} + \sqrt{\frac{1}{2}(|a|^2+|b|^2)}\sqrt{\frac{1}{2}(|c|^2+|d|^2)}} \\ &= \frac{\frac{1}{4}(|a|-|b|)(|c|-|d|)(|a|+|b|)(|c|+|d|)}{\sqrt{\frac{1}{2}(|ac|^2+|bd|^2)} + \sqrt{\frac{1}{2}(|a|^2+|b|^2)}\sqrt{\frac{1}{2}(|c|^2+|d|^2)}} \\ &\le |a-b||c-d|\frac{\frac{1}{2}(|a|+|b|)}{\sqrt{\frac{1}{2}(|a|^2+|b|^2)}}\frac{\frac{1}{2}(|c|+|d|)}{\sqrt{\frac{1}{2}(|c|^2+|d|^2)}} \\ &\le |a-b||c-d|, \end{aligned}$$

where the first inequality used the obvious estimate $\sqrt{\frac{1}{2}(|ac|^2+|bd|^2)} \ge 0$, and the second used the modified Cauchy–Schwarz inequality $(\alpha+\beta)^2 \le 2(\alpha^2+\beta^2)$.

Adding these two estimates now gives us the submultiplicativity of the norm. □

A slight variant of this norm, namely $\sqrt{\frac{1}{2}(|x|^2+|y|^2)} + \frac{1}{2}|x-y|$, is also strictly convex and submultiplicative, although the proof is more delicate in this case. We mention it here because the latter norm has found practical applications. Restricted to real variables, it is the dual of the stadium norm on $\mathbb{R}^2$ [10] (Proposition 4.2). The stadium norm arises naturally as a measure of changes to the environment in road building [10] (Sections 3.1 and 6.3); thus, an algorithm to minimize the amount of earth work required will involve calculations of the dual norm.

These norms have several natural extensions to infinite dimensions. The easiest to describe is the algebra $C[0,1]$ of continuous functions on the unit interval, with its usual pointwise multiplication, but with the norm defined by

$$\|f\| = \left(\int_0^1 |f|^2\right)^{\frac{1}{2}} + \mathrm{diam} f([0,1]).$$

Here, the diameter of a bounded set S in a metric space is denoted as usual by $\mathrm{diam} S = \sup\{d(x,y) : x,y \in S\}$. It is routine to check that this norm is strictly convex, equivalent to the standard supremum norm, and the unit element has norm one. However, we are not certain whether it is a Banach algebra norm.

Problem 1. *Is the norm just defined on* $C[0,1]$ *also submultiplicative?*

We are not even sure about the corresponding finite-dimensional norms. A routine approximation argument shows that a positive solution to Problem 2 *for all n* would imply a positive solution to Problem 1.

Problem 2. *Is the corresponding norm on* $\mathbb{C}^n$

$$\|(x_1,\ldots,x_n)\| = \sqrt{\frac{|x_1|^2+\ldots+|x_n|^2}{n}} + \max_{1\leq i,j\leq n}|x_i-x_j|$$

also submultiplicative?

It is not hard to write a small program that plugs random numbers into these formulas to test their validity. We have performed this thousands of times for $n=5$, and not found any counterexample yet. On the other hand, a rigorous proof remains elusive.

A positive solution to these problems would give easily described higher-dimensional examples of strictly convex Banach algebras. Some such examples, albeit more complicated, were produced in [9]. Let us briefly recall the construction: first, inductively define an increasing sequence $A = A_0 \subset A_1 \subset A_2 \ldots$ of Banach algebras by setting $A_{n+1} = A_n \oplus A_n$, equipped with pointwise multiplication and the norm $\|(x,y)\| = \sqrt{\frac{1}{2}(\|x\|^2+\|y\|^2)} + \|x-y\|$. Practically the same calculations show that each A_n is then a strictly convex, unital Banach algebra. Furthermore, the map $x \mapsto (x,x)$ is a natural embedding of A_n into A_{n+1}, and the resulting inductive limit $\varinjlim A_n$ is clearly an infinite-dimensional, commutative, unital Banach algebra. We do not know whether it is strictly convex, although it obviously has a strictly convex dense subalgebra.

Now, we move to the other extreme and consider a very specific class of Banach algebras. A C^*-algebra is a Banach algebra together with an involution satisfying the identity $\|x^*x\| = \|x\|^2$. This condition implies that the involution is isometric. Obvious natural examples are the algebras of bounded linear operators on Hilbert spaces, and of course their closed self-adjoint subalgebras. The celebrated theorem of Gelfand and Naimark shows that there are no others. This is non-trivial. It is not difficult to show that if J is a closed two-sided ideal in a C^*-algebra A, then A/J is also a C^*-algebra; but, even when A has a representation on a Hilbert space, there is no obvious or natural such representation for A/J. We refer to Doran and Wichmann [11] for a lucid account of basic C^*-algebra theory and the Gelfand–Naimark theorem.

C^*-algebras were first considered as a way to model algebras of physical observables in quantum mechanics. They have since become a large field of independent interest. Our purpose here is to highlight the strong relationship between geometry and algebra due to the rigid structure of C^*-algebras. To this end, let us introduce another geometric property: for a fixed natural number n, a subspace J is said to have the n-ball property in a Banach space X if, whenever $B_1,\ldots,B_n$ are open balls in X, with $\bigcap_{i=1}^n B_i$ non-empty and $J\cap B_i$ non-empty for each i, then we also have $J \cap \bigcap_{i=1}^n B_i$ non-empty. This property is **not** a property of either J or X; rather, it describes the way J sits inside X.

This concept, and the following result, are due to Alfsen and Effros [12]; see also [13] and [14] for other proofs.

Theorem 3. *If A is a C^*-algebra, and J is a closed subspace of A, then the following are equivalent.*

(i) *J is an ideal in A;*
(ii) *J has the n-ball property in A, for every n;*
(iii) *J has the 3-ball property in A;*
(iv) *the polar subspace J^0 is an L_1-summand in A^*.*

Somewhat surprisingly, we thus have an equivalence between a purely algebraic property (being an ideal), and purely geometric properties ((ii), (iii) and (iv)). In fact, (ii), (iii) and (iv), which do not refer to the multiplication in A, make sense and are equivalent in arbitrary Banach spaces; such subspaces are called M-ideals. Examples of M-ideals also include many ideals in uniform algebras, and the compact operators in $L(\ell_p)$, for $1 < p < \infty$. The literature on M-ideals is now vast, with applications to operator algebras, harmonic analysis, approximation theory and other fields; we refer to [14] for a comprehensive introduction.

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