






Article

Near-Common Fixed Point Result in Cone Interval b -Metric Spaces over Banach Algebras

Muhammad Sarwar ^{1,*} , Ziaul Islam ^{1,*} , Hijaz Ahmad ² , Hüseyin Işık ³  and Samad Noeiaghdam ^{4,5} ¹ Department of Mathematics, University of Malakand, Chakdara Dir(L) 18800, Pakistan² Department of Basic Sciences, University of Engineering and Technology, Peshawar 25000, Pakistan; hijaz555@gmail.com³ Department of Engineering Science, Bandırma Onyedi Eylül University, Bandırma 10200, Turkey; hisik@bandirma.edu.tr⁴ Industrial Mathematics Laboratory, Baikal School of BRICS, Irkutsk National Research Technical University, 664074 Irkutsk, Russia; snoei@istu.edu⁵ Department of Applied Mathematics and Programming, South Ural State University, Lenin Prospect 76, 454080 Chelyabinsk, Russia; noiagdams@susu.ru

* Correspondence: sarwar@uom.edu.pk (M.S.); ziaulislam173@gmail.com (Z.I.)

Abstract: In this article, we proposed the concept of cone interval b -metric space over Banach algebras. Furthermore, some near-fixed point and near-common fixed point results are proved in the context of cone interval b -metric space and normed interval spaces for self-mappings under different types of generalized contractions. An example is presented to validate our main outcome.

Keywords: cone interval b -metric spaces; near-fixed point; near-common fixed point; near-coincidence point



Citation: Sarwar, M.; Islam, Z.; Ahmad, H.; Işık, H.; Noeiaghdam, S. Near-Common Fixed Point Result in Cone Interval b -Metric Spaces over Banach Algebras. *Axioms* **2021**, *10*, 251. <https://doi.org/10.3390/axioms10040251>

Academic Editor: Chris Goodrich

Received: 1 August 2021

Accepted: 27 September 2021

Published: 9 October 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction and Preliminaries

Fixed point theory is without a doubt one of the most powerful techniques from nonlinear analysis. This provides a best method to study various problems in economics, science and mathematics; particularly the initial and boundary value problems involving ordinary, partial and fractional differential equations, difference equation, integral equations, functional equations, variational inequality, etc.; in particular, this provides the existence and uniqueness of solutions of such problems. If $T : X \rightarrow X$ is a mapping of a space X into itself, then the question arises whether some points are mapped onto itself, i.e., does the equation $T(x) = x$ have a solution? If so, x is called a fixed point of T . The Banach contraction principle was considered as the most fundamental result in metric fixed point theory. Several authors generalized the Banach contraction principle, and this generalization goes in two different directions or ways, as recorded below:

- i To generalize the contraction condition of Banach,
- ii Replacing complete metric space with some generalized metric space.

Some authors used both ways or directions to generalize the Banach contraction principle. In this paper, we make an attempt to cover both ways or directions to generalize the Banach contraction principle, i.e., we make an effort to generalize the contraction condition and also metric space and its axioms.

By swapping the real numbers with ordered Banach space, Huang and Zhang [1] generalized the notion of a metric space and demonstrated some fixed point results of contractive maps using the normality condition in such spaces. Rezapour and Hambarani [2] subsequently ignored the normality assumption and obtained some generalizations of the Huang and Zhang [1] results. However, it should be noted that in recent research, some scholars established an equivalence between cone metric spaces and metric spaces in the sense of the existence of fixed points of the mappings involved. See, for instance, [3–5]. Liu and Xu [6]

proposed the concept of a cone metric space over Banach algebra in order to solve these shortcomings by replacing Banach space with Banach algebra, which became an interesting discovery in the study of fixed point theory since it can be shown that cone metric spaces over Banach algebra are not equal to metric spaces in terms of the presence of the fixed points of mappings. Among these generalizations, Wu [7] introduced the concept of metric interval spaces and normed interval spaces as a generalization of metric spaces, and also studied some near-fixed point results in such spaces.

In 1973, Hardy and Rogers [8] proposed a new definition of mappings called the contraction of Hardy–Rogers that generalizes the theory of Banach contraction and the theorem of Reich [9] in metric space setting. For other related work about the concept of Hardy–Rogers contractions, see for instance ([10–12] and the references therein).

We recollect certain essential notes, definitions required and primary results consistent with the literature.

Interval Space 1. [7]

Let I denote the set of all closed and bounded intervals $[\epsilon, \epsilon]$, where $\epsilon, \epsilon \in \mathbb{R}$ and $\epsilon \leq \epsilon$, we consider $\epsilon \in \mathbb{R}$ as the element $[\epsilon, \epsilon] \in I$. The addition is given by

$$[\epsilon, \epsilon] \oplus [\rho, \rho] = [\epsilon + \rho, \epsilon + \rho]$$

and the scalar multiplication is calculated as follows:

$$k[\epsilon, \epsilon] = \begin{cases} [k\epsilon, k\epsilon], & \text{if } k \geq 0 \\ [k\epsilon, k\epsilon], & \text{if } k < 0. \end{cases}$$

It is evident that I is not a (conventional) vector space under the aforementioned addition and scalar multiplication. The main reason for this is that, as the following explanation will show, the inverse element does not exist for any non-degenerated closed interval.

It is clear to see that the zero element is $[0, 0] \in I$. However, for any $[\epsilon, \epsilon] \in I$, the subtraction

$$[\epsilon, \epsilon] \ominus [\epsilon, \epsilon] = [\epsilon, \epsilon] \oplus [-\epsilon, -\epsilon] = [\epsilon - \epsilon, \epsilon - \epsilon] = [\epsilon - \epsilon, -(\epsilon - \epsilon)]$$

is not a zero element. In other words, there is no inverse element of $[\epsilon, \epsilon]$.

By considering the zero element, the null set is defined as follows:

$$\Omega = \{[\epsilon, \epsilon] \ominus [\epsilon, \epsilon] : [\epsilon, \epsilon] \in I\}.$$

It is easy to see that

$$\Omega = \{-k, k\} : k \geq 0\}.$$

It may also be demonstrated that $[-1, 1]$ generates Ω via non-negative scalar multiplication, as demonstrated below:

$$\Omega = \{k[-1, 1] : k \geq 0\}.$$

We call $[-1, 1]$ a generator of the null set Ω in this situation. The following observations in interval space are noteworthy.

- The distributive law for scalar addition does not hold in general; that is to say,

$$(\alpha + \beta)[\epsilon, \epsilon] \neq \alpha[\epsilon, \epsilon] \oplus \beta[\epsilon, \epsilon]$$

for any $[\epsilon, \epsilon] \in I$ and $\alpha, \beta \in \mathbb{R}$.

- For positive scalar addition, the distributive law is true; that is,

$$(\alpha + \beta)[\epsilon, \epsilon] = \alpha[\epsilon, \epsilon] \oplus \beta[\epsilon, \epsilon]$$

for any $[\epsilon, \epsilon] \in I$ and $\alpha, \beta > 0$.

- For negative scalar addition, the distributive law is valid; that is,

$$(\alpha + \beta)[\epsilon, \epsilon] = \alpha[\epsilon, \epsilon] \oplus \beta[\epsilon, \epsilon]$$

for any $[\epsilon, \epsilon] \in I$ and $\alpha, \beta < 0$.

- For any $[\epsilon, \epsilon], [\rho, \rho], [\sigma, \sigma] \in I$, we have

$$[\sigma, \sigma] \ominus ([\epsilon, \epsilon] \oplus [\rho, \rho]) = [\sigma, \sigma] \ominus [\epsilon, \epsilon] \ominus [\rho, \rho] = [\sigma, \sigma] \oplus (-[\epsilon, \epsilon]) \oplus (-[\rho, \rho]). \tag{1}$$

- We write $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$ if and only if there exists $\omega_1, \omega_2 \in \Omega$ such that

$$[\epsilon, \epsilon] \oplus \omega_1 = [\rho, \rho] \oplus \omega_2.$$

It is clear to see that $[\epsilon, \epsilon] = [\rho, \rho]$ implies $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$ by taking $\omega_1 = \omega_2 = [0, 0]$. The reverse, on the other hand, is not true. According to the binary relation $\stackrel{\Omega}{=}$, for any $[\epsilon, \epsilon] \in I$, the class $\langle [\epsilon, \epsilon] \rangle$ is defined as

$$\langle [\epsilon, \epsilon] \rangle = \{[\rho, \rho] \in I : [\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]\}. \tag{2}$$

The family of all classes $\langle [\epsilon, \epsilon] \rangle$ for $[\epsilon, \epsilon] \in I$ is denoted by $\langle I \rangle$.

Proposition 1. $[7] \stackrel{\Omega}{=}$ is a binary equivalence relation.

According to the preceding Proposition, the classes defined in (2) create the equivalence classes. The family $\langle I \rangle$ is referred to as the quotient set of I in this case. It is also seen that $[\rho, \rho] \in \langle [\epsilon, \epsilon] \rangle$ implies $\langle [\epsilon, \epsilon] \rangle = \langle [\rho, \rho] \rangle$. In other words, the whole set I is partitioned by the family of all equivalence classes.

Metric Interval Space 1. [7]

Let I denote the set of all closed and bounded intervals in \mathbb{R} with the null set Ω , and d denote a mapping from $I \times I$ to non-negative real numbers that fulfills the following axioms:

- i. $d([\epsilon, \epsilon], [\rho, \rho]) = 0$ if and only if $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$ for all $[\epsilon, \epsilon], [\rho, \rho] \in I$;
- ii. $d([\epsilon, \epsilon], [\rho, \rho]) = d([\rho, \rho], [\epsilon, \epsilon])$ for all $[\epsilon, \epsilon], [\rho, \rho] \in I$;
- iii. $d([\epsilon, \epsilon], [\sigma, \sigma]) \leq d([\epsilon, \epsilon], [\rho, \rho]) + d([\rho, \rho], [\sigma, \sigma])$ for all $[\epsilon, \epsilon], [\rho, \rho], [\sigma, \sigma] \in I$.

If the following condition iv is satisfied, then we say that d satisfies the null equalities:

- iv. for any $\omega_1, \omega_2 \in \Omega$ and $[\epsilon, \epsilon], [\rho, \rho] \in I$, the following holds true:
 - $d([\epsilon, \epsilon] \oplus \omega_1, [\rho, \rho] \oplus \omega_2) = d([\epsilon, \epsilon], [\rho, \rho])$;
 - $d([\epsilon, \epsilon] \oplus \omega_1, [\rho, \rho]) = d([\epsilon, \epsilon], [\rho, \rho])$;
 - $d([\epsilon, \epsilon], [\rho, \rho] \oplus \omega_2) = d([\epsilon, \epsilon], [\rho, \rho])$.

Example 1. [7] Let I be the set of all closed bounded intervals in \mathbb{R} , and d denote the function from $I \times I$ to \mathbb{R}^+ provided by

$$d([\epsilon, \epsilon], [\rho, \rho]) = |(\epsilon + \epsilon) - (\rho + \rho)|.$$

Then, (I, d) is a metric interval space in which d fulfills the null equalities.

Normed Interval Space 1. [7] Given a non-negative real-valued function $\|\cdot\| : I \rightarrow \mathbb{R}^+$, then we say that $(I, \|\cdot\|)$ is normed interval space if the following conditions hold true:

- i $\|\alpha[\epsilon, \epsilon]\| = |\alpha| \cdot \|[\epsilon, \epsilon]\|$ for any $[\epsilon, \epsilon] \in I$ and $\alpha \in \mathbb{F}$;
- ii $\|[\epsilon, \epsilon] \oplus [\rho, \rho]\| \leq \|[\epsilon, \epsilon]\| + \|[\rho, \rho]\|$ for any $[\epsilon, \epsilon], [\rho, \rho] \in I$;
- iii $\|[\epsilon, \epsilon]\| = 0$ implies $[\epsilon, \epsilon] \in \Omega$.

We say that $\|\cdot\|$ satisfies the null equality if and only if $\|[\epsilon, \epsilon] \oplus \omega\| = \|[\epsilon, \epsilon]\|$ for any $[\epsilon, \epsilon] \in I$ and $\omega \in \Omega$.

Example 2. [7] Let us define a non-negative real-valued function $\|\cdot\|$ on I by

$$\|[\epsilon, \epsilon]\| = |\epsilon + \epsilon|.$$

Then, $(I, \|\cdot\|)$ is a normed interval space in which the null equality is satisfied by the norm $\|\cdot\|$.

Proposition 2. [7] Given the normed interval space $(I, \|\cdot\|)$ such that $\|\cdot\|$ satisfy $\|[\epsilon, \epsilon]\| \leq \|[\epsilon, \epsilon] \oplus \omega\|$ for any $[\epsilon, \epsilon] \in I$ and $\omega \in \Omega$. For any $[\epsilon, \epsilon], [\sigma, \zeta], [\rho_1, \rho_1], \dots, [\rho_n, \rho_n] \in I$, we have

$$\begin{aligned} \|[\epsilon, \epsilon] \ominus [\sigma, \zeta]\| &\leq \\ &\|[\epsilon, \epsilon] \ominus [\rho_1, \rho_1]\| + \|[\rho_1, \rho_1] \ominus [\rho_2, \rho_2]\| + \dots + \|[\rho_m, \rho_m] \ominus [\rho_{m+1}, \rho_{m+1}]\| \\ &+ \dots + \|[\rho_n, \rho_n] \ominus [\sigma, \zeta]\|. \end{aligned}$$

Proposition 3. [7] The following hold true.

- i Given the normed interval space $(I, \|\cdot\|)$ such that $\|\cdot\|$ satisfies the null equality. For any $[\epsilon, \epsilon], [\rho, \rho] \in I$, if $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$, then $\|[\epsilon, \epsilon]\| = \|[\rho, \rho]\|$.
- ii Given the normed interval space $(I, \|\cdot\|)$. For any $[\epsilon, \epsilon], [\rho, \rho] \in I$, $\|[\epsilon, \epsilon] \ominus [\rho, \rho]\| = 0$ implies $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$.

The concept of convergence and Cauchy sequence in normed interval spaces is given below.

Given the normed interval space $(I, \|\cdot\|)$. Given a sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^{\infty}$ in I , it is clear that

$$\|[\epsilon_n, \epsilon_n] \ominus [\epsilon, \epsilon]\| = \|[\epsilon, \epsilon] \ominus [\epsilon_n, \epsilon_n]\|.$$

Definition 1. [7] Given the normed interval space $(I, \|\cdot\|)$.

- i A sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^{\infty}$ in I is said to be convergent to $[\epsilon, \epsilon] \in I$ if

$$\lim_{n \rightarrow \infty} \|[\epsilon_n, \epsilon_n] \ominus [\epsilon, \epsilon]\| = \lim_{n \rightarrow \infty} \|[\epsilon, \epsilon] \ominus [\epsilon_n, \epsilon_n]\| = 0.$$

- ii If the sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^{\infty}$ in I converges to some $[\epsilon, \epsilon] \in I$, then the equivalence class $\langle [\epsilon, \epsilon] \rangle$ is called the class limit of $\{[\epsilon_n, \epsilon_n]\}_{n=1}^{\infty}$, that is,

$$\lim_{n \rightarrow \infty} [\epsilon_n, \epsilon_n] = \langle [\epsilon, \epsilon] \rangle \text{ or } [\epsilon_n, \epsilon_n] \rightarrow \langle [\epsilon, \epsilon] \rangle.$$

- iii A sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^{\infty} \subset I$ is Cauchy sequence if, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$\|[\epsilon_n, \epsilon_n] \ominus [\epsilon_m, \epsilon_m]\| = \|[\epsilon_m, \epsilon_m] \ominus [\epsilon_n, \epsilon_n]\| < \delta$$

for all $n, m > N$ with $n \neq m$. If every Cauchy sequence in I is convergent, then I is complete.

- iv If I is complete, then it is also called a Banach interval space.

Definition 2. [13] Consider \mathcal{B} a real Banach algebra, and multiplication operation is defined with the following properties: (for all $\alpha, \beta, \gamma \in \mathcal{B}$, $\eta \in \mathbb{R}$)

- (a₁). $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;
- (a₂). $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ and $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$;
- (a₃). $\eta(\alpha\beta) = (\eta\alpha)\beta = \alpha(\eta\beta)$;
- (a₄). $\|\alpha\beta\| \leq \|\alpha\| \|\beta\|$.

Unless otherwise stated, we will assume in this article that \mathcal{B} is a real Banach algebra. If $e \in \mathcal{B}$ occurs, we call e the unit of \mathcal{B} , so that $e\epsilon = \epsilon e = \epsilon$. We call \mathcal{B} a unital in this case. If an inverse element $\epsilon \in \mathcal{B}$ exists, the element $\epsilon \in \mathcal{B}$ is said to be invertible, so that $\epsilon\epsilon = \epsilon\epsilon = e$. The inverse of ϵ in such case is unique and is denoted by ϵ^{-1} . We require the following propositions in the sequel.

Lemma 1. [13] Consider the unital Banach algebra \mathcal{B} with unit e and let $\epsilon \in \mathcal{B}$ be an arbitrary element. Then, $\lim_{n \rightarrow \infty} \|\epsilon^n\|^{\frac{1}{n}}$ exists, and the spectral radius satisfies

$$r(\epsilon) = \lim_{n \rightarrow \infty} \|\epsilon^n\|^{\frac{1}{n}} = \inf \|\epsilon^n\|^{\frac{1}{n}}.$$

If $r(\epsilon) < |\lambda|$, then $\lambda e - \epsilon$ is invertible. In fact

$$(\lambda e - \epsilon)^{-1} = \sum_{k=1}^{\infty} \frac{\epsilon^k}{\lambda^{k+1}},$$

where λ is a complex constant.

Lemma 2. [14] Let \mathcal{B} be a Banach algebra with a unit e and $\epsilon \in \mathcal{B}$. If λ is a complex constant and $r(\epsilon) < |\lambda|$, then

$$r((\lambda e - \epsilon)^{-1}) \leq \frac{1}{|\lambda| - r(\epsilon)}.$$

Remark 1. [15] If $r(\epsilon) < 1$ then $\|\epsilon^n\| \rightarrow 0$ as $(n \rightarrow \infty)$.

Lemma 3. [15] Let \mathcal{B} be Banach algebra, e their unit element and $\epsilon, \varepsilon \in \mathcal{B}$. If ϵ, ε commutes, then

- (f₁). $r(\epsilon + \varepsilon) \leq r(\epsilon) + r(\varepsilon)$.
- (f₂). $r(\epsilon\varepsilon) \leq r(\epsilon)r(\varepsilon)$.

Definition 3. [6] Consider the Banach algebra \mathcal{B} with unit element e , zero element θ and $\mathcal{C} \neq \emptyset$. Then, $\mathcal{C} \subset \mathcal{B}$ is cone in \mathcal{B} if:

- (b₁). $e \in \mathcal{C}$;
- (b₂). $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$;
- (b₃). $\lambda\mathcal{C} \subset \mathcal{C}$ for all $\lambda \geq 0$;
- (b₄). $\mathcal{C}\mathcal{C} \subset \mathcal{C}$;
- (b₅). $\mathcal{C} \cap (-\mathcal{C}) = \{\theta\}$.

Defining partial order relation \preceq in \mathcal{B} w.r.t \mathcal{C} by $\epsilon \preceq \varepsilon$ if and only if $\varepsilon - \epsilon \in \mathcal{C}$ also $\epsilon \prec \varepsilon$ if $\epsilon \preceq \varepsilon$, but $\epsilon \neq \varepsilon$ while $\epsilon \ll \varepsilon$ stands for $\varepsilon - \epsilon \in \text{int}\mathcal{C}$, where $\text{int}\mathcal{C}$ is the interior of \mathcal{C} . \mathcal{C} is solid if $\text{int}\mathcal{C} \neq \emptyset$.

If there is $M > 0$ such that for all $\epsilon, \varepsilon \in \mathcal{C}$, we have

$$\theta \preceq \epsilon \preceq \varepsilon \text{ implies } \|\epsilon\| \leq M\|\varepsilon\|$$

then \mathcal{C} is normal. If M is least and positive in the above, then it is a normal constant of \mathcal{C} [1].

Definition 4. [1,6] Let mapping $d : \mathfrak{N} \times \mathfrak{N} \rightarrow \mathcal{B}$ and $\mathfrak{N} \neq \emptyset$:

- (c₁). for all $\epsilon, \varepsilon \in \mathfrak{N}$, $d(\epsilon, \varepsilon) \succeq \theta$ and $d(\epsilon, \varepsilon) = \theta$ if and only if $\epsilon = \varepsilon$,
- (c₂). for all $\epsilon, \varepsilon \in \mathfrak{N}$, $d(\epsilon, \varepsilon) = d(\varepsilon, \epsilon)$,
- (c₃). for all $\epsilon, \varepsilon, \zeta \in \mathfrak{N}$, $d(\epsilon, \zeta) \preceq d(\epsilon, \varepsilon) + d(\varepsilon, \zeta)$.

Then, (\mathfrak{N}, d) over Banach algebra \mathcal{B} with cone metric d is a cone metric space.

In [16], over Banach algebra with constant $b \geq 1$ the cone b -metric space is introduced as a generalization of cone metric space over Banach algebra.

Definition 5. [16] Let mapping $d : \mathfrak{N} \times \mathfrak{N} \rightarrow \mathcal{B}$ and $\mathfrak{N} \neq \emptyset$:

- (e₁). for all $\epsilon, \varepsilon \in \mathfrak{N}$, $\theta \preceq d(\epsilon, \varepsilon)$ and $d(\epsilon, \varepsilon) = \theta$ if and only if $\epsilon = \varepsilon$,
- (e₂). for all $\epsilon, \varepsilon \in \mathfrak{N}$, $d(\epsilon, \varepsilon) = d(\varepsilon, \epsilon)$,
- (e₃). there is $b \in \mathcal{C}$, $b \geq 1$ and for all $\epsilon, \varepsilon, \zeta \in \mathfrak{N}$, $d(\epsilon, \zeta) \preceq b[d(\epsilon, \varepsilon) + d(\varepsilon, \zeta)]$.

Then, (\mathbb{N}, d) over Banach algebra \mathcal{B} with cone b -metric d is cone b -metric space. Note that if we take $b = 1$, then it reduces to cone metric space over Banach algebra \mathcal{B} .

Definition 6. [17] Let a sequence $\{\epsilon_n\}$ be in \mathcal{B} , then sequence $\{\epsilon_n\}$ is c -sequence, if for each $c \gg \theta$ there is $N \in \mathbb{N}$ such that $\epsilon_n \ll c$ for all $n > N$.

Lemma 4. [18] Consider the Banach algebra \mathcal{B} and $\text{int}\mathcal{C} \neq \emptyset$. Furthermore, consider $\{\epsilon_n\}$ a c -sequence in \mathcal{B} and $k \in \mathcal{C}$ where k is arbitrary, then $\{k\epsilon_n\}$ is a c -sequence.

Lemma 5. [14] Consider the Banach algebra \mathcal{B} , e their unit element and $\mathcal{C} \neq \emptyset$. Let $L \in \mathcal{B}$ and $\epsilon_n = L^n$. If $r(L) < 1$, then $\{\epsilon_n\}$ is a c -sequence.

Lemma 6. [18] Consider the Banach algebra \mathcal{B} and $\text{int}\mathcal{C} \neq \emptyset$. Let $\{\epsilon_n\}$ and $\{\zeta_n\}$ be c -sequences in \mathcal{B} . Then, for arbitrary $\eta, \zeta \in \mathcal{C}$ we have $\{\eta\epsilon_n + \zeta\zeta_n\}$ is also a c -sequence.

Lemma 7. [18] Consider the Banach algebra \mathcal{B} and $\text{int}\mathcal{C} \neq \emptyset$. Let $\{\epsilon_n\} \subset \mathcal{C}$ such that $\|\epsilon_n\| \rightarrow 0$ as $n \rightarrow \infty$. Then, $\{\epsilon_n\}$ is a c -sequence.

Lemma 8. [19] Let $\mathcal{C} \subset \mathcal{B}$ be a cone.

- (l₁). If $\epsilon, \epsilon \in \mathcal{B}$, $k \in \mathcal{C}$ and $\epsilon \preceq \epsilon$, then $k\epsilon \preceq k\epsilon$.
- (l₂). If $\epsilon, \epsilon \in \mathcal{C}$, $r(\epsilon) < 1$ and $\epsilon \preceq \epsilon\epsilon$, then $\epsilon = \theta$.
- (l₃). For any $n \in \mathbb{N}$, $r(\epsilon^n) < 1$ with $\epsilon \in \mathcal{C}$ and $r(\epsilon) < 1$.

Lemma 9. [20] Consider the Banach algebra \mathcal{B} and $\text{int}\mathcal{C} \neq \emptyset$.

- (n₁). If $\epsilon, \epsilon, \zeta \in \mathcal{B}$ and $\epsilon \preceq \epsilon \ll \zeta$, then $\epsilon \ll \zeta$.
- (n₂). If $\epsilon \in \mathcal{C}$ and $\epsilon \ll c$ for $c \gg \theta$, then $\epsilon = \theta$.

2. Results and Discussion

In this section, we introduce the concept of so called cone interval b -metric space over Banach algebra in short $CIbMS$ over BA as a generalization of metric interval space. In the rest of the below discussion, we consider θ as the zero element of the Banach algebra \mathcal{B} , \mathcal{C} as the cone in \mathcal{B} and e as the unit element of \mathcal{B} .

Definition 7. Let I be the collection of all closed bounded intervals in \mathbb{R} , $s \geq 1$ a constant and \mathcal{B} with a non-normal cone \mathcal{C} as a Banach algebra. Suppose that the mapping $d : I \times I \rightarrow \mathcal{B}$ satisfies:

- i. $\theta \prec d([\epsilon, \epsilon], [\rho, \rho])$ for all $[\epsilon, \epsilon], [\rho, \rho] \in I$ with $[\epsilon, \epsilon] \stackrel{\Omega}{\neq} [\rho, \rho]$ and $d([\epsilon, \epsilon], [\rho, \rho]) = \theta$ if and only if $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$;
- ii. $d([\epsilon, \epsilon], [\rho, \rho]) = d([\rho, \rho], [\epsilon, \epsilon])$ for all $[\epsilon, \epsilon], [\rho, \rho] \in I$;
- iii. $d([\epsilon, \epsilon], [\rho, \rho]) \preceq s[d([\epsilon, \epsilon], [\sigma, \sigma]) + d([\sigma, \sigma], [\rho, \rho])]$ for all $[\epsilon, \epsilon], [\rho, \rho], [\sigma, \sigma] \in I$.

Then, the pair (I, d) is $CIbMS$ over BA \mathcal{B} with parameter $s \geq 1$. We claim that d satisfies the null equality if the following are satisfied:

- iv. for any $\omega_1, \omega_2 \in \Omega$ and $[\epsilon, \epsilon], [\rho, \rho] \in I$, the following equalities are satisfied:
 - $d([\epsilon, \epsilon] \oplus \omega_1, [\rho, \rho] \oplus \omega_2) = d([\epsilon, \epsilon], [\rho, \rho])$;
 - $d([\epsilon, \epsilon] \oplus \omega_1, [\rho, \rho]) = d([\epsilon, \epsilon], [\rho, \rho])$;
 - $d([\epsilon, \epsilon], [\rho, \rho] \oplus \omega_2) = d([\epsilon, \epsilon], [\rho, \rho])$.

Remark 2. In above definition, if we take the Banach algebra $\mathcal{B} = \mathbb{R}^+$, we can obtain an interval b -metric space ($IbMS$) with parameter $s \geq 1$. Furthermore, by taking $\mathcal{B} = \mathbb{R}^+$ and $s = 1$, then we can obtain metric interval space (MIS). By taking $s = 1$ in the above definition, we can obtain cone interval metric space ($CIMS$) over BA \mathcal{B} .

Remark 3. The class of CibMS over BA \mathcal{B} is larger than the class of MIS and class of (CIMS) over BA \mathcal{B} since the latter must be the former, but the converse is not true. We can present an example, as follows, which shows that introducing a CibMS over BA \mathcal{B} instead of a MIS and CIMS over BA \mathcal{B} is very meaningful since there exists CibMS over BA \mathcal{B} , which is not MIS and not a CIMS over BA \mathcal{B} .

Example 3. Let $\mathcal{B} = C[a, b]$ be the set of continuous functions on the closed interval $[a, b]$ with supremum norm. Define multiplication in the usual way. Then, \mathcal{B} is a Banach algebra with a unit 1. Set $\mathcal{C} = \{f \in \mathcal{B} : f(t) \geq 0, t \in [a, b]\}$ and $X = I$, where I is the collection of all closed bounded intervals in \mathbb{R} . Define a mapping $d : I \times I \rightarrow \mathcal{B}$ by

$$d([\epsilon, \epsilon], [\rho, \rho]) = |\epsilon + \epsilon - (\rho + \rho)|^p e^t$$

for all $[\epsilon, \epsilon], [\rho, \rho] \in I$, where $p > 1$ is a constant. Then, (I, d) is a CibMS over BA \mathcal{B} with parameter $s = 2^{p-1}$, but it is not a MIS nor CIMS over BA \mathcal{B} , since the triangle inequality does not hold true in both cases.

Proof. We consider the closed intervals $[\epsilon, \epsilon]$ and $[\rho, \rho]$. Then, it is easy to see that $\epsilon - \rho \leq \epsilon - \rho$.

- (i) Assume that $[\epsilon, \epsilon] \stackrel{\Omega}{\neq} [\rho, \rho]$, then we have $d([\epsilon, \epsilon], [\rho, \rho]) = |\epsilon + \epsilon - \rho - \rho|^p e^t$. As $\epsilon - \rho \leq \epsilon - \rho$. Therefore, if $\epsilon - \rho < 0$, then $|\epsilon + \epsilon - \rho - \rho|^p e^t \neq \theta$. However, by definition $|\epsilon + \epsilon - \rho - \rho|^p > 0$, therefore, $\theta \prec |\epsilon + \epsilon - \rho - \rho|^p e^t$, that is, $\theta \prec d([\epsilon, \epsilon], [\rho, \rho])$ for $[\epsilon, \epsilon] \stackrel{\Omega}{\neq} [\rho, \rho]$.
Suppose that

$$\theta = d([\epsilon, \epsilon], [\rho, \rho]) = |\epsilon + \epsilon - \rho - \rho|^p e^t.$$

We are going to claim that $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$.

As $|\epsilon + \epsilon - \rho - \rho|^p e^t = \theta$, but $e^t \neq \theta$, therefore, we must have $|\epsilon + \epsilon - \rho - \rho|^p = \theta$ and so we have $|\epsilon + \epsilon - \rho - \rho| = \theta$, which implies that $\epsilon + \epsilon = \rho + \rho$. Now we have that

$$\epsilon + \rho - \rho = 2\rho - \epsilon. \tag{3}$$

Furthermore, we have from $[\epsilon, \epsilon]$ and $[\rho, \rho]$ that $\epsilon \leq \epsilon$ and $\rho \leq \rho$, and so we have from these that

$$\epsilon + \rho - \rho \leq \epsilon + \rho - \rho. \tag{4}$$

Using (3) in (4), we have

$$2\rho - \epsilon \leq \epsilon + \rho - \rho. \tag{5}$$

From (4) and (5), we can form two identical intervals as

$$[\epsilon + \rho - \rho, \epsilon + \rho - \rho] = [2\rho - \epsilon, \epsilon + \rho - \rho].$$

Now, the closed intervals $[\epsilon + \rho - \rho, \epsilon + \rho - \rho]$ and $[2\rho - \epsilon, \epsilon + \rho - \rho]$ can be written as

$$[\epsilon + \rho - \rho, \epsilon + \rho - \rho] = [\epsilon, \epsilon] \oplus [\rho - \rho, \rho - \rho] \tag{6}$$

and

$$[2\rho - \epsilon, \epsilon + \rho - \rho] = [\rho, \rho] \oplus [\rho - \epsilon, \epsilon - \rho]. \tag{7}$$

Let

$$\omega_1 = [\rho - \rho, \rho - \rho] = (\rho - \rho)[-1, 1] \in \Omega$$

and

$$\omega_2 = [\rho - \epsilon, \epsilon - \rho] = (\epsilon - \rho)[-1, 1] \in \Omega.$$

Therefore, from (6) and (7), we obtain

$$[\epsilon, \epsilon] \oplus \omega_1 = [\rho, \rho] \oplus \omega_2,$$

which show that $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$, since $\omega_1, \omega_2 \in \Omega$. Conversely, suppose that $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\rho, \rho]$. Then, $[\epsilon, \epsilon] \oplus \omega_1 = [\rho, \rho] \oplus \omega_2$, where $\omega_1 = [-k_1, k_2], \omega_2 = [-k_2, k_2] \in \Omega$ for some positive k_1, k_2 . Therefore, we have

$$[\epsilon - k_1, \epsilon + k_1] = [\rho - k_2, \rho + k_2],$$

that is, $\epsilon - k_1 = \rho - k_2$ and $\epsilon + k_1 = \rho + k_2$. Then we obtain

$$d([\epsilon, \epsilon], [\rho, \rho]) = |(\epsilon - \rho) + (\epsilon - \rho)|^p e^t = |(k_1 - k_2) + (k_2 - k_1)|^p e^t = \theta.$$

(ii) We have

$$d([\epsilon, \epsilon], [\rho, \rho]) = |\epsilon + \epsilon - \rho - \rho|^p e^t = |\rho + \rho - \epsilon - \epsilon|^p e^t = d([\rho, \rho], [\epsilon, \epsilon]).$$

(iii) We have

$$\begin{aligned} d([\epsilon, \epsilon], [\rho, \rho]) &= |\epsilon + \epsilon - \rho - \rho|^p e^t \\ &= |(\epsilon + \epsilon - \sigma - \zeta) + (\sigma + \zeta - \rho - \rho)|^p e^t \\ &\leq s[|\epsilon + \epsilon - \sigma - \zeta|^p e^t + |\sigma + \zeta - \rho - \rho|^p e^t] \\ &= s[d([\epsilon, \epsilon], [\sigma, \zeta]) + d([\sigma, \zeta], [\rho, \rho])]. \end{aligned}$$

Note that for $[\epsilon, \epsilon] = [3, 5], [\rho, \rho] = [1, 2]$ and $[\sigma, \zeta] = [1, 5]$, we have the following:

$$\begin{aligned} d([3, 5], [1, 2]) &= |3 + 5 - 1 - 2|^p e^t = 5^p e^t, \\ d([3, 5], [1, 5]) &= |3 + 5 - 1 - 5|^p e^t = 2^p e^t, \\ d([1, 5], [1, 2]) &= |1 + 5 - 1 - 2|^p e^t = 3^p e^t. \end{aligned}$$

The above discussion show that the triangle inequality in MIS and CIMS over BA \mathcal{B} does not hold true, that is, for $p > 1$ we have $d([3, 5], [1, 2]) \not\leq d([3, 5], [1, 5]) + d([1, 5], [1, 2])$, which implies that $5^2 \not\leq 2^2 + 3^2$ for $p = 2$, that is, $13 - 25 = -12 \notin \mathcal{C}$, that implies that the triangle inequality does not hold true, but for the parameter $s = 2^{p-1}$ it is a CIBMS over BA \mathcal{B} .

(iv) Furthermore, d satisfies the null equality, that is, for any $[\epsilon, \epsilon], [\rho, \rho] \in I$ and $k_1, k_2 \in \mathbb{R}^+$, i.e., $[-k_1, k_1], [-k_2, k_2] \in \Omega$, we have

$$\begin{aligned} d([\epsilon, \epsilon] \oplus [-k_1, k_1], [\rho, \rho] \oplus [-k_2, k_2]) &= d([\epsilon - k_1, \epsilon + k_1], [\rho - k_2, \rho + k_2]) \\ &= |(\epsilon - k_1 + \epsilon + k_1) - (\rho - k_2 + \rho + k_2)|^p e^t \\ &= |(\epsilon + \epsilon) - (\rho + \rho)|^p e^t \\ &= d([\epsilon, \epsilon], [\rho, \rho]). \end{aligned}$$

That completes the verification. \square

Example 4. Let $\mathcal{B} = \{a = (a_{ij})_{3 \times 3} : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ and $\|a\| = \frac{1}{3} \sum_{1 \leq i, j \leq 3} |a_{ij}|$. Take a cone $\mathcal{C} = \{a \in \mathcal{B} : a_{ij} \geq 0, 1 \leq i, j \leq 3\}$ in \mathcal{B} . Let $I' = \{[\epsilon, \epsilon], [\epsilon, \epsilon], [\rho, \rho]\}$. Define a mapping $d : I' \times I' \rightarrow \mathcal{B}$ by $d([\epsilon, \epsilon], [\epsilon, \epsilon]) = d([\epsilon, \epsilon], [\epsilon, \epsilon]) = d([\rho, \rho], [\rho, \rho]) = \theta$ and

$$d([\epsilon, \epsilon], [\epsilon, \epsilon]) = d([\epsilon, \epsilon], [\epsilon, \epsilon]) = \begin{pmatrix} 1 & 1 & 4 \\ 4 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

$$d([\epsilon, \epsilon], [\rho, \rho]) = d([\rho, \rho], [\epsilon, \epsilon]) = \begin{pmatrix} 4 & 1 & 4 \\ 4 & 3 & 5 \\ 2 & 3 & 1 \end{pmatrix},$$

$$d([\epsilon, \epsilon], [\rho, \rho]) = d([\rho, \rho], [\epsilon, \epsilon]) = \begin{pmatrix} 9 & 5 & 6 \\ 16 & 4 & 4 \\ 3 & 4 & 2 \end{pmatrix}.$$

It is easy to verify that (I', d) is a *CIbMS* over the BA \mathcal{B} with parameter $s \geq \frac{5}{2}$, but it is not a *CIMS* nor *MIS* since $d([\epsilon, \epsilon], [\rho, \rho]) \not\leq d([\epsilon, \epsilon], [\epsilon, \epsilon]) + d([\epsilon, \epsilon], [\rho, \rho])$.

The concept of convergence, Cauchy sequence and completeness in *CIbMS* over BA \mathcal{B} is defined as follows:

Definition 8. Let (I, d) be a *CIbMS* over BA \mathcal{B} . The sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^\infty$ in I is said to be convergent if and only if

$$\lim_{n \rightarrow \infty} d([\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) = \theta \text{ for some } [\epsilon, \epsilon] \in I.$$

The element $[\epsilon, \epsilon]$ is called the limit of the sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^\infty$.
If there exists $[\rho, \rho] \in I$ such that

$$\lim_{n \rightarrow \infty} d([\epsilon_n, \epsilon_n], [\rho, \rho]) = \theta, \text{ then } [\rho, \rho] \in \langle [\epsilon, \epsilon] \rangle,$$

or

$$\lim_{n \rightarrow \infty} d([\epsilon_n, \epsilon_n], [\rho, \rho]) = \theta \text{ for all } [\rho, \rho] \in \langle [\epsilon, \epsilon] \rangle.$$

Definition 9. If the limit of the sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^\infty$ is $[\epsilon, \epsilon]$, then the class $\langle [\epsilon, \epsilon] \rangle$ is said to be the limit class for the sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^\infty$. We write

$$\lim_{n \rightarrow \infty} [\epsilon_n, \epsilon_n] = \langle [\epsilon, \epsilon] \rangle.$$

Definition 10. Consider the sequence $\{[\epsilon_n, \epsilon_n]\}_{n=1}^\infty$ in *CIbMS* (I, d) such that for any $c \gg \theta$, there exists a natural number N such that $d([\epsilon_n, \epsilon_n], [\epsilon_m, \epsilon_m]) \ll c$ for all $n, m > N$. Then, the sequence is called Cauchy sequence.

Definition 11. If every Cauchy sequence is convergent to a point in a subset M of the *CIbMS* (I, d) , then the subset M is said to be complete.

We shall investigate the so called near-fixed point, near-common fixed point and near-coincidence point in the setting of *CIbMS* over BA \mathcal{B} as follows.

Definition 12. Let T be a self-mapping of I into itself. A point $[\epsilon, \epsilon] \in I$ is a near-fixed point of T if and only if $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} [\epsilon, \epsilon]$.

Example 5. Let us consider a mapping $T : I \rightarrow I$ defined by

$$T([\epsilon, \epsilon]) = [\epsilon - 1, \epsilon + 1].$$

To show that $[\epsilon, \epsilon]$ is a near-fixed point of T , we have to show that $T([\epsilon, \epsilon]) = [\epsilon - 1, \epsilon + 1] \stackrel{\Omega}{=} [\epsilon, \epsilon]$. Since it is not difficult to verify that for $\omega_1 = [-1, 1], \omega_2 = [-2, 2] \in \Omega$ we have

$$\begin{aligned} [\epsilon - 1, \epsilon + 1] &\stackrel{\Omega}{=} [\epsilon, \epsilon] \\ \Leftrightarrow [\epsilon - 1, \epsilon + 1] \oplus [-1, 1] &= [\epsilon, \epsilon] \oplus [-2, 2] \\ \Leftrightarrow [\epsilon - 2, \epsilon + 2] &= [\epsilon - 2, \epsilon + 2], \end{aligned}$$

which shows that $[\epsilon, \epsilon]$ is a near-fixed point of T .

Definition 13. Let T, F be two self-mappings of I into itself. A point $[\epsilon, \epsilon] \in I$ is called a near-common fixed point of T and F if and only if $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} F([\epsilon, \epsilon]) \stackrel{\Omega}{=} [\epsilon, \epsilon]$.

Example 6. Let us consider the mappings $T, F : I \rightarrow I$ defined by

$$T([\epsilon, \epsilon]) = [\epsilon - 2, \epsilon + 2] \text{ and } F([\epsilon, \epsilon]) = [\epsilon - 1, \epsilon + 1].$$

To show that $[\epsilon, \epsilon]$ is a near-common fixed point of T and F , we have to show that $T([\epsilon, \epsilon]) = [\epsilon - 2, \epsilon + 2] \stackrel{\Omega}{=} [\epsilon, \epsilon]$, $F([\epsilon, \epsilon]) = [\epsilon - 1, \epsilon + 1] \stackrel{\Omega}{=} [\epsilon, \epsilon]$ and $T([\epsilon, \epsilon]) = [\epsilon - 2, \epsilon + 2] \stackrel{\Omega}{=} [\epsilon - 1, \epsilon + 1] = F([\epsilon, \epsilon])$. For $\omega_1 = [0, 0], \omega_2 = [-2, 2] \in \Omega$ we have

$$\begin{aligned} [\epsilon - 2, \epsilon + 2] &\stackrel{\Omega}{=} [\epsilon, \epsilon] \\ \Leftrightarrow [\epsilon - 2, \epsilon + 2] \oplus [0, 0] &= [\epsilon, \epsilon] \oplus [-2, 2] \\ \Leftrightarrow [\epsilon - 2, \epsilon + 2] &= [\epsilon - 2, \epsilon + 2], \end{aligned}$$

which shows that $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} [\epsilon, \epsilon]$. Similarly, for $\omega_1 = [0, 0], \omega_2 = [-1, 1] \in \Omega$, we have $F([\epsilon, \epsilon]) \stackrel{\Omega}{=} [\epsilon, \epsilon]$. Furthermore, as the relation $\stackrel{\Omega}{=}$ is an equivalence relation, therefore, $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} F([\epsilon, \epsilon]) \stackrel{\Omega}{=} [\epsilon, \epsilon]$. Hence $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} F([\epsilon, \epsilon]) \stackrel{\Omega}{=} [\epsilon, \epsilon]$. That is $[\epsilon, \epsilon]$ is the near-common fixed point of the mappings T and F .

Definition 14. Let $T, F : I \rightarrow I$ be two self mappings on I into itself. If $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} F([\epsilon, \epsilon]) \stackrel{\Omega}{=} [\rho, \rho]$ for $[\epsilon, \epsilon], [\rho, \rho] \in I$, then $[\rho, \rho]$ is called near-point of coincidence and $[\epsilon, \epsilon]$ is near-coincidence point for the mappings T and F .

Example 7. Let us consider the mappings $T, F : I \rightarrow I$ defined by

$$T([\epsilon, \epsilon]) = [\epsilon - 1, \epsilon + 1] \text{ and } F([\epsilon, \epsilon]) = [\epsilon - 2, \epsilon + 2].$$

We have to check that $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} F([\epsilon, \epsilon]) \stackrel{\Omega}{=} [\rho, \rho]$ for some $[\epsilon, \epsilon], [\rho, \rho] \in I$. Clearly, for $\omega_1 = [-1, 1], \omega_2 = [0, 0] \in \Omega$, we have

$$\begin{aligned} T([\epsilon, \epsilon]) &\stackrel{\Omega}{=} F([\epsilon, \epsilon]) \\ \Leftrightarrow [\epsilon - 1, \epsilon + 1] \oplus [-1, 1] &= [\epsilon - 2, \epsilon + 2] \oplus [0, 0] \\ \Leftrightarrow [\epsilon - 2, \epsilon + 2] &= [\epsilon - 2, \epsilon + 2], \end{aligned}$$

which shows that $[\epsilon, \epsilon]$ is a near-coincidence point of the mappings T and F . Particularly, for $[\epsilon, \epsilon] = [-1, 1]$ and $[\rho, \rho] = [-4, 4]$ we have $T([-1, 1]) = [-2, 2]$ and $F([-1, 1]) = [-3, 3]$. Clearly, $[-2, 2] \stackrel{\Omega}{=} [-4, 4]$ and $[-3, 3] \stackrel{\Omega}{=} [-4, 4]$, that is, $T([-1, 1]) \stackrel{\Omega}{=} F([-1, 1]) \stackrel{\Omega}{=} [-4, 4]$. Hence, $[-1, 1]$ is near-coincidence point of T and F , and $[-4, 4]$ is near-point of coincidence of T and F .

Definition 15. Let $T, F : I \rightarrow I$ be two self-mappings. We say that T and F are commuting at point $[\epsilon, \epsilon] \in I$ if $TF([\epsilon, \epsilon]) \stackrel{\Omega}{=} FT([\epsilon, \epsilon])$.

Example 8. Let us consider $T, F : I \rightarrow I$ defined by

$$T([\epsilon, \epsilon]) = [2\epsilon - 2, 2\epsilon + 2] \text{ and } F([\epsilon, \epsilon]) = [\epsilon - 1, \epsilon + 1] \text{ for } \epsilon, \epsilon \in \mathbb{R} \text{ with } \epsilon \leq \epsilon.$$

Then, for any $[\epsilon, \epsilon] \in I$, we have

$$TF([\epsilon, \epsilon]) = [2\epsilon - 4, 2\epsilon + 4] \text{ and } FT([\epsilon, \epsilon]) = [2\epsilon - 3, 2\epsilon + 3].$$

Now, to show that $TF([\epsilon, \epsilon]) \stackrel{\Omega}{=} FT([\epsilon, \epsilon])$, we have to show $[2\epsilon - 4, 2\epsilon + 4] \stackrel{\Omega}{=} [2\epsilon - 3, 2\epsilon + 3]$. For this, let us take $\omega_1 = [0, 0], \omega_2 = [-1, 1]$, then it is not easy to see that $[2\epsilon - 4, 2\epsilon + 4] \oplus [0, 0] = [2\epsilon - 3, 2\epsilon + 3] \oplus [-1, 1]$. Hence, we can say that the mappings T and F are commuting mappings.

Definition 16. Let $T, F : I \rightarrow I$ be two self-mappings. We say that T, F are said to be weakly compatible if they commute at their near-coincidence point. That is, for $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} F([\epsilon, \epsilon])$ we have $TF([\epsilon, \epsilon]) \stackrel{\Omega}{=} FT([\epsilon, \epsilon])$ for any $[\epsilon, \epsilon] \in I$.

Example 9. Let $T, F : I \rightarrow I$ be defined as:

$$T([\epsilon, \epsilon]) = [\epsilon - 1, \epsilon + 1] \text{ and } F([\epsilon, \epsilon]) = [\epsilon - 2, \epsilon + 2].$$

Clearly, by previous example, it is seen that $T([\epsilon, \epsilon]) \stackrel{\Omega}{=} F([\epsilon, \epsilon])$, that is, $[\epsilon, \epsilon]$ is a near-coincidence point of T and F . Now, we are showing that T and F are commuting at $[\epsilon, \epsilon]$. For this, we have to show that $TF([\epsilon, \epsilon]) \stackrel{\Omega}{=} FT([\epsilon, \epsilon])$. It is not difficult to verify that

$$TF([\epsilon, \epsilon]) = [\epsilon - 3, \epsilon + 3] \text{ and } FT([\epsilon, \epsilon]) = [\epsilon - 3, \epsilon + 3].$$

As $[\epsilon - 3, \epsilon + 3] = [\epsilon - 3, \epsilon + 3]$, we can have $[\epsilon - 3, \epsilon + 3] \stackrel{\Omega}{=} [\epsilon - 3, \epsilon + 3]$. Hence T and F are weakly compatible.

Lemma 10. Let the mappings T and F are weakly compatible self-maps of a set I . If T and F have a unique near-point of coincidence $[\rho, \rho] \stackrel{\Omega}{=} F([\epsilon, \epsilon]) \stackrel{\Omega}{=} T([\epsilon, \epsilon])$, then $[\rho, \rho]$ is the unique near-common fixed point of T and F .

Proof. Since $[\rho, \rho] \stackrel{\Omega}{=} F([\epsilon, \epsilon]) \stackrel{\Omega}{=} T([\epsilon, \epsilon])$ and T and F are weakly compatible, we have

$$T([\rho, \rho]) \stackrel{\Omega}{=} TF([\epsilon, \epsilon]) \stackrel{\Omega}{=} FT([\epsilon, \epsilon]) \stackrel{\Omega}{=} F([\rho, \rho]),$$

that is, $T([\rho, \rho]) \stackrel{\Omega}{=} F([\rho, \rho])$ is near-point of coincidence of T and F . However, $[\rho, \rho]$ is the unique near-point of coincidence of T and F , therefore we have $T([\rho, \rho]) \stackrel{\Omega}{=} F([\rho, \rho]) \stackrel{\Omega}{=} [\rho, \rho]$. Moreover, if $[\sigma, \sigma] \stackrel{\Omega}{=} F([\sigma, \sigma]) \stackrel{\Omega}{=} T([\sigma, \sigma])$, then $[\sigma, \sigma]$ is a near-point of coincidence of T and F , and therefore $[\sigma, \sigma] \stackrel{\Omega}{=} [\rho, \rho]$ by uniqueness. Thus, $[\rho, \rho]$ is a unique near-common fixed point of T and F . \square

Now, we are in the position to state and prove our first main theorems.

Theorem 1. Let (I, d) be a CIbMS over BA \mathcal{B} with the parameter $s \geq 1$ and \mathcal{C} be the underlying solid cone in \mathcal{B} . Let $a_i \in \mathcal{C} (i = 1, \dots, 5)$ be constants with $2sr(a_1) + (s + 1)r(a_2 + a_3 + sa_4 + sa_5) < 2$. Suppose that a_1 commutes with $a_2 + a_3 + sa_4 + sa_5$ and the mappings $T, F : I \rightarrow I$ satisfy that

$$d(T[\epsilon, \epsilon], T[\rho, \rho]) \preceq a_1d(F[\epsilon, \epsilon], F[\rho, \rho]) + a_2d(T[\epsilon, \epsilon], F[\epsilon, \epsilon]) + a_3d(T[\rho, \rho], F[\rho, \rho]) + a_4d(F[\epsilon, \epsilon], T[\rho, \rho]) + a_5d(T[\epsilon, \epsilon], F[\rho, \rho]) \tag{8}$$

for all $[\epsilon, \epsilon], [\rho, \rho] \in I$. If the range of F contains the range of T and $F(I)$ is a complete subspace, then T and F have a unique near-point of coincidence in I . Moreover, if T and F are weakly compatible, then T and F have a unique near-common fixed point.

Proof. Choose an initial element $[\epsilon_0, \epsilon_0] \in I$. Since $T(I) \subset F(I)$, there exists an $[\epsilon_1, \epsilon_1] \in I$ such that $T[\epsilon_0, \epsilon_0] = F[\epsilon_1, \epsilon_1]$. By induction, a sequence $\{T[\epsilon_n, \epsilon_n]\}$ can be chosen such that $T[\epsilon_n, \epsilon_n] = F[\epsilon_{n+1}, \epsilon_{n+1}] (n = 0, 1, 2, \dots)$. Thus, by (8), for any natural number n , we have

$$\begin{aligned} d(F[\epsilon_{n+1}, \epsilon_{n+1}], F[\epsilon_n, \epsilon_n]) &= d(T[\epsilon_n, \epsilon_n], T[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\preceq a_1 d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]) + a_2 d(T[\epsilon_n, \epsilon_n], F[\epsilon_n, \epsilon_n]) \\ &\quad + a_3 d(T[\epsilon_{n-1}, \epsilon_{n-1}], F[\epsilon_{n-1}, \epsilon_{n-1}]) + a_4 d(F[\epsilon_n, \epsilon_n], T[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\quad + a_5 d(T[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\preceq (a_1 + a_3 + sa_5) d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]) + (a_2 + sa_5) d(F[\epsilon_{n+1}, \epsilon_{n+1}], F[\epsilon_n, \epsilon_n]), \end{aligned}$$

which implies that

$$(e - a_2 - sa_5) d(F[\epsilon_{n+1}, \epsilon_{n+1}], F[\epsilon_n, \epsilon_n]) \preceq (a_1 + a_3 + sa_5) d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]). \tag{9}$$

On the other hand, we have

$$\begin{aligned} d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n+1}, \epsilon_{n+1}]) &= d(T[\epsilon_{n-1}, \epsilon_{n-1}], T[\epsilon_n, \epsilon_n]) \\ &\preceq a_1 d(F[\epsilon_{n-1}, \epsilon_{n-1}], F[\epsilon_n, \epsilon_n]) + a_2 d(T[\epsilon_{n-1}, \epsilon_{n-1}], F[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\quad + a_3 d(T[\epsilon_n, \epsilon_n], F[\epsilon_n, \epsilon_n]) + a_4 d(F[\epsilon_{n-1}, \epsilon_{n-1}], T[\epsilon_n, \epsilon_n]) \\ &\quad + a_5 d(T[\epsilon_{n-1}, \epsilon_{n-1}], F[\epsilon_n, \epsilon_n]) \\ &\preceq (a_1 + a_2 + sa_4) d(F[\epsilon_{n-1}, \epsilon_{n-1}], F[\epsilon_n, \epsilon_n]) + (a_3 + sa_4) d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n+1}, \epsilon_{n+1}]), \end{aligned}$$

which implies that

$$(e - a_3 - sa_4) d(F[\epsilon_{n+1}, \epsilon_{n+1}], F[\epsilon_n, \epsilon_n]) \preceq (a_1 + a_2 + sa_4) d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]). \tag{10}$$

Add up (9) and (10) yields that

$$\begin{aligned} (2e - a_2 - a_3 - sa_4 - sa_5) d(F[\epsilon_{n+1}, \epsilon_{n+1}], F[\epsilon_n, \epsilon_n]) &\preceq \\ &(2a_1 + a_2 + a_3 + sa_4 + sa_5) d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]). \end{aligned} \tag{11}$$

Denote $a_2 + a_3 + sa_4 + sa_5 = a$, then (11) becomes

$$(2e - a) d(F[\epsilon_{n+1}, \epsilon_{n+1}], F[\epsilon_n, \epsilon_n]) \preceq (2a_1 + a) d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]). \tag{12}$$

Note that for $s \geq 1$, we have

$$2r(a) \leq (s + 1)r(a) \leq 2sr(a_1) + (s + 1)r(a) < 2,$$

therefore $r(a) < 1 < 2$, then by Lemma 1 it follows that $2e - a$ is invertible. Furthermore,

$$(2e - a)^{-1} = \sum_{k=1}^{\infty} \frac{a^k}{2^{k+1}}.$$

Multiply both sides of (12) by $(2e - a)^{-1}$, we arrive at

$$d(F[\epsilon_{n+1}, \epsilon_{n+1}], F[\epsilon_n, \epsilon_n]) \preceq (2e - a)^{-1} (2a_1 + a) d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]). \tag{13}$$

Denote $B = (2e - a)^{-1}(2a_1 + a)$, then by (13) we get

$$\begin{aligned} d(F[\epsilon_{n+1}, \epsilon_{n+1}], F[\epsilon_n, \epsilon_n]) &\preceq Bd(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\preceq B^2d(F[\epsilon_{n-1}, \epsilon_{n-1}], F[\epsilon_{n-2}, \epsilon_{n-2}]) \\ &\vdots \\ &\preceq B^n d(F[\epsilon_1, \epsilon_1], F[\epsilon_0, \epsilon_0]) \\ &= B^n d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]). \end{aligned}$$

Since a_1 commutes with a , it follows that

$$\begin{aligned} (2e - a)^{-1}(2a_1 + a) &= \left(\sum_{k=0}^{\infty} \frac{a^k}{2^{k+1}}\right)(2a_1 + a) = 2\left(\sum_{k=0}^{\infty} \frac{a^k}{2^{k+1}}\right)a_1 + \sum_{k=0}^{\infty} \frac{a^{k+1}}{2^{k+1}} \\ &= 2a_1\left(\sum_{k=0}^{\infty} \frac{a^k}{2^{k+1}}\right) + a\sum_{k=0}^{\infty} \frac{a^k}{2^{k+1}} \\ &= (2a_1 + a)\sum_{k=0}^{\infty} \frac{a^k}{2^{k+1}} \\ &= (2a_1 + a)(2e - a)^{-1}, \end{aligned}$$

that is to say, $(2e - a)^{-1}$ commutes with $2a_1 + a$. Then, by Lemmas 2 and 3, we gain

$$\begin{aligned} r(B) &= r((2e - a)^{-1}(2a_1 + a)) \\ &\leq r((2e - a)^{-1})r(2a_1 + a) \\ &\leq \frac{1}{2 - r(a)}[2r(a_1) + r(a)] < \frac{1}{s}, \end{aligned}$$

which shows that $e - sB$ is invertible and also by using Remark 1 $\|B^n\| \rightarrow 0$ as $m \rightarrow \infty$. Hence, for any $n \geq 1, p \geq 1$ and $B \in \mathcal{C}$ with $r(B) < \frac{1}{s}$, we have that

$$\begin{aligned} &d(F[\epsilon_m, \epsilon_m], F[\epsilon_{m+p}, \epsilon_{m+p}]) \\ &\preceq s[d(F[\epsilon_m, \epsilon_m], F[\epsilon_{m+1}, \epsilon_{m+1}]) + d(F[\epsilon_{m+1}, \epsilon_{m+1}], F[\epsilon_{m+p}, \epsilon_{m+p}])] \\ &\preceq sd(F[\epsilon_m, \epsilon_m], F[\epsilon_{m+1}, \epsilon_{m+1}]) + s^2[d(F[\epsilon_{m+1}, \epsilon_{m+1}], F[\epsilon_{m+2}, \epsilon_{m+2}]) \\ &\quad + d(F[\epsilon_{m+2}, \epsilon_{m+2}], F[\epsilon_{m+p}, \epsilon_{m+p}])] \\ &\preceq sd(F[\epsilon_m, \epsilon_m], F[\epsilon_{m+1}, \epsilon_{m+1}]) + s^2d(F[\epsilon_{m+1}, \epsilon_{m+1}], F[\epsilon_{m+2}, \epsilon_{m+2}]) \\ &\quad + s^3d(F[\epsilon_{m+2}, \epsilon_{m+2}], F[\epsilon_{m+3}, \epsilon_{m+3}]) + s^4d(F[\epsilon_{m+3}, \epsilon_{m+3}], F[\epsilon_{m+4}, \epsilon_{m+4}]) \\ &\quad + \dots + s^{p-1}d(F[\epsilon_{m+p-2}, \epsilon_{m+p-2}], F[\epsilon_{m+p-1}, \epsilon_{m+p-1}]) \\ &\quad + s^p d(F[\epsilon_{m+p-1}, \epsilon_{m+p-1}], F[\epsilon_{m+p}, \epsilon_{m+p}]) \\ &\preceq sB^m d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]) + s^2B^{m+1}d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]) \\ &\quad + s^3B^{m+2}d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]) + s^4B^{m+3}d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]) \\ &\quad + \dots + s^{p-1}B^{m+p-2}d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]) + s^p B^{m+p-1}d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]) \\ &= sB^m[e + sB + (sB)^2 + \dots + (sB)^{p-1}]d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]). \end{aligned}$$

However, since $e - sB$ is invertible, therefore we have

$$d(F[\epsilon_m, \epsilon_m], F[\epsilon_{m+p}, \epsilon_{m+p}]) \preceq sB^m(e - sB)^{-1}d(T[\epsilon_0, \epsilon_0], F[\epsilon_0, \epsilon_0]).$$

Now, by taking advantage of Lemma 4 and Lemma 5, we find that $\{F[\epsilon_n, \epsilon_n]\}$ is a Cauchy sequence. Since $F(I)$ is complete, therefore every Cauchy sequence is convergent due to Definition 11, and so by Definition 8, it is easy to see that $\lim_{n \rightarrow \infty} d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) = \theta$ for

some $[\epsilon, \epsilon] \in F(I)$, but according to Definition 9 we have that $\lim_{n \rightarrow \infty} F[\epsilon_n, \epsilon_n] = \langle [\epsilon, \epsilon] \rangle$, therefore there must be some $[\rho, \rho] \in \langle [\epsilon, \epsilon] \rangle$ such that $F[\rho, \rho] \stackrel{\Omega}{=} [\epsilon, \epsilon]$. Next, we will prove that $T[\rho, \rho] \stackrel{\Omega}{=} [\epsilon, \epsilon]$. In order to do this, for one thing,

$$\begin{aligned} d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) &= d(T[\epsilon_{n-1}, \epsilon_{n-1}], T[\rho, \rho]) \\ &\leq a_1 d(F[\epsilon_{n-1}, \epsilon_{n-1}], F[\rho, \rho]) + a_2 d(T[\epsilon_{n-1}, \epsilon_{n-1}], F[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\quad + a_3 d(T[\rho, \rho], F[\rho, \rho]) + a_4 d(F[\epsilon_{n-1}, \epsilon_{n-1}], T[\rho, \rho]) \\ &\quad + a_5 d(T[\epsilon_{n-1}, \epsilon_{n-1}], F[\rho, \rho]) \\ &\leq a_1 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + sa_2 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) \\ &\quad + sa_2 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + sa_3 d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) \\ &\quad + sa_3 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) + sa_4 [d(F[\epsilon_{n-1}, \epsilon_{n-1}], F[\epsilon_n, \epsilon_n]) \\ &\quad + d(F[\epsilon_n, \epsilon_n], T[\rho, \rho])] + a_5 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) \\ &\leq a_1 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + sa_2 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) \\ &\quad + sa_2 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + sa_3 d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) \\ &\quad + sa_3 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) + s^2 a_4 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) \\ &\quad + s^2 a_4 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) + sa_4 d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) \\ &\quad + a_5 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]), \end{aligned}$$

which implies that

$$\begin{aligned} (e - sa_3 - sa_4) d(d(F[\epsilon_n, \epsilon_n], T[\rho, \rho])) &\leq (a_1 + sa_2 + s^2 a_4) d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) \\ &\quad + (sa_2 + sa_3 + s^2 a_4 + a_5) d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]). \end{aligned} \tag{14}$$

For another thing,

$$\begin{aligned} d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) &= d(T[\epsilon_{n-1}, \epsilon_{n-1}], T[\rho, \rho]) = d(T[\rho, \rho], T[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\leq a_1 d(F[\rho, \rho], F[\epsilon_{n-1}, \epsilon_{n-1}]) + a_2 d(T[\rho, \rho], F[\rho, \rho]) \\ &\quad + a_3 d(F[\epsilon_{n-1}, \epsilon_{n-1}], T[\epsilon_{n-1}, \epsilon_{n-1}]) + a_4 d(F[\rho, \rho], T[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\quad + a_5 d(T[\rho, \rho], F[\epsilon_{n-1}, \epsilon_{n-1}]) \\ &\leq a_1 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + sa_2 d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) \\ &\quad + sa_2 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) + sa_3 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) \\ &\quad + sa_3 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + a_4 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) \\ &\quad + sa_5 [d(T[\rho, \rho], F[\epsilon_n, \epsilon_n]) + d(F[\epsilon_n, \epsilon_n], F[\epsilon_{n-1}, \epsilon_{n-1}])] \\ &\leq a_1 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + sa_2 d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) \\ &\quad + sa_2 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) + sa_3 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) \\ &\quad + sa_3 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + a_4 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) \\ &\quad + s^2 a_5 d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) + s^2 a_5 d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) \\ &\quad + sa_5 d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]), \end{aligned}$$

which we have that

$$\begin{aligned} (e - sa_2 - sa_5) d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) &\leq (a_1 + sa_3 + s^2 a_5) d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) \\ &\quad + (sa_2 + sa_3 + a_4 + s^2 a_5) d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]). \end{aligned} \tag{15}$$

As we can see that

$$(2e - sa) d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) \leq (2e - sa_2 - sa_3 - sa_4 - sa_5) d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]),$$

therefore, by combining (14) and (15), we have

$$\begin{aligned} (2e - sa)d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) &\preceq (2e - sa_2 - sa_3 - sa_4 - sa_5)d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) \\ &\preceq (2a_1 + sa)d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) \\ &\quad + (sa_2 + sa_3 + a_4 + a_5 + sa)d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]). \end{aligned} \tag{16}$$

Now, we can see

$$r(sa) = sr(a) \leq (s + 1)r(a) \leq 2sr(a) + (s + 1)r(a) < 2,$$

thus by Lemma 1, it concludes that $2e - sa$ is invertible. As a result, it follows from (16) that

$$\begin{aligned} d(F[\epsilon_n, \epsilon_n], T[\rho, \rho]) &\preceq (2e - sa)^{-1} \left[(2a_1 + sa)d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon]) \right. \\ &\quad \left. + (sa_2 + sa_3 + a_4 + a_5 + sa)d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon]) \right]. \end{aligned}$$

Since the sequences $\{d(F[\epsilon_{n-1}, \epsilon_{n-1}], [\epsilon, \epsilon])\}$ and $\{d(F[\epsilon_n, \epsilon_n], [\epsilon, \epsilon])\}$ are c -sequences, then by Lemma 6, we acquire that $\{d(F[\epsilon_n, \epsilon_n], T[\rho, \rho])\}$ is a c -sequence, thus $F[\epsilon_n, \epsilon_n] \rightarrow T[\rho, \rho]$ as $n \rightarrow \infty$. Hence $T[\rho, \rho] \stackrel{\Omega}{=} [\epsilon, \epsilon] \stackrel{\Omega}{=} F[\rho, \rho]$. Which shows that $[\epsilon, \epsilon]$ is the near-point of coincidence and $[\rho, \rho]$ is the near-coincidence point of the mappings T and F . Next, we shall prove that T and F have a unique near-point of coincidence.

For this, let $[\rho', \rho'] \in I$ be another point such that $T[\rho', \rho'] \stackrel{\Omega}{=} F[\rho', \rho'] \stackrel{\Omega}{=} [\epsilon', \epsilon']$ and assume that $[\epsilon, \epsilon] \stackrel{\Omega}{\neq} [\epsilon', \epsilon']$. Thus, we get

$$\begin{aligned} d(F[\rho', \rho'], F[\rho, \rho]) &= d(T[\rho', \rho'], T[\rho, \rho]) \\ &\preceq a_1d(F[\rho', \rho'], F[\rho, \rho]) + a_2d(T[\rho', \rho'], F[\rho', \rho']) + a_3d(T[\rho, \rho], F[\rho, \rho]) \\ &\quad + a_4d(F[\rho', \rho'], T[\rho, \rho]) + a_5d(T[\rho', \rho'], F[\rho, \rho]) \\ &= (a_1 + a_4 + a_5)d(F[\rho', \rho'], F[\rho, \rho]). \end{aligned}$$

In the light of Lemma 8, we speculate that $d(F[\rho', \rho'], F[\rho, \rho]) = \theta$, that is, $F[\rho', \rho'] \stackrel{\Omega}{=} F[\rho, \rho]$. Hence, $[\epsilon, \epsilon] \stackrel{\Omega}{=} [\epsilon', \epsilon']$.

Finally, if T and F are weakly compatible, then by using Lemma 10, we claim that T and F have a unique near-common fixed point. \square

From the above theorem, we can obtain the following series of corollaries.

Corollary 1. Let (I, d) be a $CIbMS$ over $BA \mathcal{B}$ with the parameter $s \geq 1$ and \mathcal{C} be the underlying solid cone in \mathcal{B} . Let $a \in \mathcal{C}$ be constant with $r(a) < \frac{1}{s}$. Suppose that the mappings $T, F : I \rightarrow I$ satisfy that

$$d(T[\epsilon, \epsilon], T[\rho, \rho]) \preceq ad(F[\epsilon, \epsilon], F[\rho, \rho]) \tag{17}$$

for all $[\epsilon, \epsilon], [\rho, \rho] \in I$. If the range of F contains the range of T and $F(I)$ is a complete subspace, then T and F have a unique near-point of coincidence in I . Moreover, if T and F are weakly compatible, then T and F have a unique near-common fixed point.

Proof. Choose $a_1 = a$ and $a_2 = a_3 = a_4 = a_5 = \theta$ in Theorem 1, the proof is valid. \square

Corollary 2. Let (I, d) be a $CIbMS$ over $BA \mathcal{B}$ with the parameter $s \geq 1$ and \mathcal{C} be the underlying solid cone in \mathcal{B} . Let $a \in \mathcal{C}$ be constant with $r(a) < \frac{1}{s+1}$. Suppose that the mappings $T, F : I \rightarrow I$ satisfy that

$$d(T[\epsilon, \epsilon], T[\rho, \rho]) \preceq a[d(T[\epsilon, \epsilon], F[\epsilon, \epsilon]) + d(T[\rho, \rho], F[\rho, \rho])] \tag{18}$$

for all $[\epsilon, \epsilon], [\rho, \rho] \in I$. If the range of F contains the range of T and $F(I)$ is a complete subspace, then T and F have a unique near-point of coincidence in I . Moreover, if T and F are weakly compatible, then T and F have a unique near-common fixed point.

Proof. Putting $a_1 = a_4 = a_5 = \theta$ and $a_2 = a_3 = a$ in Theorem 1, we complete the proof. \square

Corollary 3. Let (I, d) be a CIbMS over BA \mathcal{B} with the parameter $s \geq 1$ and \mathcal{C} be the underlying solid cone in \mathcal{B} . Let $a \in \mathcal{C}$ be constant with $r(a) < \frac{1}{s(s+1)}$. Suppose that the mappings $T, F : I \rightarrow I$ satisfy that

$$d(T[\epsilon, \epsilon], T[\rho, \rho]) \preceq a[d(F[\epsilon, \epsilon], T[\rho, \rho]) + d(T[\epsilon, \epsilon], F[\rho, \rho])] \tag{19}$$

for all $[\epsilon, \epsilon], [\rho, \rho] \in I$. If the range of F contains the range of T and $F(I)$ is a complete subspace, then T and F have a unique near-point of coincidence in I . Moreover, if T and F are weakly compatible, then T and F have a unique near-common fixed point.

Proof. Set $a_1 = a_2 = a_3 = \theta$ and $a_4 = a_5 = a$ in Theorem 1, the claim holds true. \square

Example 10. Let $\mathcal{B} = C[0, 1]$ be the set of continuous functions on the closed interval $[0, 1]$ with supremum norm. Define multiplication in the usual way. Then, \mathcal{B} is a Banach algebra with a unit 1. Set $\mathcal{C} = \{f \in \mathcal{B} : f(t) \geq 0, t \in [0, 1]\}$ and $X = I$, where I is the collection of all closed bounded intervals in \mathbb{R} . Define a mapping $d : I \times I \rightarrow \mathcal{B}$ by

$$d([\epsilon, \epsilon], [\rho, \rho]) = |\epsilon + \epsilon - (\rho + \rho)|^2 e^t$$

for all $[\epsilon, \epsilon], [\rho, \rho] \in I$. Then, (I, d) is a CIbMS over BA \mathcal{B} with parameter $s = 2$, which we claim is not a MIS nor CIMS over BA \mathcal{B} , since the triangle inequality does not hold true in both cases. Now, define the mappings $T, F : I \rightarrow I$ by

$$T[\epsilon, \epsilon] = \left[\frac{\epsilon}{8}, \frac{\epsilon}{8}\right], \quad F[\epsilon, \epsilon] = \left[\frac{\epsilon}{2}, \frac{\epsilon}{2}\right].$$

Choose $a_1 = \frac{1}{8} + \frac{1}{8}t, a_2 = \frac{1}{12} + \frac{1}{12}t, a_3 = \frac{1}{16} + \frac{1}{16}t$ and $a_4 = a_5 = \theta$. Then, we have

$$\begin{aligned} 2sr(a_1) + (s + 1)r(a_2 + a_3 + sa_4 + sa_5) &= 2sr(a_1) + (s + 1)r(a_2 + a_3) \\ &= 4r\left(\frac{1}{8} + \frac{1}{8}t\right) + 3r\left(\frac{1}{12} + \frac{1}{12}t + \frac{1}{16} + \frac{1}{16}t\right) \\ &= 4\left(\frac{1}{4}\right) + 3\left(\frac{7}{24}\right) \\ &= \frac{15}{8} < 2. \end{aligned}$$

Additionally, a_1 commute with $a_2 + a_3 + sa_4 + sa_5$. Furthermore, by simple calculations the contractive condition can easily be verified. Hence, all the conditions of Theorem 1 are fulfilled. Therefore, it is seen that $[0, 0]$ is the unique near-common fixed point of the mappings T and F .

Theorem 2. Let $(I, \|\cdot\|)$ be a Banach interval space with Ω as the null set, and $\|\cdot\|$ satisfy the null equality. Let $T : I \rightarrow I$ be a mapping satisfying:

$$\|T([\epsilon, \epsilon]) \ominus T([\rho, \rho])\| \leq \alpha\|[\epsilon, \epsilon] \ominus [\rho, \rho]\| + \beta\|[\epsilon, \epsilon] \ominus T([\epsilon, \epsilon])\| + \gamma\|[\rho, \rho] \ominus T([\rho, \rho])\| \tag{20}$$

for all $[\epsilon, \epsilon], [\rho, \rho] \in I$, where α, β, γ are non-negative constants with $\alpha + \beta + \gamma < 1$. Then, T has a unique near-fixed point.

Proof. Let $[\epsilon_0, \epsilon_0] \in I$ be arbitrary initial element. Then, define a sequence $[\epsilon_n, \epsilon_n]_{n=1}^\infty$ by $[\epsilon_{n+1}, \epsilon_{n+1}] = T[\epsilon_n, \epsilon_n]$ for all $n \geq 0$. We shall show that $[\epsilon_n, \epsilon_n]_{n=1}^\infty$ is a Cauchy sequence.

If $[\epsilon_n, \epsilon_n] \stackrel{\Omega}{=} [\epsilon_{n+1}, \epsilon_{n+1}]$, then $[\epsilon_n, \epsilon_n]$ is a near-fixed point of T . So we suppose that, for all $n \geq 0$, $[\epsilon_n, \epsilon_n] \stackrel{\Omega}{\neq} [\epsilon_{n+1}, \epsilon_{n+1}]$. It follows from (20) that

$$\begin{aligned} \|[\epsilon_n, \epsilon_n] \ominus [\epsilon_{n+1}, \epsilon_{n+1}]\| &= \|T([\epsilon_{n-1}, \epsilon_{n-1}]) \ominus T([\epsilon_n, \epsilon_n])\| \\ &\leq \alpha \|[\epsilon_{n-1}, \epsilon_{n-1}] \ominus [\epsilon_n, \epsilon_n]\| + \beta \|[\epsilon_{n-1}, \epsilon_{n-1}] \ominus T([\epsilon_{n-1}, \epsilon_{n-1}])\| \\ &\quad + \gamma \|[\epsilon_n, \epsilon_n] \ominus T([\epsilon_n, \epsilon_n])\| \\ &= \alpha \|[\epsilon_{n-1}, \epsilon_{n-1}] \ominus [\epsilon_n, \epsilon_n]\| + \beta \|[\epsilon_{n-1}, \epsilon_{n-1}] \ominus [\epsilon_n, \epsilon_n]\| \\ &\quad + \gamma \|[\epsilon_n, \epsilon_n] \ominus [\epsilon_{n+1}, \epsilon_{n+1}]\| \end{aligned}$$

that is

$$\|[\epsilon_n, \epsilon_n] \ominus [\epsilon_{n+1}, \epsilon_{n+1}]\| \leq \delta \|[\epsilon_{n-1}, \epsilon_{n-1}] \ominus [\epsilon_n, \epsilon_n]\|, \tag{21}$$

where $\delta = \frac{\alpha+\beta}{1-\gamma}$. Now, using (21), we have

$$\begin{aligned} \|[\epsilon_n, \epsilon_n] \ominus [\epsilon_{n+1}, \epsilon_{n+1}]\| &\leq \delta \|[\epsilon_{n-1}, \epsilon_{n-1}] \ominus [\epsilon_n, \epsilon_n]\| \\ &\leq \delta^2 \|[\epsilon_{n-2}, \epsilon_{n-2}] \ominus [\epsilon_{n-1}, \epsilon_{n-1}]\| \\ &\leq \dots \leq \delta^n \|[\epsilon_0, \epsilon_0] \ominus [\epsilon_1, \epsilon_1]\|. \end{aligned}$$

Thus, for $n < m$, using Proposition 2, we obtain

$$\begin{aligned} \|[\epsilon_m, \epsilon_m] \ominus [\epsilon_n, \epsilon_n]\| &\leq \|[\epsilon_m, \epsilon_m] \ominus [\epsilon_{m-1}, \epsilon_{m-1}]\| + \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\epsilon_{m-2}, \epsilon_{m-2}]\| \\ &\quad + \dots + \|[\epsilon_{n+1}, \epsilon_{n+1}] \ominus [\epsilon_n, \epsilon_n]\| \\ &\leq (\delta^{m-1} + \delta^{m-2} + \dots + \delta^n) \|[\epsilon_1, \epsilon_1] \ominus [\epsilon_0, \epsilon_0]\| \\ &= \delta^n \cdot \frac{1 - \delta^{m-n}}{1 - \delta} \|[\epsilon_1, \epsilon_1] \ominus [\epsilon_0, \epsilon_0]\|. \end{aligned}$$

Since $0 < \delta < 1$, we have $1 - \delta^{m-n} < 1$, which we have

$$\|[\epsilon_m, \epsilon_m] \ominus [\epsilon_n, \epsilon_n]\| \leq \frac{\delta^n}{1 - \delta} \|[\epsilon_1, \epsilon_1] \ominus [\epsilon_0, \epsilon_0]\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves that $\{[\epsilon_n, \epsilon_n]\}_{n=1}^\infty$ is a Cauchy sequence. Since I is complete, there exists $[\epsilon, \epsilon] \in I$ such that

$$\|[\epsilon_n, \epsilon_n] \ominus [\epsilon, \epsilon]\| = \|[\epsilon, \epsilon] \ominus [\epsilon_n, \epsilon_n]\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Assume that $\|\cdot\|$ satisfies the null equality. We are going to show that any point $[\bar{\epsilon}, \bar{\epsilon}] \in \langle[\epsilon, \epsilon]\rangle$ is a near-fixed point. Since $[\bar{\epsilon}, \bar{\epsilon}] \in \langle[\epsilon, \epsilon]\rangle$ therefore, for $\omega_1, \omega_2 \in \Omega$, we have $[\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1 = [\epsilon, \epsilon] \oplus \omega_2$. Now

$$\begin{aligned}
 \|[\bar{\epsilon}, \bar{\epsilon}] \ominus T([\bar{\epsilon}, \bar{\epsilon}])\| &= \|([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1) \ominus T([\bar{\epsilon}, \bar{\epsilon}])\| \text{ (since } \|\cdot\| \text{ satisfies the null equality)} \\
 &\leq \|([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1) \ominus [\epsilon_m, \epsilon_m]\| + \|[\epsilon_m, \epsilon_m] \ominus T([\bar{\epsilon}, \bar{\epsilon}])\| \\
 &\quad \text{(using Proposition 2)} \\
 &= \|([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1) \ominus [\epsilon_m, \epsilon_m]\| + \|T([\epsilon_{m-1}, \epsilon_{m-1}]) \ominus T([\bar{\epsilon}, \bar{\epsilon}])\| \\
 &\leq \|([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1) \ominus [\epsilon_m, \epsilon_m]\| + \alpha \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\bar{\epsilon}, \bar{\epsilon}]\| \\
 &\quad + \beta \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus T([\epsilon_{m-1}, \epsilon_{m-1}])\| + \gamma \|[\bar{\epsilon}, \bar{\epsilon}] \ominus T([\bar{\epsilon}, \bar{\epsilon}])\| \\
 &\quad \text{(using (20))} \\
 &\leq \|([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1) \ominus [\epsilon_m, \epsilon_m]\| + \frac{\alpha}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\bar{\epsilon}, \bar{\epsilon}]\| \\
 &\quad + \frac{\beta}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\epsilon_m, \epsilon_m]\| \\
 &= \|([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1) \ominus [\epsilon_m, \epsilon_m]\| + \frac{\alpha}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\bar{\epsilon}, \bar{\epsilon}] \oplus (-\omega_1)\| \\
 &\quad + \frac{\beta}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\epsilon_m, \epsilon_m]\| \\
 &\quad \text{(since } -\omega_1 \in \Omega \text{ and } \|\cdot\| \text{ satisfies the null equality)} \\
 &= \|([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1) \ominus [\epsilon_m, \epsilon_m]\| + \frac{\alpha}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus ([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_1)\| \\
 &\quad + \frac{\beta}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\epsilon_m, \epsilon_m]\| \text{ (using (1))} \\
 &\leq \|([\epsilon, \epsilon] \oplus \omega_2) \ominus [\epsilon_m, \epsilon_m]\| + \frac{\alpha}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus ([\epsilon, \epsilon] \oplus \omega_2)\| \\
 &\quad + \frac{\beta}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\epsilon, \epsilon]\| + \frac{\beta}{1-\gamma} \|[\epsilon, \epsilon] \ominus [\epsilon_m, \epsilon_m]\| \\
 &\quad \text{(using Proposition 2)} \\
 &= \|[\epsilon, \epsilon] \ominus [\epsilon_m, \epsilon_m]\| + \frac{\alpha}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\epsilon, \epsilon]\| \\
 &\quad + \frac{\beta}{1-\gamma} \|[\epsilon_{m-1}, \epsilon_{m-1}] \ominus [\epsilon, \epsilon]\| + \frac{\beta}{1-\gamma} \|[\epsilon, \epsilon] \ominus [\epsilon_m, \epsilon_m]\| \\
 &\quad \text{(using the null equality and (1)),}
 \end{aligned}$$

which implies that $\|[\bar{\epsilon}, \bar{\epsilon}] \ominus T([\bar{\epsilon}, \bar{\epsilon}])\| = 0$ as $m \rightarrow \infty$. Now, by part (ii) of Proposition 3, we conclude that $T([\bar{\epsilon}, \bar{\epsilon}]) \stackrel{\Omega}{=} [\bar{\epsilon}, \bar{\epsilon}]$ for any $[\bar{\epsilon}, \bar{\epsilon}] \in \langle[\epsilon, \epsilon]\rangle$.

Now, assume that there is another near-fixed point $[\rho, \rho]$ of T with $[\rho, \rho] \notin \langle[\epsilon, \epsilon]\rangle$, that is, $[\rho, \rho] \stackrel{\Omega}{=} T([\rho, \rho])$ and $[\bar{\epsilon}, \bar{\epsilon}] \stackrel{\Omega}{=} T([\bar{\epsilon}, \bar{\epsilon}])$. Then,

$$[\rho, \rho] \oplus \omega_1 = T([\rho, \rho]) \oplus \omega_2 \text{ and } [\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_3 = T([\bar{\epsilon}, \bar{\epsilon}]) \oplus \omega_4$$

for some $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega$. We obtain

$$\begin{aligned}
 \|[\rho, \rho] \ominus [\bar{\epsilon}, \bar{\epsilon}]\| &= \|([\rho, \rho] \oplus \omega_1) \ominus ([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_2)\| \text{ (using the null equality and (1))} \\
 &= \|T([\rho, \rho] \oplus \omega_2) \ominus T([\bar{\epsilon}, \bar{\epsilon}] \oplus \omega_4)\| = \|T([\rho, \rho]) \ominus T([\bar{\epsilon}, \bar{\epsilon}])\| \\
 &\quad \text{(using the null equality and (1))} \\
 &\leq \alpha \|[\rho, \rho] \ominus [\bar{\epsilon}, \bar{\epsilon}]\| + \beta \|[\rho, \rho] \ominus T([\rho, \rho])\| + \gamma \|[\bar{\epsilon}, \bar{\epsilon}] \ominus T([\bar{\epsilon}, \bar{\epsilon}])\| \\
 &= \alpha \|[\rho, \rho] \ominus [\bar{\epsilon}, \bar{\epsilon}]\| + \beta \|[\rho, \rho] \ominus [\rho, \rho]\| + \gamma \|[\bar{\epsilon}, \bar{\epsilon}] \ominus [\bar{\epsilon}, \bar{\epsilon}]\| \\
 &\quad \text{(using the null equality and (1))} \\
 &\leq \alpha \|[\rho, \rho] \ominus [\bar{\epsilon}, \bar{\epsilon}]\|.
 \end{aligned}$$

Since $0 < \alpha < 1$, we conclude that $\|[\rho, \varrho] \ominus [\bar{\varepsilon}, \bar{\varepsilon}]\| = 0$, that is, $[\rho, \varrho] \stackrel{\Omega}{=} [\bar{\varepsilon}, \bar{\varepsilon}]$, which contradicts $[\rho, \varrho] \notin \langle [\bar{\varepsilon}, \bar{\varepsilon}] \rangle = \langle [\varepsilon, \varepsilon] \rangle$. Therefore, any $[\rho, \varrho] \notin \langle [\varepsilon, \varepsilon] \rangle$ cannot be the near-fixed point. Equivalently, if $[\rho, \varrho]$ is a near-fixed point of T , then $[\rho, \varrho] \in \langle [\varepsilon, \varepsilon] \rangle$. \square

Remark 4. Theorem 2 of this paper generalize Theorem 4 of [7].

Corollary 4. Let $(I, \|\cdot\|)$ be a Banach interval space with Ω as the null set, and $\|\cdot\|$ satisfy the null equality. Let $T : I \rightarrow I$ be a mapping satisfying:

$$\|T([\varepsilon, \varepsilon]) \ominus T([\rho, \varrho])\| \leq \alpha \|[\varepsilon, \varepsilon] \ominus [\rho, \varrho]\|$$

for all $[\varepsilon, \varepsilon], [\rho, \varrho] \in I$, where α is non-negative constant with $\alpha < 1$. Then T has a unique near-fixed point.

Proof. On taking $\beta = \gamma = 0$ in Theorem 2, we can obtain the desired result. \square

Corollary 5. Let $(I, \|\cdot\|)$ be a Banach interval space with the null set Ω such that $\|\cdot\|$ satisfies the null equality. Let $T : I \rightarrow I$ be a mapping satisfying:

$$\|T([\varepsilon, \varepsilon]) \ominus T([\rho, \varrho])\| \leq \beta \left[\|[\varepsilon, \varepsilon] \ominus T([\varepsilon, \varepsilon])\| + \|[\rho, \varrho] \ominus T([\rho, \varrho])\| \right] \quad (22)$$

for all $[\varepsilon, \varepsilon], [\rho, \varrho] \in I$, where β is non-negative constant with $\beta < \frac{1}{2}$. Then, T has a unique near-fixed point.

Proof. On taking $\alpha = 0$ and $\beta = \gamma$ in Theorem 2, we can obtain the desired result. \square

Author Contributions: Conceptualization, writing—original draft preparation, writing—review and editing, supervision, investigation, M.S. and Z.I.; methodology, formal analysis, H.A. and Z.I.; funding acquisition, visualization H.I. and S.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: We are very grateful to the editor and unbiased arbitrator for his/her prudent interpretation and proposition which refined the excellency of this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Huang, L.G.; Zhang, X. Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl.* **2007**, *332*, 1468–1476. [CrossRef]
- Rezapour, S.; Hambarani, R. Some notes on the paper “Cone metric spaces and fixed point theorems of contractive mappings”. *J. Math. Anal. Appl.* **2008**, *345*, 719–724. [CrossRef]
- Arandelović, I.D.; Kečkić, D.J. TVS-cone metric spaces as a special case of metric spaces. *arXiv* **2012**, arXiv:1202.5930.
- Çakallı, H.; Sönmez, A.; Genç, Ç. On an equivalence of topological vector space valued cone metric spaces and metric spaces. *Appl. Math. Lett.* **2012**, *25*, 429–433. [CrossRef]
- Du, W.S. A note on cone metric fixed point theory and its equivalence. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 2259–2261. [CrossRef]
- Liu, H.; Xu, S. Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings. *Fixed Point Theory Appl.* **2013**, *2013*, 320. [CrossRef]
- Wu, H.C. A new concept of fixed point in metric and normed interval spaces. *Mathematics* **2018**, *6*, 219. [CrossRef]
- Hardy, G.E.; Rogers, T. A generalization of a fixed point theorem of Reich. *Can. Math. Bull.* **1973**, *16*, 201–206. [CrossRef]
- Reich, S. Fixed point theorem. *Acts Natl. Acad. Lincei-Acc.-Cl.-Phys.-Math. Nat. Sci.* **1971**, *51*, 26.
- Islam, Z.; Sarwar, M.; de la Sen, M. Fixed-Point Results for Generalized α -Admissible Hardy-Rogers’ Contractions in Cone b2-Metric Spaces over Banach’s Algebras with Application. *Adv. Math. Phys.* **2020**. [CrossRef]

11. Mitrovic, Z.D.; Hussain, N. On results of Hardy-Rogers and Reich in cone b-metric space over Banach algebra and applications. *UPB Sci. Bull. Ser. A* **2019**, *81*, 147–154.
12. Rangamma, M.; Murthy, P.R.B. Hardy and Rogers type Contractive condition and common fixed point theorem in Cone 2-metric space for a family of self-maps. *Glob. J. Pure Appl. Math.* **2016**, *12*, 2375–2383.
13. Rudin, W. Functional analysis 2nd ed. *Int. Ser. Pure Appl. Math.* **1991**, *45*, 1.10–1.11.
14. Huang, H.; Radenovic, S. Common fixed point theorems of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras and applications. *J. Nonlinear Sci. Appl.* **2015**, *8*, 787–799. [[CrossRef](#)]
15. Xu, S.; Radenović, S. Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality. *Fixed Point Theory Appl.* **2014**, *2014*, 102. [[CrossRef](#)]
16. Huang, H.; Radenović, S. Some fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras. *J. Comput. Anal. Appl.* **2016**, *20*, 566–583.
17. Huang, H.; Deng, G.; Radenović, S. Some topological properties and fixed point results in cone metric spaces over Banach algebras. *Positivity* **2019**, *23*, 21–34. [[CrossRef](#)]
18. Huang, H.; Radenovic, S. Common fixed point theorems of generalized Lipschitz mappings in cone metric spaces over Banach algebras. *Appl. Math. Inf. Sci.* **2015**, *9*, 2983. [[CrossRef](#)]
19. Shukla, S.; Balasubramanian, S.; Pavlović, M. A generalized Banach fixed point theorem. *Bull. Malays. Math. Sci. Soc.* **2016**, *39*, 1529–1539. [[CrossRef](#)]
20. Janković, S.; Kadelburg, Z.; Radenović, S. On cone metric spaces: A survey. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 2591–2601. [[CrossRef](#)]