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On New Generalizations of Hermite-Hadamard Type Inequalities via Atangana-Baleanu Fractional Integral Operators

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Abstract: Fractional operators are one of the frequently used tools to obtain new generalizations of clasical inequalities in recent years and many new fractional operators are defined in the literature. This development in the field of fractional analysis has led to a new orientation in various branches of mathematics and in many of the applied sciences. Thanks to this orientation, it has brought a whole new dimension to the field of inequality theory as well as many other disciplines. In this study, a new lemma has been proved for the fractional integral operator defined by Atangana and Baleanu. Later with the help of this lemma and known inequalities such as Young, Jensen, Hölder, new generalizations of Hermite-Hadamard inequality are obtained. Many reduced corollaries about the main findings are presented for classical integrals.

Keywords: convex function; hölder inequality; young inequality; power mean inequality; Atangana-Baleanu fractional integral operators

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1. Introduction

First of all, let us recall the concept of convex function which is the basic concept of convex analysis.

Definition 1. *The function $\kappa : [\mu, \nu] \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds*

$$\kappa(\omega x + (1-\omega)y) \leq \omega\kappa(x) + (1-\omega)\kappa(y) \tag{1}$$

for all $x, y \in [\mu, \nu]$ and $\omega \in [0,1]$. We say that κ is concave if $(-\kappa)$ is convex.

There are many inequalities in the literature for convex functions. But among these inequalities the most take attention of researchers is the Hermite-Hadamard inequality on which hundreds of studies have been conducted. The clasical Hermite-Hadamard integral inequalities are as the following.

Theorem 1. *Assume that $\kappa : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$ where $\mu < \nu$. The following statement;*

$$\kappa\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\kappa(x)dx \leq \frac{\kappa(\mu)+\kappa(\nu)}{2} \tag{2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if κ is concave.

Several new results have been proved related different kinds of convex functions and associated integral inequalities. In [1], Bakula et al. gave some new integral inequalities of Hadamard type for m-convex and (α, m)-convex functions. A similar paper has been written by Kirmaci et al. for s-convex functions in [2]. Besides, in [3], Kavurmaci et al. proved some new inequalities for convex functions. In [4], the authors have given several new results for co-ordinated convexity which is a modification of convexity on the co-ordinates. In [5], Özdemir et al. have defined a generalization of convexity and proved some Hadamard type inequalities. On all of these, in [6], Sarikaya et al. gave a different perspective to the inequality (2) by using the Riemann-Liouville fractional integral operators as follows:

Theorem 2 ([6]). *Let $\kappa : [\mu, \nu] \to \mathbb{R}$ be a positive function with $0 \leq \mu < \nu$ and $\kappa \in L_1[\mu, \nu]$. If κ is a convex function on $[\mu, \nu]$, then the following inequalities for fractional integrals hold:*

$$\kappa\left(\frac{\mu+\nu}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left[\mathrm{J}^{\alpha}_{\mu^+}\kappa(\nu) + \mathrm{J}^{\alpha}_{\nu^-}\kappa(\mu)\right] \leq \frac{\kappa(\mu)+\kappa(\nu)}{2}$$

with $\alpha > 0$.

In here, $J^{\alpha}_{\mu^+}$ and $J^{\alpha}_{\nu^-}$ are respectly right and left side of Riemann-Liouville fractional integral operators, as follows:

Definition 2. *Let $\kappa \in L[\mu, \nu]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{\mu^+}$ and $J^{\alpha}_{\nu^-}$ of order $\alpha > 0$ with $\mu \geq 0$ are defined by*

$$J^{\alpha}_{\mu^+}\kappa(x) = \frac{1}{\Gamma(\alpha)}\int_{\mu}^{x}(x-\rho)^{\alpha-1}\kappa(\rho)d\rho, \ x > \mu$$

and

$$J^{\alpha}_{\nu^-}\kappa(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\nu}(\rho-x)^{\alpha-1}\kappa(\rho)d\rho, \ x < \nu$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-\rho}\rho^{\alpha-1}d\rho$ and $J^{0}_{\mu^+}\kappa(x) = J^{0}_{\nu^-}\kappa(x) = \kappa(x)$.

This study played a key role in generalizing, expanding and obtaining variations of classical integral inequalities with the help of fractional integrals. On the other hand, by defining different versions of Riemann-Liouville fractional integral operators in the last decade, new versions and generalizations of inequalities on both convex functions and differentiable functions have been obtained (see the paper [7]). Studies in the field of fractional analysis have brought a new perspective and orientation to many fields of applied sciences and mathematics in addition to the theory of inequality in recent years. It has shed light on many real world problems with the applications of newly defined fractional integral and derivative operators. In these new operators, several important criteria have differentiated them and have made some advantageous in applications compared to others. Exponentially or power law expressions used in the kernel of fractional operators revealed their features such as locality and singularity, and it became important to obtain the initial conditions for the special versions of the parameters used in the definition. Another important detail is to reveal the spaces where the operators are defined and to show the suitability for the real world problems. In [8], Atangana and Baleanu has offered a non-singular and non-local fractional derivative and proved properties of this interesting operator. Due to this new definition, many real world problems has been solved again with time memory effect. In [9], the authors have given right sided Atangana-Baleanu integral operators and proved some new results that depend to this non-singular operator.

In [10], Awan et al. have established Hadamard type inequalities for preinvex functions via conformable fractional integral operators. In [11], further motivated results have been performed by using fractional integrals. Similarly, Tariboon et al. have proved inequalities via Riemann-Liouville fractional integrals in [12]. The readers can find collection of the fractional derivative and integral operators in [13] and also we refer to the interested readers the following papers [14–22]. In this sense, let's examine some fractional derivative and integral operators with many applications and features. We first start with the Atangana-Baleanu derivative operator as following.

Definition 3 ([8]). *Let $\kappa \in H^1(\mu,\nu)$, $\nu > \mu$, $\alpha \in [0,1]$ then, the definition of the new fractional derivative is given:*

$$ {}^{ABC}_{\mu}D^{\alpha}_{\rho}[\kappa(\rho)] = \frac{B(\alpha)}{1-\alpha}\int_{\mu}^{\rho}\kappa'(x)E_{\alpha}\left[-\alpha\frac{(\rho-x)^{\alpha}}{(1-\alpha)}\right]dx. \tag{3}$$

Here $H^1(\mu,\nu)$ can be defined as $H^1(\mu,\nu) = \{\kappa : \kappa \in L[\mu,\nu] \text{ and } \kappa' \in L[\mu,\nu]\}$.

Definition 4 ([8]). *Let $\kappa \in H^1(\mu,\nu)$, $\nu > \mu$, $\alpha \in [0,1]$ then, the definition of the new fractional derivative is given:*

$$ {}^{ABR}_{\mu}D^{\alpha}_{\rho}[\kappa(\rho)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{d\rho}\int_{\mu}^{\rho}\kappa(x)E_{\alpha}\left[-\alpha\frac{(\rho-x)^{\alpha}}{(1-\alpha)}\right]dx. \tag{4}$$

Equations (3) and (4) have a non-local kernel. Also in Equation (4) when the function is constant we get zero.

With the help of Laplace transform and convolution theorem, Atangana-Baleanu described the fractional integral operator as follows.

Definition 5 ([8]). *The fractional integral associate to the new fractional derivative with non-local kernel of a function $\kappa \in H^1(\mu,\nu)$ as defined:*

$$ {}^{AB}_{\mu}I^{\alpha}\{\kappa(\rho)\} = \frac{1-\alpha}{B(\alpha)}\kappa(\rho) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\mu}^{\rho}\kappa(y)(\rho-y)^{\alpha-1}dy$$

where $\nu > \mu, \alpha \in [0,1]$.

In [9], Abdeljawad and Baleanu introduced right hand side of integral operator as following; The right fractional new integral with ML kernel of order $\alpha \in [0,1]$ is defined by

$$ {}^{AB}I^{\alpha}_{\nu}\{\kappa(\rho)\} = \frac{1-\alpha}{B(\alpha)}\kappa(\rho) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\rho}^{\nu}\kappa(y)(y-\rho)^{\alpha-1}dy.$$

The main purpose of this article is to obtain an integral identity that includes the Atangana-Baleanu integral operator and to prove Hermite-Hadamard type integral inequalities for differentiable convex functions with the help of this identity. The main motivation point of the study is to prove a new integral identity with the potential to produce Hermite-Hadamard type inequalities for Atangana-Baleanu fractional integral operators based on a non-singular and non-local derivative operator. It is aimed to bring more general and effective results to the inequality theory thanks to the kernel structure and properties of the operator.

2. Main Results

Let $\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$. Throughout this section we will take

$$\begin{aligned}
& {}^{AB}I_{\kappa}(\rho, \alpha, \mu, \nu) \\
= \; & {}^{AB}_{\mu}I^{\alpha}\left\{\kappa\left(\frac{\mu+\rho}{2}\right)\right\} + {}^{AB}I^{\alpha}_{\rho}\left\{\kappa\left(\frac{\mu+\rho}{2}\right)\right\} \\
& + {}^{AB}_{\rho}I^{\alpha}\left\{\kappa\left(\frac{\nu+\rho}{2}\right)\right\} + {}^{AB}I^{\alpha}_{\nu}\left\{\kappa\left(\frac{\nu+\rho}{2}\right)\right\} \\
& - \frac{(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[\kappa(\rho)+\kappa(\mu)] - \frac{(\nu-\rho)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[\kappa(\rho)+\kappa(\nu)] \\
& - \frac{2(1-\alpha)}{B(\alpha)}\left[\kappa\left(\frac{\mu+\rho}{2}\right) + \kappa\left(\frac{\nu+\rho}{2}\right)\right].
\end{aligned}$$

We will start with a new integral identity that will be used the proofs of our main findings as following:

Lemma 1. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$. Then we have the following identity for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
& {}^{AB}_{\mu}I^{\alpha}\left\{\kappa\left(\tfrac{\mu+\rho}{2}\right)\right\} + {}^{AB}I^{\alpha}_{\rho}\left\{\kappa\left(\tfrac{\mu+\rho}{2}\right)\right\} \\
& + {}^{AB}_{\rho}I^{\alpha}\left\{\kappa\left(\tfrac{\nu+\rho}{2}\right)\right\} + {}^{AB}I^{\alpha}_{\nu}\left\{\kappa\left(\tfrac{\nu+\rho}{2}\right)\right\} \\
& - \tfrac{(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[\kappa(\rho)+\kappa(\mu)] - \tfrac{(\nu-\rho)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[\kappa(\rho)+\kappa(\nu)] \\
& - \tfrac{2(1-\alpha)}{B(\alpha)}\left[\kappa\left(\tfrac{\mu+\rho}{2}\right) + \kappa\left(\tfrac{\nu+\rho}{2}\right)\right] \\
= \; & \tfrac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \tfrac{\omega^{\alpha}}{2}\kappa'\left(\tfrac{1+\omega}{2}\rho + \tfrac{1-\omega}{2}\mu\right)d\omega - \int_0^1 \tfrac{\omega^{\alpha}}{2}\kappa'\left(\tfrac{1+\omega}{2}\rho + \tfrac{1-\omega}{2}\mu\right)d\omega\right] \\
& + \tfrac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \tfrac{\omega^{\alpha}}{2}\kappa'\left(\tfrac{1+\omega}{2}\rho + \tfrac{1-\omega}{2}\nu\right)d\omega - \int_0^1 \tfrac{\omega^{\alpha}}{2}\kappa'\left(\tfrac{1-\omega}{2}\rho + \tfrac{1+\omega}{2}\nu\right)d\omega\right]
\end{aligned} \tag{5}$$

where $\alpha \in [0,1]$, $\rho \in [\mu, \nu]$, $B(\alpha)$ is the normalization function and $\Gamma(.)$ is Gamma function.

Proof. By using integration by parts for the right hand side of equation, we have

$$\begin{aligned}
& \tfrac{1-\alpha}{B(\alpha)}\kappa\left(\tfrac{\mu+\rho}{2}\right) - \tfrac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\int_0^1 \tfrac{\omega^{\alpha}}{2}\kappa'\left(\tfrac{1+\omega}{2}\rho + \tfrac{1-\omega}{2}\mu\right)d\omega \\
= \; & \tfrac{1-\alpha}{B(\alpha)}\kappa\left(\tfrac{\mu+\rho}{2}\right) - \tfrac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\tfrac{\omega^{\alpha}}{2}\tfrac{2\kappa\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)}{\rho-\mu}\Big|_0^1\right. \\
& \left. - \int_0^1 \tfrac{2\kappa\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)}{\rho-\mu}\tfrac{\alpha\omega^{\alpha-1}}{2}d\omega\right] \\
= \; & \tfrac{1-\alpha}{B(\alpha)}\kappa\left(\tfrac{\mu+\rho}{2}\right) - \tfrac{(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\kappa(\rho) + \tfrac{\alpha(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\int_0^1 \omega^{\alpha-1}\kappa\left(\tfrac{1+\omega}{2}\rho + \tfrac{1-\omega}{2}\mu\right)d\omega \\
= \; & -\tfrac{(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\kappa(\rho) + \tfrac{1-\alpha}{B(\alpha)}\kappa\left(\tfrac{\mu+\rho}{2}\right) + \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\frac{\mu+\rho}{2}}^{\rho}\left(u - \tfrac{\mu+\rho}{2}\right)^{\alpha-1}\kappa(u)du \\
= \; & {}^{AB}I^{\alpha}_{\rho}\left\{\kappa\left(\tfrac{\mu+\rho}{2}\right)\right\} - \tfrac{(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\kappa(\rho).
\end{aligned}$$

Then we can write following identity

$$\begin{aligned}
& \tfrac{1-\alpha}{B(\alpha)}\kappa\left(\tfrac{\mu+\rho}{2}\right) - \tfrac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\int_0^1 \tfrac{\omega^{\alpha}}{2}\kappa'\left(\tfrac{1+\omega}{2}\rho + \tfrac{1-\omega}{2}\mu\right)d\omega \\
= \; & {}^{AB}I^{\alpha}_{\rho}\left\{\kappa\left(\tfrac{\mu+\rho}{2}\right)\right\} - \tfrac{(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\kappa(\rho).
\end{aligned} \tag{6}$$

Similarly, we have following identities.

$$\begin{aligned} & \frac{1-\alpha}{B(\alpha)}\kappa\left(\frac{\mu+\rho}{2}\right)+\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\int_0^1\frac{\omega^{\alpha}}{2}\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)d\omega \\ = & \ {}^{AB}_{\mu}I^{\alpha}\left\{\kappa\left(\frac{\mu+\rho}{2}\right)\right\}-\frac{(\rho-\mu)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\kappa(\mu), \end{aligned} \tag{7}$$

$$\begin{aligned} & \frac{1-\alpha}{B(\alpha)}\kappa\left(\frac{\rho+\nu}{2}\right)+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\int_0^1\frac{\omega^{\alpha}}{2}\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)d\omega \\ = & \ {}^{AB}_{\rho}I^{\alpha}\left\{\kappa\left(\frac{\rho+\nu}{2}\right)\right\}-\frac{(\nu-\rho)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\kappa(\rho) \end{aligned} \tag{8}$$

and

$$\begin{aligned} & \frac{1-\alpha}{B(\alpha)}\kappa\left(\frac{\rho+\nu}{2}\right)-\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\int_0^1\frac{\omega^{\alpha}}{2}\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)d\omega \\ = & \ {}^{AB}I^{\alpha}_{\nu}\left\{\kappa\left(\frac{\rho+\nu}{2}\right)\right\}-\frac{(\nu-\rho)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\kappa(\nu). \end{aligned} \tag{9}$$

By adding identities (6)–(9), we obtain desired result. □

Corollary 1. *In Lemma 1, if we take $\alpha = 1$, then the identity (5) reduces to the identity*

$$\begin{aligned} & \int_{\mu}^{\nu}\kappa(x)dx-\frac{(\rho-\mu)}{2}[\kappa(\rho)+\kappa(\mu)]-\frac{(\nu-\rho)}{2}[\kappa(\rho)+\kappa(\nu)] \\ = & \ \frac{(\rho-\mu)^2}{2}\left[\int_0^1\frac{\omega}{2}\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)d\omega-\int_0^1\frac{\omega}{2}\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)d\omega\right] \\ & +\frac{(\nu-\rho)^2}{2}\left[\int_0^1\frac{\omega}{2}\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)d\omega-\int_0^1\frac{\omega}{2}\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)d\omega\right]. \end{aligned}$$

Theorem 3. *$\kappa:[\mu,\nu]\to\mathbb{R}$ be differentiable function on (μ,ν) with $\mu<\nu$ and $\kappa'\in L_1[\mu,\nu]$. If $|\kappa'|$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} & \left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\ \leq & \ \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left[\frac{|\kappa'(\rho)|+|\kappa'(\mu)|}{\alpha+1}\right]+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left[\frac{|\kappa'(\rho)|+|\kappa'(\nu)|}{\alpha+1}\right] \end{aligned} \tag{10}$$

where $\rho\in[\mu,\nu]$, $\alpha\in[0,1]$, $B(\alpha)$ is normalization function.

Proof. By using the identity that is given in Lemma 1, we can write

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
= &\left|\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \frac{\omega^{\alpha}}{2}\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)d\omega-\int_0^1 \frac{\omega^{\alpha}}{2}\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)d\omega\right]\right. \\
&\left.+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \frac{\omega^{\alpha}}{2}\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)d\omega-\int_0^1 \frac{\omega^{\alpha}}{2}\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)d\omega\right]\right| \\
\leq &\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)\right|d\omega+\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)\right|d\omega\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)\right|d\omega+\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)\right|d\omega\right].
\end{aligned}$$

By using convexity of $|\kappa'|$, we get

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
\leq &\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \frac{\omega^{\alpha}}{2}\left[\left(\frac{1-\omega}{2}\right)|\kappa'(\rho)|+\left(\frac{1+\omega}{2}\right)|\kappa'(\mu)|\right]d\omega\right. \\
&\left.+\int_0^1 \frac{\omega^{\alpha}}{2}\left[\left(\frac{1+\omega}{2}\right)|\kappa'(\rho)|+\left(\frac{1-\omega}{2}\right)|\kappa'(\mu)|\right]d\omega\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \frac{\omega^{\alpha}}{2}\left[\left(\frac{1+\omega}{2}\right)|\kappa'(\rho)|+\left(\frac{1-\omega}{2}\right)|\kappa'(\nu)|\right]d\omega\right. \\
&\left.+\int_0^1 \frac{\omega^{\alpha}}{2}\left[\left(\frac{1-\omega}{2}\right)|\kappa'(\rho)|+\left(\frac{1+\omega}{2}\right)|\kappa'(\nu)|\right]d\omega\right] \\
= &\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\frac{|\kappa'(\rho)|}{4}\int_0^1\left(\omega^{\alpha}-\omega^{\alpha+1}\right)d\omega+\frac{|\kappa'(\mu)|}{4}\int_0^1\left(\omega^{\alpha}+\omega^{\alpha+1}\right)d\omega\right. \\
&\left.+\frac{|\kappa'(\rho)|}{4}\int_0^1\left(\omega^{\alpha}+\omega^{\alpha+1}\right)d\omega+\frac{|\kappa'(\mu)|}{4}\int_0^1\left(\omega^{\alpha}-\omega^{\alpha+1}\right)d\omega\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\frac{|\kappa'(\rho)|}{4}\int_0^1\left(\omega^{\alpha}+\omega^{\alpha+1}\right)d\omega+\frac{|\kappa'(\nu)|}{4}\int_0^1\left(\omega^{\alpha}-\omega^{\alpha+1}\right)d\omega\right. \\
&\left.+\frac{|\kappa'(\rho)|}{4}\int_0^1\left(\omega^{\alpha}-\omega^{\alpha+1}\right)d\omega+\frac{|\kappa'(\nu)|}{4}\int_0^1\left(\omega^{\alpha}+\omega^{\alpha+1}\right)d\omega\right] \\
= &\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left[\frac{|\kappa'(\rho)|+|\kappa'(\mu)|}{\alpha+1}\right]+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left[\frac{|\kappa'(\rho)|+|\kappa'(\nu)|}{\alpha+1}\right]
\end{aligned}$$

and the proof is completed. □

Corollary 2. *In Theorem 3, if we take $\alpha=1$, then the inequality (10) reduces to the inequality*

$$\begin{aligned}
&\left|\int_{\mu}^{\nu}\kappa(x)dx-\frac{(\rho-\mu)}{2}[\kappa(\rho)+\kappa(\mu)]-\frac{(\nu-\rho)}{2}[\kappa(\rho)+\kappa(\nu)]\right| \\
\leq &\frac{(\rho-\mu)^2}{2^3}\left[\left|\kappa'(\rho)\right|+\left|\kappa'(\mu)\right|\right]+\frac{(\nu-\rho)^2}{2^3}\left[\left|\kappa'(\rho)\right|+\left|\kappa'(\nu)\right|\right].
\end{aligned}$$

Theorem 4. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu, \nu]$. If $|\kappa'|^q$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned} & \left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^{p}(\alpha p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\kappa'(\rho)|^{q}+3|\kappa'(\mu)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\kappa'(\rho)|^{q}+|\kappa'(\mu)|^{q}}{4}\right)^{\frac{1}{q}}\right] \\ & +\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^{p}(\alpha p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|\kappa'(\rho)|^{q}+|\kappa'(\nu)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|\kappa'(\rho)|^{q}+3|\kappa'(\nu)|^{q}}{4}\right)^{\frac{1}{q}}\right] \end{aligned} \tag{11}$$

where $p^{-1}+q^{-1}=1$, $\rho \in [\mu, \nu]$, $\alpha \in [0,1]$, $q>1$, $B(\alpha)$ is normalization function.

Proof. By Lemma 1, we can write

$$\begin{aligned} & \left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)\right|d\omega+\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)\right|d\omega\right] \\ & +\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)\right|d\omega+\int_0^1 \frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)\right|d\omega\right]. \end{aligned}$$

By appliying Hölder inequality, we have

$$\begin{aligned} & \left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^{p}d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)\right|^{q}d\omega\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^{p}d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)\right|^{q}d\omega\right)^{\frac{1}{q}}\right] \\ & +\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^{p}d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)\right|^{q}d\omega\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^{p}d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)\right|^{q}d\omega\right)^{\frac{1}{q}}\right]. \end{aligned}$$

By using convexity of $|\kappa'|^q$, we have

$$\begin{aligned} & \left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^{p}d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{1-\omega}{2}\right)|\kappa'(\rho)|^{q}+\left(\frac{1+\omega}{2}\right)|\kappa'(\mu)|^{q}\right]d\omega\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^{p}d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{1+\omega}{2}\right)|\kappa'(\rho)|^{q}+\left(\frac{1-\omega}{2}\right)|\kappa'(\mu)|^{q}\right]d\omega\right)^{\frac{1}{q}}\right] \\ & +\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^{p}d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{1+\omega}{2}\right)|\kappa'(\rho)|^{q}+\left(\frac{1-\omega}{2}\right)|\kappa'(\nu)|^{q}\right]d\omega\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^{p}d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{1-\omega}{2}\right)|\kappa'(\rho)|^{q}+\left(\frac{1+\omega}{2}\right)|\kappa'(\nu)|^{q}\right]d\omega\right)^{\frac{1}{q}}\right] \\ =\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^{p}(\alpha p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\kappa'(\rho)|^{q}+3|\kappa'(\mu)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\kappa'(\rho)|^{q}+|\kappa'(\mu)|^{q}}{4}\right)^{\frac{1}{q}}\right] \\ & +\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^{p}(\alpha p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|\kappa'(\rho)|^{q}+|\kappa'(\nu)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|\kappa'(\rho)|^{q}+3|\kappa'(\nu)|^{q}}{4}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

So, the proof is completed. □

Corollary 3. *In Theorem 4, if we take $\alpha = 1$, then the inequality (11) reduces to the inequality*

$$\begin{aligned}
&\left|\int_{\mu}^{\nu}\kappa(x)dx - \frac{(\rho-\mu)}{2}[\kappa(\rho)+\kappa(\mu)] - \frac{(\nu-\rho)}{2}[\kappa(\rho)+\kappa(\nu)]\right| \\
\leq\ & \frac{(\rho-\mu)^2}{2}\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\kappa'(\rho)|^q+3|\kappa'(\mu)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\kappa'(\rho)|^q+|\kappa'(\mu)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&+\frac{(\nu-\rho)^2}{2}\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|\kappa'(\rho)|^q+|\kappa'(\nu)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\kappa'(\rho)|^q+3|\kappa'(\nu)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 5. *$\kappa : [\mu,\nu] \to \mathbb{R}$ be differentiable function on (μ,ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu,\nu]$. If $|\kappa'|^q$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\frac{1}{2^{p-1}\ p\ (\alpha p+1)}+\frac{|\kappa'(\rho)|^q+|\kappa'(\mu)|^q}{q}\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\frac{1}{2^{p-1}\ p\ (\alpha p+1)}+\frac{|\kappa'(\rho)|^q+|\kappa'(\nu)|^q}{q}\right]
\end{aligned} \tag{12}$$

where $p^{-1}+q^{-1}=1$, $\rho \in [\mu,\nu]$, $\alpha \in [0,1]$, $q>1$, $B(\alpha)$ is normalization function.

Proof. By Lemma 1, we can write

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)\right|d\omega+\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)\right|d\omega\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)\right|d\omega+\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)\right|d\omega\right].
\end{aligned}$$

By appliying Young inequality as $xy \leq \frac{1}{p}x^p+\frac{1}{q}y^q$, we have

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^p d\omega+\frac{1}{q}\int_0^1\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)\right|^q d\omega\right. \\
&\left.+\frac{1}{p}\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^p d\omega+\frac{1}{q}\int_0^1\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)\right|^q d\omega\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^p d\omega+\frac{1}{q}\int_0^1\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)\right|^q d\omega\right. \\
&\left.+\frac{1}{p}\int_0^1\left(\frac{\omega^{\alpha}}{2}\right)^p d\omega+\frac{1}{q}\int_0^1\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)\right|^q d\omega\right].
\end{aligned}$$

By using convexity of $|\kappa'|^q$ and by a simple computation, we have the desired result. □

Corollary 4. *In Theorem 5, if we take $\alpha = 1$, then the inequality (12) reduces to the inequality*

$$\begin{aligned}&\left|\int_{\mu}^{\nu}\kappa(x)dx-\frac{(\rho-\mu)}{2}[\kappa(\rho)+\kappa(\mu)]-\frac{(\nu-\rho)}{2}[\kappa(\rho)+\kappa(\nu)]\right|\\ \leq\ &\frac{(\rho-\mu)^2}{2}\left[\frac{1}{2^{p-1}\ p\ (p+1)}+\frac{|\kappa'(\rho)|^q+|\kappa'(\mu)|^q}{q}\right]\\ &+\frac{(\nu-\rho)^2}{2}\left[\frac{1}{2^{p-1}\ p\ (p+1)}+\frac{|\kappa'(\rho)|^q+|\kappa'(\nu)|^q}{q}\right].\end{aligned}$$

Theorem 6. *$\kappa:[\mu,\nu]\to\mathbb{R}$ be differentiable function on (μ,ν) with $\mu<\nu$ and $\kappa'\in L_1[\mu,\nu]$. If $|\kappa'|^q$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right|\\ \leq\ &\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\kappa'(\rho)|^q+(2\alpha+3)|\kappa'(\mu)|^q}{4(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{(2\alpha+3)|\kappa'(\rho)|^q+|\kappa'(\mu)|^q}{4(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right]\\ &+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{(2\alpha+3)|\kappa'(\rho)|^q+|\kappa'(\nu)|^q}{4(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{|\kappa'(\rho)|^q+(2\alpha+3)|\kappa'(\nu)|^q}{4(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right]\end{aligned}$$

where $\rho\in[\mu,\nu]$, $\alpha\in[0,1]$, $q\geq 1$, $B(\alpha)$ is normalization function.

Proof. By Lemma 1, we can write

$$\begin{aligned}&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right|\\ \leq\ &\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)\right|d\omega+\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)\right|d\omega\right]\\ &+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)\right|d\omega+\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)\right|d\omega\right].\end{aligned}$$

By using power mean inequality, we have

$$\begin{aligned}&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right|\\ \leq\ &\frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\frac{\omega^{\alpha}}{2}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)\right|^q d\omega\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1\frac{\omega^{\alpha}}{2}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)\right|^q d\omega\right)^{\frac{1}{q}}\right]\\ &+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\frac{\omega^{\alpha}}{2}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)\right|^q d\omega\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1\frac{\omega^{\alpha}}{2}d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)\right|^q d\omega\right)^{\frac{1}{q}}\right].\end{aligned}$$

By using convexity of $|\kappa'|^q$ and by a simple computation, we have the desired result. □

Corollary 5. *In Theorem 6, if we take $\alpha = 1$, then the inequality (13) reduces to the inequality*

$$\begin{aligned}
&\left|\int_{\mu}^{\nu}\kappa(x)dx-\frac{(\rho-\mu)}{2}[\kappa(\rho)+\kappa(\mu)]-\frac{(\nu-\rho)}{2}[\kappa(\rho)+\kappa(\nu)]\right| \\
\leq\ & \frac{(\rho-\mu)^2}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{|\kappa'(\rho)|^q+5|\kappa'(\mu)|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{5|\kappa'(\rho)|^q+|\kappa'(\mu)|^q}{24}\right)^{\frac{1}{q}}\right] \\
&+\frac{(\nu-\rho)^2}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{5|\kappa'(\rho)|^q+|\kappa'(\nu)|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{|\kappa'(\rho)|^q+5|\kappa'(\nu)|^q}{24}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 7. *$\kappa:[\mu,\nu]\to\mathbb{R}$ be differentiable function on (μ,ν) with $\mu<\nu$ and $\kappa'\in L_1[\mu,\nu]$. If $|\kappa'|$ is a concave function for $q>1$, then we have*

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2(\alpha+1)}\right)\left[\left|\kappa'\left(\frac{\rho+(2\alpha+3)\mu}{2(\alpha+2)}\right)\right|+\left|\kappa'\left(\frac{(2\alpha+3)\rho+\mu}{2(\alpha+2)}\right)\right|\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2(\alpha+1)}\right)\left[\left|\kappa'\left(\frac{(2\alpha+3)\rho+\nu}{2(\alpha+2)}\right)\right|+\left|\kappa'\left(\frac{\rho+(2\alpha+3)\nu}{2(\alpha+2)}\right)\right|\right]
\end{aligned}\tag{13}$$

where $\rho\in[\mu,\nu]$, $\alpha\in[0,1]$, $B(\alpha)$ is normalization function.

Proof. From Lemma 1 and the Jensen integral inequality, we have

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\tfrac{1-\omega}{2}\rho+\tfrac{1+\omega}{2}\mu\right)\right|d\omega+\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\tfrac{1+\omega}{2}\rho+\tfrac{1-\omega}{2}\mu\right)\right|d\omega\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\tfrac{1+\omega}{2}\rho+\tfrac{1-\omega}{2}\nu\right)\right|d\omega+\int_0^1\frac{\omega^{\alpha}}{2}\left|\kappa'\left(\tfrac{1-\omega}{2}\rho+\tfrac{1+\omega}{2}\nu\right)\right|d\omega\right] \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\frac{\omega^{\alpha}}{2}dw\right)\left|\kappa'\left(\frac{\int_0^1\frac{\omega^{\alpha}}{2}\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\mu\right)dw}{\int_0^1\frac{\omega^{\alpha}}{2}dw}\right)\right|\right. \\
&\left.+\left(\int_0^1\frac{\omega^{\alpha}}{2}dw\right)\left|\kappa'\left(\frac{\int_0^1\frac{\omega^{\alpha}}{2}\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\mu\right)dw}{\int_0^1\frac{\omega^{\alpha}}{2}dw}\right)\right|\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1\frac{\omega^{\alpha}}{2}dw\right)\left|\kappa'\left(\frac{\int_0^1\frac{\omega^{\alpha}}{2}\left(\frac{1+\omega}{2}\rho+\frac{1-\omega}{2}\nu\right)dw}{\int_0^1\frac{\omega^{\alpha}}{2}dw}\right)\right|\right. \\
&\left.+\left(\int_0^1\frac{\omega^{\alpha}}{2}dw\right)\left|\kappa'\left(\frac{\int_0^1\frac{\omega^{\alpha}}{2}\left(\frac{1-\omega}{2}\rho+\frac{1+\omega}{2}\nu\right)dw}{\int_0^1\frac{\omega^{\alpha}}{2}dw}\right)\right|\right].
\end{aligned}$$

By computing the above integrals, we have desired result. □

Corollary 6. *In Theorem 7, if we take $\alpha = 1$, then the inequality (13) reduces to the inequality*

$$\begin{aligned}
&\left|\int_{\mu}^{\nu}\kappa(x)dx-\frac{(\rho-\mu)}{2}[\kappa(\rho)+\kappa(\mu)]-\frac{(\nu-\rho)}{2}[\kappa(\rho)+\kappa(\nu)]\right| \\
\leq\ & \frac{(\rho-\mu)^2}{2}\left(\frac{1}{4}\right)\left[\left|\kappa'\left(\frac{\rho+5\mu}{6}\right)\right|+\left|\kappa'\left(\frac{5\rho+\mu}{6}\right)\right|\right] \\
&+\frac{(\nu-\rho)^2}{2}\left(\frac{1}{4}\right)\left[\left|\kappa'\left(\frac{5\rho+\nu}{6}\right)\right|+\left|\kappa'\left(\frac{\rho+5\nu}{6}\right)\right|\right].
\end{aligned}$$

Theorem 8. *$\kappa : [\mu, \nu] \to \mathbb{R}$ be differentiable function on (μ, ν) with $\mu < \nu$ and $\kappa' \in L_1[\mu, \nu]$. If $|\kappa'|^q$ is a concave function, we have*

$$\begin{aligned} & \left| {}^{AB}I_{\kappa}(\rho, \alpha, \mu, \nu) \right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)} \left(\frac{1}{2^{p}(\alpha p+1)} \right)^{\frac{1}{p}} \left[\left| \kappa' \left(\frac{\rho+3\mu}{4} \right) \right| + \left| \kappa' \left(\frac{3\rho+\mu}{4} \right) \right| \right] \\ & + \frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)} \left(\frac{1}{2^{p}(\alpha p+1)} \right)^{\frac{1}{p}} \left[\left| \kappa' \left(\frac{3\rho+\nu}{4} \right) \right| + \left| \kappa' \left(\frac{\rho+3\nu}{4} \right) \right| \right] \end{aligned} \tag{14}$$

where $p^{-1} + q^{-1} = 1$, $\rho \in [\mu, \nu]$, $\alpha \in [0, 1]$, $q > 1$, $B(\alpha)$ is normalization function.

Proof. By using Lemma 1 and Hölder integral inequality, we can write

$$\begin{aligned} & \left| {}^{AB}I_{\kappa}(\rho, \alpha, \mu, \nu) \right| \\ \leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 \left(\frac{\omega^{\alpha}}{2} \right)^{p} d\omega \right)^{\frac{1}{p}} \left(\int_0^1 \left| \kappa' \left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\mu \right) \right|^{q} d\omega \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 \left(\frac{\omega^{\alpha}}{2} \right)^{p} d\omega \right)^{\frac{1}{p}} \left(\int_0^1 \left| \kappa' \left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\mu \right) \right|^{q} d\omega \right)^{\frac{1}{q}} \right] \\ & + \frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 \left(\frac{\omega^{\alpha}}{2} \right)^{p} d\omega \right)^{\frac{1}{p}} \left(\int_0^1 \left| \kappa' \left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\nu \right) \right|^{q} d\omega \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 \left(\frac{\omega^{\alpha}}{2} \right)^{p} d\omega \right)^{\frac{1}{p}} \left(\int_0^1 \left| \kappa' \left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\nu \right) \right|^{q} d\omega \right)^{\frac{1}{q}} \right]. \end{aligned}$$

By using concavity of $|\kappa'|^q$ and Jensen integral inequality, we get

$$\begin{aligned} \int_0^1 \left| \kappa' \left(\tfrac{1-\omega}{2}\rho + \tfrac{1+\omega}{2}\mu \right) \right|^{q} d\omega &= \int_0^1 w^0 \left| \kappa' \left(\tfrac{1-\omega}{2}\rho + \tfrac{1+\omega}{2}\mu \right) \right|^{q} d\omega \\ &\leq \left(\int_0^1 w^0 dw \right) \left| \kappa' \left(\frac{\int_0^1 w^0 \left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\mu \right) dw}{\int_0^1 w^0 dw} \right) \right|^{q} \\ &= \left| \kappa' \left(\tfrac{\rho+3\mu}{4} \right) \right|^{q}. \end{aligned}$$

Similarly

$$\begin{aligned} \int_0^1 \left| \kappa' \left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\mu \right) \right|^{q} d\omega &\leq \left| \kappa' \left(\frac{3\rho+\mu}{4} \right) \right|^{q}, \\ \int_0^1 \left| \kappa' \left(\frac{1+\omega}{2}\rho + \frac{1-\omega}{2}\nu \right) \right|^{q} d\omega &\leq \left| \kappa' \left(\frac{3\rho+\nu}{4} \right) \right|^{q} \end{aligned}$$

and

$$\int_0^1 \left| \kappa' \left(\frac{1-\omega}{2}\rho + \frac{1+\omega}{2}\nu \right) \right|^{q} d\omega \leq \left| \kappa' \left(\frac{\rho+3\nu}{4} \right) \right|^{q}.$$

So we obtain,

$$\begin{aligned}
&\left|{}^{AB}I_{\kappa}(\rho,\alpha,\mu,\nu)\right| \\
\leq\ & \frac{(\rho-\mu)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^{p}(\alpha p+1)}\right)^{\frac{1}{p}}\left[\left|\kappa'\left(\frac{\rho+3\mu}{4}\right)\right|+\left|\kappa'\left(\frac{3\rho+\mu}{4}\right)\right|\right] \\
&+\frac{(\nu-\rho)^{\alpha+1}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^{p}(\alpha p+1)}\right)^{\frac{1}{p}}\left[\left|\kappa'\left(\frac{3\rho+\nu}{4}\right)\right|+\left|\kappa'\left(\frac{\rho+3\nu}{4}\right)\right|\right]
\end{aligned}$$

and the proof is completed. □

Corollary 7. *In Theorem 8, if we take $\alpha = 1$, then the inequality (14) reduces to the inequality*

$$\begin{aligned}
&\left|\int_{\mu}^{\nu}\kappa(x)dx-\frac{(\rho-\mu)}{2}[\kappa(\rho)+\kappa(\mu)]-\frac{(\nu-\rho)}{2}[\kappa(\rho)+\kappa(\nu)]\right| \\
\leq\ & \frac{(\rho-\mu)^{2}}{2}\left(\frac{1}{2^{p}(p+1)}\right)^{\frac{1}{p}}\left[\left|\kappa'\left(\frac{\rho+3\mu}{4}\right)\right|+\left|\kappa'\left(\frac{3\rho+\mu}{4}\right)\right|\right] \\
&+\frac{(\nu-\rho)^{2}}{2}\left(\frac{1}{2^{p}(p+1)}\right)^{\frac{1}{p}}\left[\left|\kappa'\left(\frac{3\rho+\nu}{4}\right)\right|+\left|\kappa'\left(\frac{\rho+3\nu}{4}\right)\right|\right].
\end{aligned}$$

3. Conclusions

In the introduction part, a historical background in the field of inequality theory and fractional analysis is presented, and in the main results part, a new integral equation is produced based on fractional integral operators. Then, new Hermite-Hadamard type inequalities are obtained by using various auxiliary inequalities for functions whose absolute values of derivatives are convex and concave. In the main findings, it is emphasized that the results obtained are general versions of classical integral inequalities, considering the particular case of the parameter such as $\alpha = 1$. The special cases of the main theorms can be applied to numerical integration to give new approaches for error estimation of the mid-point and trapezoidal formula.

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