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Series with Commuting Terms in Topologized Semigroups

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Abstract: We show that the following general version of the Riemann–Dirichlet theorem is true: if every rearrangement of a series with pairwise commuting terms in a Hausdorff topologized semigroup converges, then its sum range is a singleton.

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1. Introduction

In 1827, Peter Lejeune-Dirichlet was the first to notice that it is possible to rearrange the terms of certain convergent series of real numbers so that the sum changes [1]. According to [2] (Ch. 2, §2.4), In 1833, Augustin-Louis Cauchy also noticed this in his "Resumes analytiques".

Later, in 1837, Dirichlet showed that this cannot happen if the series converges absolutely: if a series formed by absolute values of a term of series of real numbers converges, then the series itself converges and every rearrangement also converges to the same sum. A series in which every rearrangement converges is called unconditionally convergent. Let us define the sum range of series as the set of all sums of all its convergent rearrangements.

It is not clear in advance that an unconditionally convergent series of real numbers is also absolutely convergent, and hence its sum range is a singleton. This is in fact true thanks to the following *Riemann rearrangement theorem*: if a convergent series of real numbers **is not** absolutely convergent, then some rearrangement **is not** convergent, and its sum range is the set of **all real numbers**.

These results depend heavily on the structure of the set of real numbers. However, the concepts of unconditional convergence and sum range make sense even in general topologized semigroups. An abelian version of the statement in the abstract appears in (unpublished) [3]. A non-abelian version for topological groups appears in [4].

Section 2 focuses on 'finite series' and Section 3 treats the general case. Section 4 contains additional comments.

2. Algebraic Part

We write $\mathbb{N}$ for the set $\{1, 2, \dots\}$ of natural numbers with its usual order and

$$\mathbb{N}_n := \{k \in \mathbb{N} : k \leq n\}, \ n = 1, 2, \dots$$

A non-empty set, X, endowed with a binary operation $+ : X \times X \to X$ is called a groupoid or a magma. For a groupoid, $(X, +)$, the value of $+$ at $(x_1, x_2) \in X \times X$ will be denoted as $x_1 + x_2$.

For a finite non-empty $I \subset \mathbb{N}$ and a family $(x_i)_{i\in I}$ of elements of a groupoid $(X,+)$, following Bourbaki, we define the (ordered) sum

$$\sum_{i\in I} x_i \in X \qquad \text{(OS)}$$

inductively as follows:

(1) If I consists of a single element, $I = \{j\}$, then $\sum_{i\in I} x_i = x_j$;

(2) If I has more than one element, j is the least element of I and $I' = I \setminus \{j\}$, then

$$\sum_{i\in I} x_i = x_j + \left(\sum_{i\in I'} x_i\right).$$

Note that:

If I consists of two elements, then $\sum_{i\in I} x_i = x_j + x_k$, where j is the least element of I and k is the last element of I;

If I consists of three elements, then $\sum_{i\in I} x_i = x_j + (x_m + x_k)$, where again, j is the least element of I, k is the last element of I and $j < m < k$.

If $I = \mathbb{N}_n$, then instead of $\sum_{i\in I} x_i$ we write also $\sum_{i=1}^{n} x_i$.

A groupoid, $(X,+)$, is a *semigroup* if its binary operation $+$ is associative, i.e., for every $(x_1, x_2, x_3) \in X \times X \times X$ we have $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.

For a finite non-empty $I \subset \mathbb{N}$ and a family $(x_i)_{i\in I}$ of elements of a *semigroup* $(X,+)$ the above given definition of (OS) can be reformulated as follows:

(1r) if I consists of a single element, $I = \{k\}$, then $\sum_{i\in I} x_i = x_k$,

(2r) if I has more than one element, k is the last element of I and $I' = I \setminus \{k\}$, then

$$\sum_{i\in I} x_i = \left(\sum_{i\in I'} x_i\right) + x_k .$$

For a set I a bijection $\sigma : I \to I$ called a permutation of I; the set of all permutations of I is denoted by $\mathbb{S}(I)$.

For a finite non-empty $I \subset \mathbb{N}$ and a family $(x_i)_{i\in I}$ of elements of a groupoid $(X,+)$,we define its *sum range*

$$SR((x_i)_{i\in I})$$

as follows:

$$SR((x_i)_{i\in I}) := \{s \in X : \exists\sigma \in \mathbb{S}(I),\ s = \sum_{i\in I} x_{\sigma(i)}\} .$$

In a case where the multiplicative notation $\cdot$ is applied for the binary operation, it would be natural to use the word 'product' instead of 'sum'; 'ordered product' (OP) *instead of 'ordered sum'* (OS)*; 'product range'* (PR) *instead 'sum range'* (SR) *and $\prod$ instead of $\sum$.*

Two elements, x_1 and x_2, of a groupoid, $(X,+)$, are said to commute (or to be permutable) if $x_1 + x_2 = x_2 + x_1$; i.e., if $SR\big((x_i)_{i\in\mathbb{N}_2}\big)$ is a singleton.

A family $(x_i)_{i\in I}$ of elements of a groupoid $(X,+)$ is *commuting* if for each $i \in I$ and $j \in I$, the elements x_i and x_j commute.

An element a of a groupoid $(X,+)$ is *left cancellable* if the left translation mapping $x \mapsto a + x$ is injective; *right cancellable* is defined similarly. An element is *cancellable* if it is both left and right cancellable.

Theorem 1 (Commutativity theorem)**.** *For a finite non-empty $I \subset \mathbb{N}$ and a family $(x_i)_{i\in I}$ of elements of a semigroup $(X,+)$ the following statements are true.*

(a) If $(x_i)_{i\in I}$ is a **commuting** *family, then $SR((x_i)_{i\in I})$ is a singleton.*

(b) If $SR((x_i)_{i\in I})$ is a singleton and either $\mathrm{Card(I)} \leq 2$ *or for every $i \in I$ the element x_i is right (resp. left) cancellable, then $(x_i)_{i\in I}$ is a* **commuting** *family.*

Proof. (a) See [5] [Ch.1, §1.5, Theorem 2 (p. 9)].

(b) For the case $\mathrm{Card}(\mathrm{I}) \leq 2$ the statement is evident. Now, let $n = \mathrm{Card}(\mathrm{I}) > 2$ and for every $i \in I$ the element x_i is right cancellable. Fix $i, j \in I,\ i \neq j$, write $I'' = I \setminus \{i, j\}$. Also write $I = \{k_1, k_2, \ldots, k_n\}$, where $k_1 < k_2 < \cdots < k_n$. Moreover, consider permutations σ and π of I such that $\sigma(k_1) = i$, $\sigma(k_2) = j$, $\sigma(\{k_3, \ldots, k_n\}) = I''$ and $\pi(k_1) = j$, $\pi(k_2) = i$, $\pi(\{k_3, \ldots, k_n\}) = I''$. As $SR((x_i)_{i \in I})$ is a singleton, we can write:

$$x_i + x_j + \left(\sum_{r \in I''} x_r \right) = \sum_{i \in I} x_{\sigma(i)} = \sum_{i \in I} x_{\pi(i)} = x_j + x_i + \left(\sum_{r \in I''} x_r \right).$$

From this equality, as $\sum_{r \in I''} x_r$ is right cancellable, we obtain $x_i + x_j = x_j + x_i$. The case where $\mathrm{Card}(\mathrm{I}) > 2$ and for every $i \in I$ the element x_i is left cancellable is considered similarly. $\square$

Our next claim is to find an analog of Theorem 1 when $I = \mathbb{N}$.

3. Series

A (formal) *series corresponding to a sequence* $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of elements of a groupoid $(X, +)$ is the sequence

$$\left(\sum_{k \in \mathbb{N}_n} x_k \right)_{n \in \mathbb{N}}. \tag{S1}$$

The 'multiplicative' counterpart is: a (formal) *infinite product corresponding to a sequence* $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of elements of a groupoid $(X, \cdot)$ is the sequence

$$\left(\prod_{k \in \mathbb{N}_n} x_k \right)_{n \in \mathbb{N}}. \tag{P1}$$

We use the additive notation herein.

Let $(X, +)$ be a groupoid and τ be a topology in X; such a triplet $(X, +, \tau)$ will be called *a topologized groupoid*.

A topologized groupoid $(X, +, \tau)$ is *a topological groupoid* if its binary operation $+$ is continuous as mapping from $(X \times X, \tau \otimes \tau)$ to (X, τ) (where $\tau \otimes \tau$ stands for the product topology).

A series corresponding to a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of elements of a topologized groupoid $(X, +, \tau)$ is said to be convergent in $(X, +, \tau)$ if the sequence (S1) converges to an element $s \in X$ in the topology τ; in such a case, we write

$$s = \sum_{k=1}^{\infty} x_k$$

and call s a sum of the series.

To a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of elements of a topologized groupoid $(X, +, \tau)$, we associate a subset $\mathfrak{P}(\mathbf{x})$ of $\mathbb{S}(\mathbb{N})$ as follows: a permutation $\pi : \mathbb{N} \to \mathbb{N}$ belongs to $\mathfrak{P}(\mathbf{x})$ if and only if the series corresponding to $(x_{\pi(n)})_{n \in \mathbb{N}}$ is convergent in $(X, +, \tau)$ and define *the sum range* of the series corresponding to $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$

$$SR(\mathbf{x})$$

as follows (cf. [6] (Definition 2.1.1)):

$$SR(\mathbf{x}) := \{ t \in X : \exists \pi \in \mathfrak{P}(\mathbf{x}),\ t = \sum_{k=1}^{\infty} x_{\pi(k)} \}.$$

It may happen that for a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ the set $\mathfrak{P}(\mathbf{x})$ is empty; in which case, $SR(\mathbf{x}) = \varnothing$ as well.

The series corresponding to $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is called *unconditionally convergent* (Bourbaki says *commutatively convergent* [7]) in $(X, +, \tau)$ if

$$\mathfrak{P}(\mathbf{x}) = \mathbb{S}(\mathbb{N})\,;$$

i.e., *if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$ the series corresponding to $\mathbf{x}_\sigma = (x_{\sigma(n)})_{n\in\mathbb{N}}$ is convergent in $(X, +, \tau)$.*

We proceed to our main result, extending to topologized semigroups the results for topological groups in [4] (Theorem 2 and Theorem 1).

Theorem 2 (Commutativity Theorem 2)**.** *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of a Hausdorff topologized semigroup $(X, +, \tau)$, the following statements are true.*

(a') If the series corresponding to $\mathbf{x}$ is convergent in $(X, +, \tau)$, $\mathbf{x}$ is a **commuting** *family and $SR(\mathbf{x})$* **is not** *a singleton, then there is a permutation $\lambda : \mathbb{N} \to \mathbb{N}$ such that the series corresponding to $\mathbf{x}_\lambda = (x_{\lambda(n)})_{n\in\mathbb{N}}$ is not convergent in $(X, +, \tau)$.*

(a) If the series corresponding to $\mathbf{x}$ is **unconditionally convergent** *in $(X, +, \tau)$ and $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is a* **commuting** *family, then $SR(\mathbf{x})$ is a singleton.*

(b) If $SR(\mathbf{x})$ is a singleton, $(X, +)$ is a group and for every $n \in \mathbb{N}$ the left translation determined by x_n is sequentially continuous, then $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is a **commuting** *family.*

Proof. (a').

To prove (a'), denote by s the limit in $(X, +, \tau)$ of the sequence (S1), i.e.,

$$(\tau)\lim_n \sum_{k\in\mathbb{N}_n} x_k = s\,. \tag{1}$$

Since $SR(\mathbf{x})$ **is not** a singleton, there is $t \in SR(\mathbf{x})$ such that $t \neq s$. Hence, there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the series corresponding to $\mathbf{x}_\pi = (x_{\pi(n)})_{n\in\mathbb{N}}$ is convergent to t in $(X, +, \tau)$, i.e.,

$$(\tau)\lim_n \sum_{k\in\mathbb{N}_n} x_{\pi(k)} = t\,. \tag{2}$$

Construction of a permutation $\lambda : \mathbb{N} \to \mathbb{N}$.

Find and fix a **strictly increasing** sequence of natural numbers $(m_k)_{k\in\mathbb{N}}$ such that

$$1 = m_1,\ \mathbb{N}_{m_{2k-1}} \subset \pi(\mathbb{N}_{m_{2k}}) \subset \mathbb{N}_{m_{2k+1}},\ k = 1, 2, \ldots \tag{3}$$

Now, define a mapping $\lambda : \mathbb{N} \to \mathbb{N}$ as follows:

$$\begin{aligned} &\lambda(1) = 1; \quad \lambda(\mathbb{N}_{m_{2k}} \setminus \mathbb{N}_{m_{2k-1}}) = \pi(\mathbb{N}_{m_{2k}}) \setminus \mathbb{N}_{m_{2k-1}}; \\ &\qquad\qquad \lambda(\mathbb{N}_{m_{2k+1}} \setminus \mathbb{N}_{m_{2k}}) = \mathbb{N}_{m_{2k+1}} \setminus \pi(\mathbb{N}_{m_{2k}}), \quad k = 1, 2, \ldots \end{aligned} \tag{4}$$

It is easy to see that $\lambda \in \mathbb{S}(\mathbb{N})$.

From (3) and (4), we can conclude that

$$\lambda(\mathbb{N}_{m_{2k+1}}) = \mathbb{N}_{m_{2k+1}}\,, k = 1, 2, \ldots \tag{5}$$

and

$$\lambda(\mathbb{N}_{m_{2k}}) = \pi(\mathbb{N}_{m_{2k}})\,, k = 1, 2, \ldots \tag{6}$$

From (5) and (6) together with Theorem 1(a) (which is applicable because $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is a commuting family), we conclude that the following relations are true:

$$\sum_{i=1}^{m_{2k+1}} x_{\lambda(i)} = \sum_{i=1}^{m_{2k+1}} x_i,\ k = 1, 2, \ldots \tag{7}$$

and
$$\sum_{i=1}^{m_{2k}} x_{\lambda(i)} = \sum_{i=1}^{m_{2k}} x_{\pi(i)}, \; k = 2,3,\ldots \tag{8}$$

The equality (7) implies:
$$\lim_k \sum_{i=1}^{m_{2k+1}} x_{\lambda(i)} = s\,, \tag{9}$$

while the equality (8) implies:
$$\lim_k \sum_{i=1}^{m_{2k}} x_{\lambda(i)} = t\,. \tag{10}$$

From (9) and (10), since $t \neq s$ and τ is a *Hausdorff topology*, we conclude that $(\sum_{i=1}^{n} x_{\lambda(i)})_{n\in\mathbb{N}}$ is not a convergent sequence. Therefore, we found a permutation $\lambda : \mathbb{N} \to \mathbb{N}$ such that the series corresponding to $\mathbf{x}_\lambda = (x_{\lambda(n)})_{n\in\mathbb{N}}$ is not convergent in $(X, +, \tau)$ and (a') is proved.

(a) follows from (a').

(b) In view of Theorem 1(b), it is sufficient to show that for a fixed natural number $n > 1$ we find that $SR((x_i)_{i\in\mathbb{N}_n})$ is a singleton.

We can suppose without loss of generality that the series corresponding to $\mathbf{x}$ is convergent in $(X, +, \tau)$ to $s \in X$. This implies:
$$\lim_{m>n} \left(\sum_{i\in\mathbb{N}_n} x_i + \sum_{i\in\mathbb{N}_m\setminus\mathbb{N}_n} x_i \right) = s\,.$$

From this, since the left translations are continuous, we obtain:
$$\lim_{m>n} \sum_{i\in\mathbb{N}_m\setminus\mathbb{N}_n} x_i = -\sum_{i\in\mathbb{N}_n} x_i + s\,.$$

Now, fix an arbitrary permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(k) = k$, $k = n+1, n+2, \ldots$ From the above equality, since the left translations are continuous, we can now write
$$\lim_{m>n} \left(\sum_{i\in\mathbb{N}_n} x_{\pi(i)} + \sum_{i\in\mathbb{N}_m\setminus\mathbb{N}_n} x_i \right) = \sum_{i\in\mathbb{N}_n} x_{\pi(i)} + (-\sum_{i\in\mathbb{N}_n} x_i + s)\,.$$

Hence, since $SR(\mathbf{x})$ is a singleton, we conclude:
$$\sum_{i\in\mathbb{N}_n} x_{\pi(i)} + (-\sum_{i\in\mathbb{N}_n} x_i + s) = s\,.$$

Therefore,
$$\sum_{i\in\mathbb{N}_n} x_{\pi(i)} = \sum_{i\in\mathbb{N}_n} x_i$$
and, as π is arbitrary, we prove that $SR((x_i)_{i\in\mathbb{N}_n})$ is a singleton. $\square$

Remark 1. *Theorem 2(a) for a Banach space was first proved in [8], where the term "B-space" was used and it was also noticed that this term is credited to M. Frechet. In [9], where the term 'Banach space' is already used, one finds a nice discussion of equivalent characterizations of unconditional convergence.*

4. Additional Comments

4.1. On Theorem 2

The statement (b) of Theorem 2 **is not** a complete converse of statement (a) of Theorem 2; in the case of Hausdorff topological groups, such a complete converse can be formulated as follows:

If for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *of elements of a Hausdorff topological group* X *the set* $SR(\mathbf{x})$ *is a singleton, then the series corresponding to* $\mathbf{x}$ *is* **unconditionally convergent** *in* X *and* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *is a* **commuting** *family.*

Let us say that a Hausdorff topological group X *has property* (HM) if whenever for a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ the set $SR(\mathbf{x})$ is a singleton, then the series corresponding to $\mathbf{x}$ is **unconditionally convergent** in X.

The Riemann rearrangement theorem implies that $X = \mathbb{R}$ *has property* (HM). In [10], it was shown that if X is an infinite-dimensional Hilbert space, then X **does not have** property (HM); a similar result was obtained in [11] for infinite-dimensional Banach spaces. From the general result of [12], we conclude that the finite-dimensional real normed spaces, as well as the countable product of real lines $\mathbb{R}^{\mathbb{N}}$, have property (HM).

4.2. On Sum Ranges

A subset A of a topological group X is *a sum range* if a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X exists such that $A = SR(\mathbf{x})$. Known results and the history of the study of the structure of sum ranges in Banach spaces are found in [6]; see also, [12–18].

A subset A of a real vector space X is called *affine* if

$$x_1 \in A,\ x_2 \in A,\ t \in \mathbb{R},\ \Longrightarrow\ tx_1 + (1-t)x_2 \in A\,.$$

It is known that:

- A subset of a finite-dimensional real Banach space is a sum range if and only if it is affine (Steinitz's theorem, see [6]);
- A subset of a real nuclear Frechet space is a sum range if and only if it is closed and affine [13];
- Every closed affine subset of a *separable* real Frechet space can be a sum range (cf. [19], where the following question is left open: *is every separable infinite-dimensional complete metrizable real topological vector space a sum range?*);
- An arbitrary *finite* subset of an *infinite-dimensional* Banach space can be a sum range [20];
- A *non-analytic* subset of an infinite-dimensional *separable* Banach space cannot be a sum range [21];
- A *non-closed* subset of an infinite-dimensional *separable* Banach space can be a sum range (see [6,22]; however, it is unknown whether a *non-closed vector subspace* of an infinite-dimensional *separable* Banach space can be a sum range [16]) .

Finally, note that it would be interesting to:

(1) Investigate, in connection with Theorem 2(a), the question of how rich the sum range $SR(\mathbf{x})$ can be for a **non-commuting** sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, the series corresponding to which is **unconditionally convergent**; may happen that $SR(\mathbf{x}) = X$?

(2) Find a "semigroup version" of Theorem 2(b).

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