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Review

The Rigging Technique for Null Hypersurfaces

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Abstract: Starting from the main definitions, we review the rigging technique for null hypersurfaces theory and most of its main properties. We make some applications to illustrate it. On the one hand, we show how we can use it to show properties of null hypersurfaces, with emphasis in null cones, totally geodesic, totally umbilic, and compact null hypersurfaces. On the other hand, we show the interplay with the ambient space, including its influence in causality theory.

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1. Introduction

Null hypersurfaces are present in general relativity in several fundamental theories. For example, the thermodynamic of black holes stated an unexpected link between classical thermodynamic theory and black hole theory using the horizon of the black hole, which is a null hypersurface [1–3]. On the other hand, the set of points that can be reached by a future null geodesic from a fixed point (the vertex) is called the future null cone. It appears in causality theory and it is a null hypersurface near the vertex, but maybe not far away because of the presence of null conjugate points to the vertex or self-intersections. Null cones as null hypersurfaces can be used as characteristic initial value problem in general relativity, see [4], and they are important to improve properties of the solution of wave equations, so it is interesting to know a lower bound of the null injectivity radius [5–7]. Two transversely intersecting null hypersurfaces can be used also as characteristic initial value problem in general relativity [8]. The problem of the formation of trapped surfaces and black holes is another fundamental theory where the use of null hypersurfaces is important [9], as well as the problem of the stability of the Minkowski space [10]. These examples show the importance of the study of null hypersurfaces in physical applications.

From a geometric point of view, null hypersurfaces are natural submanifolds in Lorentzian, or more generally, semi-Riemannian manifolds, except in the Riemannian case because they have not a Riemannian counterpart. The family of null hypersurfaces in Lorentzian geometry is a distinguished family of submanifolds, and this fact is enough to think that their properties are in part a consequence of the general properties of the ambient space, and conversely, their properties can say something about the ambient space. In a certain sense, a null hypersurface resembles Lagrange submanifolds in symplectic geometry or Legendrian submanifolds in contact geometry, because they are submanifolds where a non-degenerate structure in the ambient space degenerates when it is induced on the submanifold.

Another interesting point is the study of an individual null hypesurface, that is, its geometry as a submanifold. For this purpose, the most important object is its null second fundamental form, obtained through the choice of a null vector field tangent to the null hypersurface. This allows us to classify them in totally umbilic, totally geodesic, or other. These ideas correspond to its extrinsic geometry. The induced metric from the ambient space in the null hypersurface is not useful because it is degenerated on it, thus its intrinsic geometry is essentially non-metric.

There are two classical techniques to study a null hypersurface L using some kind of intrinsic geometry that can be implemented on it. The first one, introduced in [11], takes a null vector field ξ tangent to L (in fact it is only needed its null direction) and study the quotient $TL/span(\xi)$, see also [12–14]. The second one needs a geometric data $(\xi, \mathcal{S})$, where ξ is a null vector field tangent to L, and $\mathcal{S}$ a distribution in L transverse to ξ, called the screen distribution. The vector field ξ allows us to define the null second fundamental form, and it and the screen distribution determine N, the unique null vector field defined on L, transverse to L, orthogonal to $\mathcal{S}$, and suitably normalized [15].

The first technique uses the null direction and the second one the choice of a null tangent vector field ξ on L. Both obtain a null second fundamental form as a fundamental tool, but the first one furnishes the orbit space with a "quotient metric" and a "quotient connection". The choice of a null vector field is essentially the way physicists study null hypersurfaces and it seems today that it can be considered as the minimum geometric choice needed to study them.

The second technique provides more geometric objects which allows us to ask new questions and solve new problems, but since the geometric data $(\xi, \mathcal{S})$ are chosen arbitrarily and independently, they are not tuned enough and it generates difficulties handling them. Moreover, they do not provide any reasonable metric and connection on the null hypersurface, so the questions that can be asked and solved are, in general, not well linked to other classical properties. An example is the lack of information about compact null hypersurfaces using this technique. Anyway, it is the most used in the mathematical literature but, due to the mentioned difficulties, the geometric point of view is much less developed than the physical counterpart.

There are more geometric points of view like [16] where the authors study the degenerated induced metric itself obtaining a partial classification of them, but they have not been further developed.

The most recent geometric point of view is called the rigging technique introduced by the authors in [17]. It is based on the choice of a vector field transverse to the null hypersurface, called rigging vector field, from which we can introduce the rigged geometric data $(\xi, \mathcal{S}, N, \widetilde{g})$ of the second technique plus a Riemannian metric $\widetilde{g}$. All these rigged geometric data are tuned together with the rigging vector field. In some cases we can choose the rigging vector field to link the geometry introduced in L with the ambient space. This solves, in part, the difficulties arising with previous techniques and it provides new perspective to link with the properties of the ambient space, especially in the presence of symmetries.

The aim of this paper is to survey this technique showing its main features and some of its applications. We are mainly interested in using it to relate the property of the family of null hypersurfaces with properties of the ambient space, even those of global nature such as in the causality theory. We are also interested in studying null hypersurfaces individually, mainly by means of the rigged Riemannian metric to explore its properties. For example, we study compact null hypersurfaces, totally geodesic or totally umbilic ones and null cones.

Almost all the results presented here use the rigging technique in a fundamental way in the sense that they was found and proved thanks to this technique. There are two remarkable exceptions, Theorem 16, which was previously proved in [13,18,19], has been used here to check the new technique, and a version of the zeroth law of the black hole's thermodynamic (Section 3.7). We are interested in the ideas more than their details, so we will not include proofs of the results, which can be checked in the references.

2. The Rigging Technique

In this section, we introduce the main ideas of the rigging technique and its properties. First, we show how a rigging vector field induces the geometric objects needed to study a null hypersurface, the most important of which is a Riemannian metric. After that, we establish important relations between the new Riemannian structure and the geometry of the ambient space. The main reference for this section is [17].

2.1. Rigged Vector Field and Rigged Metric

Along this review (M, g) is a connected n-dimensional Lorentzian manifold with signature $(-, +, ..., +)$. We always consider embedded null hypersurfaces L unless otherwise stated and denote $i : L \to M$ the canonical inclusion, although it will be usually avoided.

Definition 1. *A hypersurface L of (M, g) is null if the inherit metric tensor i^*g is degenerate at every point of L.*

The radical of L is defined as $Rad_p(L) = T_pL \cap T_pL^\perp$ for all $p \in L$. Since T_pL is a null hyperplane and the ambient metric is Lorentzian, $T_pL^\perp \subset T_pL$ and $\dim T_pL^\perp = 1$. So there is a unique null direction in L which is orthogonal to any direction in L. In particular, it does not contain timelike directions and it is foliated by null curves.

Observe that being foliated by null curves is not sufficient for a hypersurface to be null. An easy counterexample is a timelike plane in the Minkowski space. However, an additional condition on the causality of the hypersurface implies that it is null, as the following result shows.

Lemma 1 ([14])**.** *If a locally achronal hypersurface of a Lorentzian manifold is foliated by null curves, then it is a null hypersurface.*

Locally, we can pick a null vector field ξ tangent to the null hypersurface. Since $\xi \in TL^\perp$, given $U \in \mathfrak{X}(L)$ we have

$$0 = g(\xi, [\xi, U]) = g(\xi, \nabla_U \xi) - g(\xi, \nabla_\xi U) = g(\nabla_\xi \xi, U),$$

so $\nabla_\xi \xi$ is proportional to ξ and the null hypersurface is locally foliated by null geodesics.

Definition 2. *A rigging vector field (a rigging for short) for a null hypersuface L is a vector field ζ defined on some open set containing L and transverse to it, that is, $\zeta_p \notin T_pL$ for each $p \in L$. If the rigging is defined only on the null hypersurface we say that it is a restricted rigging.*

The notion of a rigging can be found in [20] and its use combined with an associated Riemannian metric was done in [21–24]. In our approach we define the rigging in an open set containing the null hypersurface to link its properties to those of the ambient space. This is especially relevant in the presence of symmetries. The freedom to choose the rigging vector field can be used to take an adapted rigging for every kind of problem, much like we choose an adapted orthonormal basis to solve problems in affine Euclidean spaces. For example, for null cones there exists a rigging vector field, such that the rigged associated data $(\xi, \mathcal{S}, N, \widetilde{g})$ have the salient property that ξ is a geodesic vector field for both metrics g and $\widetilde{g}$. This allows us to study the localization of null conjugate points to the vertex using the Riemannian metric $\widetilde{g}$, see Section 3.3.

It is well known that we cannot project ζ on L in a canonical way due to the degeneracy of the induced metric i^*g. In the non-degenerate case, the orthogonal projection of ζ is the vector field metrically equivalent to $i^*\alpha$ where α is the one-form metrically equivalent to ζ. This construction can also be done in the degenerate case when ζ is transverse to L.

Take α the 1-form metrically equivalent to ζ and consider $\omega = i^*\alpha$. Observe that

$$\omega(U) = \alpha(U) \tag{1}$$

for any $U \in \mathfrak{X}(L)$. We will use systematically this identity.

Since ζ is transverse to L, the bilinear form $\widetilde{g} = \omega \otimes \omega + i^* g$ is a Riemmanian metric on L, which is called the *rigged metric* associated to ζ. The $\widetilde{g}$-metrically equivalent vector field to ω is denoted by ξ and it is called the *rigged vector field* associated to ζ.

Lemma 2. *The rigged vector field ξ is the unique null vector field tangent to L normalized by $g(\zeta, \xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary.*

Of course we can define the rigged vector field directly with the characterization provided in Lemma 2, but by doing so we miss the desired tuning with the ambient space. For example, if the rigging is closed (resp. conformal), the rigged ξ is $\widetilde{g}$-geodesic (resp. g-geodesic) as we will see below.

We can consider the screen distribution on L given by $\zeta^{\perp} = \ker \omega$, which is denoted by $\mathcal{S}^{\zeta}$ or $\mathcal{S}$ if there is no confusion. The null transverse vector field to L is given by

$$N = \zeta - \frac{1}{2} g(\zeta, \zeta)\xi,$$

which is the unique null vector field in $\mathfrak{X}(i)$, the $C^{\infty}(L)$-module of vector fields on M defined on points of L, normalized so that $g(N, \xi) = 1$. Moreover, we have the following decompositions

$$T_p M = T_p L \oplus span(N_p), \tag{2}$$

$$T_p L = \mathcal{S}_p \oplus span(\xi_p), \tag{3}$$

for all $p \in L$. We call $P : TL \longrightarrow \mathcal{S}$ the projection associated to the decomposition (3) above.

In this way, we have associated with the rigging ζ the classical geometric data $(\xi, \mathcal{S}, N)$ needed to study the geometry of a null hypersurface. Additionally, we have two important improvements: the tuning of the geometric data with the rigging vector field (and, therefore, between themselves), which provides a link of the properties of the hypersurfaces with the ambient space, and the associated rigged Riemannian metric $\widetilde{g}$ which will allow us to use Riemannian techniques in null hypersurfaces.

Rigging vector fields are abundant in examples. A time-orientable Lorentzian manifold is a Lorentzian manifold furnished with a timelike vector field, which will be a rigging for any null hypersurface. Recall that if a Lorentzian manifold is not time-orientable, then it admits a double covering which is. Time orientability is a hypothesis usually assumed in physics.

The existence of a rigging vector field for a null hypersurface is the generic situation. In fact, despite the above comment on timelike vector fields, a rigging does not need to be timelike at all, so we have a great freedom to choose it. However, although the local existence of a rigging is guaranteed, it may not exist globally because it implies the existence of the rigged vector field on L which is a never zero vector field, a non trivial topological property. The following example shows another difficulty due to the lack of orientability.

Example 1. *Consider the Minkowski space $(\mathbb{L}^3, g) = (\mathbb{R}^3, dx^2 + dy^2 + dydz)$ and call M the quotient by the isometry group generated by $\Phi(x, y, z) = (x - 1, -y, -z)$, which induces a Lorentzian metric on M. The projection of the plane $y = 0$ is a null hypersurface diffeomorphic to a Möbius band. It does not exist a globally defined rigging for this hypersurface, since it does not admit a globally defined null section ξ.*

In the above example, M is orientable but L is not. In general, if M is orientable and there is a rigging for a null hypersurface L, then L is also orientable.

From now on, we suppose that L is a null hypersurface which admits a rigging vector field ζ. We will denote U, V, W vector fields in L and X, Y, Z vector fields in $\mathcal{S}$.

We review the classical equations associated to the geometric data $(\xi, \mathcal{S}, N)$ [15]. According to the decomposition (2) we have

$$\nabla_U V = \nabla^L_U V + B(U,V)N, \tag{4}$$
$$\nabla_U N = \tau(U)N - A(U),$$

where $\nabla^L_U V, A(U) \in TL$. The induced connection ∇^L is torsion free but, in general, is not metric, so its use is limited. The one-form τ defined on L is determined by the second equation above as

$$\tau(U) = g(\nabla_U N, \xi) = g(\nabla_U \zeta, \xi), \tag{5}$$

and in physics literature it is usually called rotation-one form.

A is the *shape operator* of L and we have $A(U) \in \mathcal{S}$, since $g(\nabla_U N, N) = 0$. B is a symmetric tensor, called the *null second fundamental form* of L, determined by the first equation above as

$$B(U,V) = -g(\nabla_U \xi, V).$$

Using that $[U,V] \in \mathfrak{X}(L)$ for any $U,V \in \mathfrak{X}(L)$, it is straightforward to see that B is symmetric and $B(\xi, U) = 0$ for all $U \in \mathfrak{X}(L)$.

Since $g(\xi, \xi)$ is constant, the vector field $\nabla_U \xi$ in tangent to L, so we can decompose it according to the direct sum decomposition (3) as

$$\nabla_U \xi = -\tau(U)\xi - A^*(U),$$

where $A^*(U) \in \mathcal{S}$. The endomorphism A^* is called the *shape operator* of $\mathcal{S}$ and it satisfies

$$B(U,V) = g(A^*(U),V) = g(U, A^*(V)), \tag{6}$$
$$B(A^*(U),V) = B(U, A^*(V)), \tag{7}$$

for all $U,V \in \mathfrak{X}(L)$.

The trace of A^* is the *null mean curvature* of L. If $\{e_3, \dots, e_n\}$ is an orthonormal basis of $\mathcal{S}_p$, then it can be written as

$$H_p = \sum_{i=3}^n g(A^*(e_i), e_i) = \sum_{i=3}^n B(e_i, e_i).$$

If we define the tensor field

$$C(U,V) = -g(\nabla_U N, P(V)) \tag{8}$$

for any $U,V \in \mathfrak{X}(L)$, then we have

$$\nabla^L_U X = \nabla^*_U X + C(U,X)\xi \tag{9}$$

for any $U \in \mathfrak{X}(L)$ and $X \in \mathcal{S}$, where $\nabla^*_U V = P(\nabla^L_U V)$. The tensor C also holds the equations

$$C(U,X) = g(A(U),X), \tag{10}$$
$$C(X,Y) - C(Y,X) = g(N,[X,Y]), \tag{11}$$

so $\mathcal{S}$ is integrable if, and only if, C restricted to $\mathcal{S}$ is symmetric. In this case, given a leaf S of $\mathcal{S}$, ∇^* restricted to it is the induced Levi–Civita connection from the ambient space, and Equations (4) and (9) show that its second fundamental form as a codimension two submanifold of M is

$$\mathbb{I}^S(X,Y) = C(X,Y)\xi + B(X,Y)N, \tag{12}$$

where $X, Y \in TS$.

The tensors B, C and τ depend on the chosen rigging but we know its behavior under a rigging change, see [25]. Suppose ζ' is another rigging for L and decompose it as

$$\zeta' = \Phi N + X_0 + g(\zeta', N)\xi,$$

where $\Phi = g(\zeta', \xi) \in C^\infty(L)$ never vanishes because ζ' is a rigging and $X_0 \in \mathcal{S}$. We can check that the rigged vector field and the null transverse vector field induced from ζ' are

$$\xi' = \frac{1}{\Phi}\xi,$$
$$N' = \Phi N + X_0 - \frac{1}{2\Phi}g(X_0, X_0)\xi.$$

Moreover, if we denote B', C' and τ' the corresponding tensors induced from ζ', and ∇'^L the induced connection, then

- $B' = \frac{1}{\Phi}B$. In particular, $H' = \frac{1}{\Phi}H$;
- $\tau'(U) = \tau(U) + \frac{1}{\Phi}B(X_0, U) + d(\ln|\Phi|)(U)$;
- $C'(U, V) = \Phi C(U, V) + g(X_0, V)\tau'(U) - g(\nabla^L_U X_0, V) - \frac{1}{2\Phi}g(X_0, X_0)B(U, V)$;
- $\nabla'^L_U V - \nabla^L_U V = -\frac{1}{\Phi}B(U, V)V_0$, where $V_0 = X_0 - \frac{1}{2\Phi}g(X_0, X_0)\xi$.

The notions of totally geodesic and totally umbilic hypersurface can also be defined in the degenerate case. We say that L is *totally geodesic* if $B \equiv 0$ and *totally umbilic* if $B = \rho g$ for certain $\rho \in C^\infty(L)$. From the above formulas we see that these definitions do not depend on the rigging.

Other geometric conditions that do not depend on the chosen rigging are having zero null mean curvature and having screen non-degenerate second fundamental form, i.e., $B(X, Y) = 0$ for all $Y \in \mathcal{S}$ implies $X = 0$. In particular, having screen definite second fundamental form, which means $B(X, X) \neq 0$ for all non-zero $X \in \mathcal{S}$, is also independent on the rigging.

Since $A^* : TL \to TL$ is selfadjoint, see (6), $A^*(\xi) = 0$ and g restricted to $\mathcal{S}$ is Riemannian, A^* is diagonalizable. The eigenvalues of A^* are called the *principal curvatures* of L respect to ζ and they only depend on the rigged vector field, not on the screen distribution, (Lemma 3.1.1 [26]). Indeed, if $k_2, \ldots, k_n$ are the principal curvatures respect to ζ, then $\frac{k_2}{\Phi}, \ldots, \frac{k_n}{\Phi}$ are the principal curvatures respect to another rigging $\zeta' = \Phi N + X_0 + g(\zeta', N)\xi$. Observe that 0 is always a principal curvature and the multiplicities of the principal curvatures are independent of the chosen rigging.

The *curvature tensor of* ∇^L is defined as

$$R^L_{UV}W = \nabla^L_U \nabla^L_V W - \nabla^L_V \nabla^L_U W - \nabla^L_{[U,V]}W.$$

It satisfies $R^L_{UV}\xi = R_{UV}\xi$ and the so called *Gauss–Codazzi equations*, see [15].

$$g(R_{UV}W, X) = g(R^L_{UV}W, X) + B(U, W)g(A(V), X) - B(V, W)g(A(U), X), \tag{13}$$

$$g(R_{UV}W, \xi) = \left(\nabla^L_U B\right)(V, W) - \left(\nabla^L_V B\right)(U, W) + \tau(U)B(V, W) - \tau(V)B(U, W), \tag{14}$$

$$g(R_{UV}W, N) = g(R^L_{UV}W, N).$$

Other important equations which follows from the above ones are the following.

$$g(R_{UV}X, N) = \left(\nabla^{*L}_U C\right)(V, X) - \left(\nabla^{*L}_V C\right)(U, X) + \tau(V)C(U, X) - \tau(U)C(V, X), \tag{15}$$

$$g(R_{UV}\xi, N) = C(V, A^*(U)) - C(U, A^*(V)) - d\tau(U, V), \tag{16}$$

where $\nabla^{*L}_U C$ is defined as

$$\left(\nabla^{*L}_U C\right)(V,X) = U(C(V,X)) - C(\nabla^L_U V, X) - C(V, \nabla^*_U X).$$

The usual sectional curvature is not defined for null planes. In [13], Harris introduced the null sectional curvature for a null plane Π as follows. Fix $u \in \Pi$ a null vector and write $\Pi = span(u,v)$ for some vector $v \in \Pi$ (necessarily spacelike). The null sectional curvature of Π respect to u is

$$\mathcal{K}_u(\Pi) = \frac{g(R_{uv}v,u)}{g(v,v)}.$$

It is easy to check that $\mathcal{K}_u(\Pi)$ does not depend on the chosen spacelike vector v, but if we take another null vector $u' = \lambda u \in \Pi$ for some non-zero $\lambda \in \mathbb{R}$, then $\mathcal{K}_{u'}(\Pi) = \lambda^2 \mathcal{K}_u(\Pi)$. Therefore, although the null sectional curvature of a null plane does depend on the chosen null vector in Π, its sign is independent on any choice. In particular, it has sense to say zero null sectional curvature without any explicit mention to the chosen null vector.

Using Equation (14), we can compute the null sectional curvature respect to ξ of a null plane Π tangent to L at a point p. If we take a unitary $v \in \mathcal{S}_p$, such that $\Pi = span(\xi_p, v)$, then

$$\mathcal{K}_\xi(\Pi) = \left(\nabla^L_\xi B\right)(v,v) - \left(\nabla^L_v B\right)(\xi,v) + \tau(\xi)B(v,v). \tag{17}$$

This equation has the following interesting consequence.

Proposition 1. *Let (M,g) be a Lorentzian manifold and $p \in M$, such that $\mathcal{K}(\Pi) \neq 0$ for any null plane $\Pi \subset T_pM$. Then, it does not exist any totally geodesic null hypersurface through p.*

This proposition applies to the Friedmann cosmological models. Since its null sectional curvature never vanishes, (Corollary 6.5 [27]), it does not admit totally geodesic null hypersurfaces.

2.2. Rigged Connection

We want to develop all the usual tools from the Riemannian structure on L provided by $\widetilde{g}$. The first step is to understand the relationship between the Levi–Civita connections ∇ and $\widetilde{\nabla}$ of g and $\widetilde{g}$, respectively, and its dependence on the rigging. After that, we will do the same for their curvatures. Observe that the null Gauss–Codazzi equations shown above relate the curvatures of ∇ and ∇^L. On the other hand, ∇^L and $\widetilde{\nabla}$ are both connections on the same manifold L, so their relationships can be established using the difference tensor $D^L = \nabla^L - \widetilde{\nabla}$. We also take $D = \nabla - \widetilde{\nabla}$ for convenience. Recall that D^L is a symmetric tensor on $\mathfrak{X}(L)$.

Proposition 2. *If L_ξ denotes the Lie derivative along ξ, then we have*

$$g(D(U,V),W) = -\frac{1}{2}\left(\omega(W)(L_\xi\widetilde{g})(U,V) + \omega(U)d\omega(V,W) + \omega(V)d\omega(U,W)\right)$$

for all $U,V,W \in \mathfrak{X}(L)$.

Since $D - D^L = B \cdot N$ we also have the following relation.

$$\begin{aligned} g(D^L(U,V),W) = &-\frac{1}{2}\left(\omega(W)(L_\xi\widetilde{g})(U,V) + \omega(U)d\omega(V,W) + \omega(V)d\omega(U,W)\right) \\ &- B(U,V)\omega(W) \end{aligned} \tag{18}$$

for all $U,V,W \in \mathfrak{X}(L)$.

The following identity will be useful in the theory. It shows a nice tuning between the rigging and B, C, ω, and τ.

Proposition 3. *Given $U \in \mathfrak{X}(L)$ and $X \in \mathcal{S}$ it holds*

$$-2C(U,X) = d\omega(U,X) + \big(L_\zeta g\big)(U,X) + g(\zeta,\zeta)B(U,X),$$
$$C(\xi,X) = -\tau(X) - \widetilde{g}(\widetilde{\nabla}_\xi \xi, X).$$

Note that we can switch $d\omega$ with $d\alpha$ in the above identity.

Another useful information is the following result which establishes a link between the intrinsic and the extrinsic geometry of $(L, \widetilde{g})$.

Proposition 4. *Take $X, Y, Z \in \mathcal{S}$. It holds*

1. $\widetilde{\nabla}_X Y = \nabla^*_X Y - \widetilde{g}(\widetilde{\nabla}_X \xi, Y)\xi$ *and, thus,* $\widetilde{g}(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z)$.
2. $\big(L_\xi \widetilde{g}\big)(X,Y) = -2B(X,Y)$. *In particular* $H = -\widetilde{\mathrm{div}}\xi$.

Definition 3. *Let (M, g) be a semi-Riemannian manifold. A vector field U is orthogonally conformal (resp. orthogonally Killing) if $(L_U g)(X,Y) = \rho g(X,Y)$ (resp. $(L_U \widetilde{g})(X,Y) = 0$) for any $X, Y \in U^\perp$. We call ρ the conformal function of U.*

If, moreover, (M, g) is Lorentzian and U is timelike and unitary, then it is called spatially conformal stationary (resp. spatially Killing stationary).

The above definition is usual in Lorentzian geometry, see [27,28] for example, since it codifies a symmetry type of the ambient space.

The first point of the above proposition and the definition of ∇^L allows us to get an explicit formula for D^L restricted to the screen distribution, simplifying the computations for the difference of curvatures. In fact, for all $X, Y \in \mathcal{S}$ we get

$$D^L(X,Y) = \Big(C(X,Y) + \widetilde{g}(\widetilde{\nabla}_X \xi, Y)\Big)\xi. \tag{19}$$

The second point in Proposition 4 implies that L is totally geodesic if, and only if, ξ is orthogonally Killing and it is totally umbilic if, and only if, ξ is orthogonally conformal. Moreover, we have an explicit expression for H as a divergence of a vector field. Although it is not surprising taking into account the expression of H in a local orthonormal adapted basis, this fact allows us to apply Stoke's Theorem. In fact, if M is orientable and L is a null hypersurface admitting a rigging, it is also orientable. If it is compact, then Stoke's theorem implies that $\int_L H d\widetilde{g} = 0$. Therefore, the null mean curvature (associated to any rigging) distributes on any compact null hypersurface so that the above integral is zero, in a way that remembers the curvature of a Torus in surface theory. In particular, it has to vanish at some point, which is a remarkable difference with respect to non-null hypersurfaces.

Another important remark is that we can express B in terms of the rigged data since for any $X, Y \in \mathcal{S}$, point 2 of Proposition 4 means

$$B(X,Y) = -\frac{1}{2}(\widetilde{g}(\widetilde{\nabla}_X \xi, Y) + \widetilde{g}(\widetilde{\nabla}_Y \xi, X)).$$

If $\mathcal{S}$ is integrable, then $d\omega(X,Y) = 0$ for any $X, Y \in \mathcal{S}$ and we have

$$B(X,Y) = -\widetilde{g}(\widetilde{\nabla}_X \xi, Y).$$

Moreover, Equation (19) above simplifies to

$$D^L(X,Y) = (C(X,Y) - B(X,Y))\xi. \tag{20}$$

Let us now explain how a suitable choice of the rigging helps us to get information from the symmetries of the ambient space. The symmetries we consider are through the existence of a rigging with special properties.

Recall that $\zeta \in \mathfrak{X}(M)$ is called a *conformal vector field* with *conformal factor* σ if $L_\zeta g = \sigma g$ for some $\sigma \in C^\infty(M)$. If $\sigma = constant \neq 0$ then it is called *homothetic vector field* and if $\sigma = 0$ then it is called a *Killing vector field*.

We will consider another kind of vector fields with suitable properties concerning the differentiable structure of the ambient space. We say that ζ is a *closed vector field* if its equivalent one-form α is a closed form. In terms of the connection all the three kind of vector fields are formally similar. In fact, if $U, V \in \mathfrak{X}(M)$, then ζ is

1. Conformal if, and only if, $g(\nabla_U \zeta, V) + g(U, \nabla_V \zeta) = 2\sigma g(U, V)$ where σ is the conformal function;
2. Killing if, and only if, $g(\nabla_U \zeta, V) + g(U, \nabla_V \zeta) = 0$;
3. Closed if, and only if, $g(\nabla_U \zeta, V) - g(U, \nabla_V \zeta) = 0$.

Although the existence of a conformal or a Killing vector field is a severe restriction on the metric, codifying a strong symmetry, the existence of a timelike closed vector field is always ensured locally, so there always exists a closed rigging in a neighborhood of any point in any null hypersurface.

Recall that the existence of a timelike, closed and conformal vector field is equivalent to the local decomposition of (M, g) as a generalized Robertson–Walker space, see [27].

The existence of a conformal rigging carries some advantages.

Proposition 5. *If ζ is a conformal rigging for a null hypersurface, then*

- *ξ is g-geodesic, that is $\tau(\xi) = 0$.;*
- *$\tau(X) = C(\xi, X) = -\frac{1}{2}\widetilde{g}(\widetilde{\nabla}_\xi \xi, X)$ for all $X \in \mathcal{S}$.*

In physics, it is usual to suppose that a null hypersurface admits a geodesic null section, that is, a geodesic null vector field. However, this can be only achieved locally. For this, we reparametrize a fixed null section to obtain a geodesic one, which is equivalent to solve an ordinary differential equation for the new parameter. Globally is not possible in general, but the region where it can be solved is enough for physical purposes. An example where it does not exist a global geodesic null section is $\mathbb{T} \times \mathbb{R}$, being $\mathbb{T}$ the Clifton–Pohl torus, see [29], where there is a null geodesic flow spinning around a $\mathbb{S}^1$ factor with increasing speed.

Another result ensuring that we can obtain a geodesic null section under a strong topological condition can be found in (Theorem 18 [14]).

The existence of a closed rigging has also some advantages, besides the integrability of the screen distribution. We call $\mathbb{I}$ the second fundamental form of a leaf of $\mathcal{S}$ as a submanifold of $(L, \widetilde{g})$.

Proposition 6. *If ζ is a closed rigging, then*

- *$\widetilde{\nabla}_X Y = \nabla^*_X Y + B(X, Y)\xi$ for all $X, Y \in \mathcal{S}$;*
- *$\widetilde{\nabla}_U \xi = -A^*(U)$ for all $U \in \mathfrak{X}(L)$;*
- *$\widetilde{\nabla}_\xi \xi = 0$ and $\widetilde{\mathbb{I}}(X, Y) = B(X, Y)\xi$;*
- *$\tau(X) = -C(\xi, X)$.*

The third point above says that the rigged vector field ξ is $\widetilde{g}$-geodesic. It is immediate since ξ is $\widetilde{g}$-unitary and the rigged one-form ω is also closed.

Observe that a conformal rigging provides us with a g-geodesic rigged vector field, whereas a closed rigging gives us a $\widetilde{g}$-geodesic one.

If L is a totally geodesic null hypersurface, then any geodesic of the ambient which is tangent to L at some point remains locally in L. This geometric interpretation of the totally geodesic condition is the same for non-degenerate hypersurfaces, however being totally umbilic has not a clear geometric interpretation for null hypersurfaces, since 0 is always a principal curvature. On the other hand, observe from Equation (12) that even if L is totally umbilic with integrable screen, the leaves of $\mathcal{S}$ are not totally umbilic codimension two submanifolds of the ambient space in general. The following corollary gives a geometric interpretation of being totally geodesic or umbilic if we choose a closed rigging.

Corollary 1. *Let L be a null hypersurface and ζ a closed rigging for it.*

1. *L is totally geodesic if, and only if, the rigged vector field ξ is $\widetilde{g}$-parallel;*
2. *L is totally geodesic (resp. umbilic) if, and only if, each leaf of $\mathcal{S}$ is totally geodesic (resp. umbilic) as a hypersurface in $(L, \widetilde{g})$.*

Another benefit of taking a closed rigging is that we have an explicit expression for the rigged connection. For this, we introduce a tensor $\bar{C}$ which contains the information of C and τ.

$$\bar{C}(U,V) = C(P(U),P(V)) - \omega(V)\tau(P(U)) - \omega(U)\tau(P(V)) - \omega(U)\omega(V)\tau(\xi)$$

for all $U, V \in \mathfrak{X}(L)$.

Proposition 7. *Suppose that ζ is a closed rigging for a null hypersurface and take $U, V, W \in \mathfrak{X}(L)$. Then*

- $\widetilde{g}(\widetilde{\nabla}_U V, W) = g(\nabla_U V, W) + \omega(W)U(\omega(V))$;
- $\widetilde{\nabla}_U V = \nabla^L_U V + (B(U,V) - \bar{C}(U,V))\xi$.

Point 2 of this result completes the information in Equation (20) to all possible directions in L.

It would be interesting to establish a simple formula for the rigged connection under a rigging change as those formulas for B, C, τ, and ∇^L. We do not know it, but we have the following related result. Recall that the shape operator $A^* : TL \to TL$ of $\mathcal{S}$ is symmetric and it has real eigenvalues called principal curvatures.

Theorem 1 ([25]). *Suppose ζ is a rigging for a null hypersurface L and ζ' is another rigging such that their rigged connections coincide, $\widetilde{\nabla} = \widetilde{\nabla}'$. We decompose*

$$\zeta' = \Phi N + X_0 + g(\zeta', N)\xi$$

with $X_0 \in \mathcal{S}$.

- *If X_0 vanishes and $\Phi^2 \neq 1$ everywhere in L, then L is totally geodesic, $d\omega = 0$ and Φ is a constant;*
- *If X_0 does not vanish at any point, then the multiplicity of the 0-principal curvature is at least* $\dim L - 2$.

As a corollary, we have that for a totally umbilic null hypersurface with non zero null mean curvature, the unique rigging that induces the same rigged connection as a given rigging ζ, is of the form $\zeta' = \pm N + g(\zeta', N)\xi$. This is because if X_0 does not vanish at some point, then we can apply the second point of the above theorem in a neighborhood of this point to obtain a contradiction. Therefore, we have that X_0 vanishes and we can apply the first point of the theorem to obtain $\Phi^2 = 1$.

We finish this subsection with the following question. Under what conditions do the rigged connections $\widetilde{\nabla}$ and the induced connection ∇^L coincide? In this case, we say that $\widetilde{\nabla}$ is a preferred connection and the following result gives us necessary and sufficient conditions, see [25].

Theorem 2. *Given a rigging ζ for a null hypersurface, it holds $\widetilde{\nabla} = \nabla^L$ if and only if $B = \bar{C}$, that is, $B = C$ and $\tau = 0$.*

2.3. Curvature Relations

The aim of this section is to establish a link between the curvature invariants of the ambient space and those of the Riemann rigged metric on a null hypersurface. The strategy is to use the Gauss–Codazzi equations given in Section 2.1 that relate the curvature of the ambient space with the curvature of the induced connection ∇^L. After that, we use the fact that ∇^L and $\widetilde{\nabla}$ are both linear connections on L to relate their curvatures. The computations are tedious and they need the results of the above section, see [17] for details. We achieve only partial results, but it is enough to show the potential of its use and to obtain new valuable information.

If we have two arbitrary symmetric connections ∇^L and $\widetilde{\nabla}$ on a manifold L, then their curvature tensors R^L and $\widetilde{R}$ are related by

$$\begin{aligned} R^L_{UV}W &= \widetilde{R}_{UV}W + (\widetilde{\nabla}_U D^L)(V,W) - (\widetilde{\nabla}_V D^L)(U,W) \\ &\quad + D^L(U, D^L(V,W)) - D^L(V, D^L(U,W)), \end{aligned}$$

for all $U, V, W \in \mathfrak{X}(L)$, where $D^L = \nabla^L - \widetilde{\nabla}$. Using this formula, we can obtain the difference of the sectional curvatures, for g and $\widetilde{g}$, of planes in the screen distribution.

Theorem 3. *Let M be a Lorentzian manifold, L a null hypersurface and ζ a rigging for it. If $\Pi = span(X,Y)$, being $X, Y \in \mathcal{S}$ unitary and orthogonal vectors, then*

$$\begin{aligned} K(\Pi) - \widetilde{K}(\Pi) &= -C(Y,Y)B(X,X) - C(X,X)B(Y,Y) \\ &\quad + (C(X,Y) + C(Y,X))B(X,Y) \\ &\quad + B(X,X)B(Y,Y) - B(X,Y)^2 + \frac{3}{4}d\omega(X,Y)^2. \end{aligned}$$

Observe that in the case of a totally geodesic hypersurface, we have the inequality $K(\Pi) \geq \widetilde{K}(\Pi)$ for any tangent plane contained in $\mathcal{S}$.

We are now interested in a formula relating the curvature for a null plane in L. In this case we need an additional hypothesis. Call $S = \widetilde{\nabla}\xi : \mathfrak{X}(L) \to \mathfrak{X}(L)$ the operator defined by $S(U) = \widetilde{\nabla}_U \xi$ and $S^* : \mathfrak{X}(L) \to \mathfrak{X}(L)$ its adjoint endomorphism respect to $\widetilde{g}$, that is, S^* is the unique endomorphism such that $\widetilde{g}(S(U), V) = \widetilde{g}(U, S^*(V))$ for all $U, V \in \mathfrak{X}(L)$.

Since $\widetilde{g}(S^*(U), \xi) = \widetilde{g}(U, S(\xi)) = \widetilde{g}(U, \widetilde{\nabla}_\xi \xi)$, we can decompose $S^*(U)$ as

$$S^*(U) = S^{*\perp}(U) + \widetilde{g}(U, \widetilde{\nabla}_\xi \xi)\xi, \quad (21)$$

where $S^{*\perp}(U)$ is $\widetilde{g}$-orthogonal to ξ. On the other hand, observe that $\widetilde{g}(U, S^*(\xi)) = \widetilde{g}(S(U), \xi) = 0$ for all $U \in \mathfrak{X}(L)$, so $S^*(\xi) = 0$.

Definition 4. *We say that the rigged vector field ξ is orthogonally normal if*

$$\widetilde{g}(S(X), S(X)) = \widetilde{g}(S^{*\perp}(X), S^{*\perp}(X)) \quad (22)$$

for all $X \in \mathcal{S}$.

Orthogonally normal rigged vector fields appear in two important cases: integrable screen distributions and totally umbilic null hypersurfaces. Indeed, if the screen distribution $\mathcal{S}$ is integrable, a straightforward computation shows $S^{*\perp}(X) = S(X)$ for all $X \in \mathcal{S}$ and obviously Equation (22) is satisfied. If L is totally umbilic, then point 2 of Proposition 4 implies that ξ is orthogonally conformal and an easy computation gives us $S^{*\perp}(X) = 2\rho X - S(X)$ for certain $\rho \in C^\infty(L)$ and all $X \in \mathcal{S}$. Now it is trivial to see that ξ is orthogonally normal.

Theorem 4. *Let (M, g) be a Lorentzian manifold, L a null hypersurface and ζ a rigging for L. Suppose that its rigged vector field ξ is orthogonally normal. If $\Pi = span(X, \xi)$, where $X \in \mathcal{S}$ is a unitary vector, then*

$$\mathcal{K}_\xi(\Pi) - \widetilde{K}(\Pi) = \tau(\xi)B(X,X) - \widetilde{g}(\widetilde{\nabla}_X \widetilde{\nabla}_\xi \xi, X) + \widetilde{g}(X, \widetilde{\nabla}_\xi \xi)^2 + \frac{1}{2}\Big(\widetilde{g}(S^2(X), X) - \widetilde{g}(S(X), S(X)\Big).$$

Corollary 2. *Let L be a null hypersurface and ζ a rigging for it. Suppose that its rigged vector field ξ is orthogonally normal. Then*

$$Ric(\xi,\xi) = \widetilde{Ric}(\xi,\xi) + \tau(\xi)H - \widetilde{\operatorname{div}}\widetilde{\nabla}_\xi \xi + \frac{1}{2}\Big(\operatorname{trace}(S^2) - |S^\perp|^2\Big),$$

where $|S^\perp|^2 = \sum_{i=3}^n \widetilde{g}(S(e_i), S(e_i))$ and $\{e_3, \ldots, e_n\}$ is an orthonormal basis of $\mathcal{S}$.

Using that ξ is orthogonally normal and the Cauchy–Schwarz inequality it is easy to see that

$$\widetilde{g}(S^2(X), X) \leq \widetilde{g}(S(X), S(X)),$$

so the last part of the formula in Theorem 4 and Corollary 2 has sign.

If the rigging is closed, the formulas above symplifies thanks to Proposition 6 and we can obtain the following corollary.

Corollary 3. *Let L be a null hypersurface and ζ a closed rigging for it. Then*

1. $\mathcal{K}_\xi(\Pi) = \widetilde{K}(\Pi) + \tau(\xi)\frac{B(X,X)}{g(X,X)}$, *where $\Pi = span(\xi, X)$ and $X \in \mathcal{S}$.*
2. $Ric(\xi,\xi) = \widetilde{Ric}(\xi,\xi) + \tau(\xi)H$.

Suppose now that the screen distribution $\mathcal{S}$ is integrable. A leaf of $\mathcal{S}$ can be considered as a submanifold of both (M, g) and $(L, \widetilde{g})$. In the first case, we know that the induced Levi–Civita connection is ∇^* and its second fundamental form is $\mathbb{I}^S(X,Y) = C(X,Y)\xi + B(X,Y)N$. In the second case, the induced connection from $(L, \widetilde{g})$ is also ∇^* but its second fundamental form is $\mathbb{I}(X,Y) = B(X,Y)\xi$.

We can relate the curvatures of the leaf corresponding to the different Levi–Civita connections involved. If S is a leaf of $\mathcal{S}$ and we call K^S and $\widetilde{K}^S$ the sectional curvature induced as a submanifold in (M, g) and $(L, \widetilde{g})$, respectively, then

$$\begin{aligned} K^S(\Pi) &= \widetilde{K}^S(\Pi), \\ K(\Pi) &= K^S(\Pi) - C(X,X)B(Y,Y) - B(X,X)C(Y,Y) \\ &\quad + 2C(X,Y)B(X,Y), \\ \widetilde{K}(\Pi) &= \widetilde{K}^S(\Pi) - B(X,X)B(Y,Y) + B(X,Y)^2, \end{aligned}$$

for any tangent plane $\Pi = span(X,Y)$ with $X, Y \in \mathcal{S}$.

3. Applications

There are several families of null hypersurfaces which have great interest. We begin with the natural classification that provides the null second fundamental form, that is, totally umbilic and totally geodesic null hypersurfaces. The significance of totally umbilic null hypersurface is not the same than its Riemannian counterpart, but it means some degree of symmetry that should be determined. An example is Theorem 19 which informs us of a kind of symmetry of a totally umbilic null cone. Another example is Theorem 9 where we claim that any totally umbilic null hypersurface admits locally a twisted rigged metric.

The hypothesis of being totally geodesic is very restrictive, for example, null cones cannot fulfill it. Nevertheless, it is an important hypothesis because isolated black hole horizons are totally geodesic. From a physical point of view, there are a strong development of this kind of horizons but from a geometric point of view we know few properties.

Null cones and black hole horizons are important families of null hypersurfaces that should be analyzed using the rigging technique. Compact null hypersurfaces are also of great interest, since they are exotic objects and their existence carries some restriction on the causality of the ambient manifold.

In the above section, we have shown most of the development of the rigging technique. Our aim now is to provide us with a good relation with the geometry of the ambient manifold. In this section, we want to show the potential of the technique. We believe that most of the problems that we will present here can not be treated with previous techniques, because we will use results most of them derived from the use of the rigged metric, or exploit the tuning of the rigged data with the rigging, but of course there are some exceptions.

We will use several subsections to present different kind of null hypersurfaces more or less close to the tittle, but in fact the division cannot be rigid.

3.1. Totally Umbilic and Totally Geodesic Null Hypersurfaces

Recall that the classification of null hypersurfaces in totally umbilic, totally geodesic, and other is made using the null second fundamental form and it does not depend on the chosen rigging. The main reference for this subsection is [17].

We have seen in Proposition 1 that the existence of totally geodesic null hypersurfaces is not guaranteed in general. The same occurs for totally umbilic null hypersurface. To see it we need the following theorem.

Theorem 5. *Let $I \times_f F$ be a generalized Robertson–Walker space. If L is a totally umbilic null hypersurface, then for each $(t_0, x_0) \in L$ there exists a decomposition of F in a neighborhood of x_0 as a twisted product with one dimensional base*

$$\left(J \times S, ds^2 + \mu(s,z)^2 g_S\right),$$

where x_0 is identified with $(0, z_0)$ for some $z_0 \in S$ and L is given by

$$\{(t,s,z) \in I \times J \times S : s = \int_{t_0}^{t} \frac{1}{f(r)} dr\}.$$

Moreover, if H is the null mean curvature of L, then

$$\mu(s,z) = \frac{f(t_0)}{f(t)} exp\left(\int_0^s \frac{H(t,r,z)f(t)^2}{n-2} dr\right)$$

for all $(t,s,z) \in L$.

Conversely, if F admits a twisted decomposition in a neighborhood of x_0 as above, then $L = \{(t,s,z) \in I \times J \times S : s = \int_{t_0}^{t} \frac{1}{f(r)} dr\}$ is a totally umbilic null hypersurface with null mean curvature

$$H = \frac{n-2}{f(t)^2}\left(f'(t) + \frac{\mu_s(s,z)}{\mu(s,z)}\right).$$

As an immediate consequence we can give the following obstruction to the existence of totally umbilic (geodesic) null hypersurfaces in a generalized Robertson–Walker space.

Corollary 4. *If the fibre of a generalized Robertson–Walker space does not admit any local decomposition as a twisted (warped) product with one dimensional base, then it does not exist any totally umbilic (geodesic) null hypersurface.*

The non-existence of a local decomposition of a Riemannian manifold as a twisted or warped product with one-dimensional base can be usually deduced from a curvature analysis. For example, in a twisted product with a one dimensional factor, any plane containing the tangent direction to the one-dimensional base has the same sectional curvature. Thus, for example $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit any local decomposition as a twisted product with one-dimensional base since for any tangent vector we can find two planes with different sectional curvatures. Therefore, using the above corollary, in the generalized Robertson–Walker space $I \times_f (\mathbb{S}^2 \times \mathbb{S}^2)$ there are not totally umbilic null hypersurfaces.

We now obtain a consequence of the relations on curvatures obtained in the above subsections, that can be more accurate for a totally geodesic null hypersurface.

Theorem 6. *Let L be a totally geodesic null hypersurface and ζ a closed rigging for it. Given $U, V, W \in \mathfrak{X}(L)$ and $X, Y \in \mathcal{S}$, the following holds.*

1. $R_{UV}W - \widetilde{R}_{UV}W = g(R_{UV}W, N)\xi$ *for all* $U, V, W \in \mathfrak{X}(L)$;
2. *If* $\Pi = span\{X, U\}$ *is a tangent plane to L, then*

$$K(\Pi) = \left(1 + \frac{g(X,X)\widetilde{g}(U,\xi)^2}{g(X,X)g(U,U) - g(X,U)^2}\right)\widetilde{K}(\Pi) \text{ if } \Pi \text{ is spacelike,}$$
$$\mathcal{K}_\xi(\Pi) = \widetilde{K}(\Pi) = 0 \text{ if } \Pi \text{ is null;}$$

3. *The Ricci tensor of* $\widetilde{g}$ *is given by*

$$\widetilde{Ric}(X,Y) = Ric(X,Y) - g(R_{\xi X}Y, N) - g(R_{\xi Y}X, N),$$
$$\widetilde{Ric}(\xi, U) = 0;$$

4. *If* $\widetilde{s}$ *and s denote the scalar curvature of* $(L, \widetilde{g})$ *and* (M, g), *respectively, then*

$$s - \widetilde{s} = 4Ric(\xi, N) - 2K(span(\xi, N)).$$

In [17], point 3 of this theorem also claims that $Ric(\xi, U) = 0$, but it is wrong (we thank R. Hounnonkpe for pointing out this). However, if additionally M holds the null dominant energy condition, then this claim is true, see Section 3.7.

We say that a Lorentzian manifold satisfies the reverse null convergence condition if $Ric(u, u) \leq 0$ for any null vector $u \in TM$. Although the opposite inequality is the usual one in physical applications, observe that the reverse null convergence condition includes the important family of Ricci-flat spacetimes.

Theorem 7. *Let (M, g) be an orientable Lorentzian manifold with $n \geq 3$ which obeys the reverse null convergence condition. If there exists a timelike conformal vector field on M, then any compact totally umbilic null hypersurface is totally geodesic.*

The proof of this theorem is a very good example of the philosophy of the rigging technique. Since the rigging is conformal, we know that the rigged vector field is g-geodesic. On the other hand, it is also orthogonally normal because L is totally umbilic, so we can apply Corollary 2 and Proposition 4 to obtain

$$Ric(\xi,\xi) - \widetilde{Ric}(\xi,\xi) = -\widetilde{div}\widetilde{\nabla}_\xi \xi + tr(S^2) - (n-2)\rho^2,$$

where $B = \rho g$. Integrating on the null hypersurface L respect to $\widetilde{g}$ and using a classical Bochner formula we obtain

$$\int_L Ric(\xi,\xi) d\widetilde{g} = (n-2)(n-3) \int_L \rho^2 d\widetilde{g}.$$

If $n \geq 4$ the conclusion of the theorem easily follows from the fact $Ric(\xi, \xi) \leq 0$. In the case $n = 3$ this last argument is not valid but we can prove the following stronger result based on the Gauss–Bonnet Theorem.

Theorem 8. *Let (M, g) be a three dimensional Lorentzian manifold furnished with a timelike conformal vector field. If it holds the null convergence condition or the reverse null convergence condition, then any compact and orientable null surface is totally geodesic.*

Moreover, if $Ric(u, u) \neq 0$ for all null vector u, then it does not exist any compact orientable null surface.

An example where the hypotheses of the Theorem 7 are fulfilled is the Lorentzian torus

$$\mathbb{T}^n = \left(\mathbb{S}^1 \times \ldots \times \mathbb{S}^1, dx_1 dx_2 + \sum_{i=3}^{n} dx_i^2\right).$$

and the null hypersurface $L = \{x \in \mathbb{T}^n : x_2 = 0\}$. In this case, $\mathbb{T}^n$ is flat, L is a compact and totally geodesic null hypersurface and $\zeta = \partial x_1 - \partial x_2$ is a timelike and conformal (in fact parallel) rigging.

The following theorem is one of the most important in this theory. It shows how the presence of a closed rigging unveils information on the local rigged metric structure of a totally umbilic null hypersurface.

Theorem 9. *Let (M, g) be a Lorentzian manifold, L a totally umbilic null hypersurface and ζ a closed rigging for L. Given $p \in L$, $(L, \widetilde{g})$ is locally isometric to a twisted product $(\mathbb{R} \times S, dr^2 + \lambda^2 g|_S)$, where the rigged vector field ξ is identified with ∂_r, S is the leaf of $\mathcal{S}$ through p and*

$$\lambda(r, q) = \exp\left(-\int_0^r \frac{H(\phi_s(q))}{n-2} ds\right),$$

being ϕ the flow of ξ. In particular, dH is proportional to ω if, and only if, $(L, \widetilde{g})$ is locally isometric to a warped product and L is totally geodesic if, and only if, $(L, \widetilde{g})$ is locally isometric to a direct product.

Moreover, if L is simply connected and ξ is complete, the above decomposition is global.

We can also obtain a global decomposition assuming the existence of a timelike gradient vector field on M instead of the simply connectedness because in this case the integral curves of ξ intersect any leaf of $\mathcal{S}$ at only one point and the flow of ξ splits L globally as $\mathbb{R} \times S$. Recall that in a stably causal spacetime it always exists a timelike gradient vector field.

Remark 1. *Compactness is an obstruction to obtain the global decomposition of a totally umbilic null hypersuperface. Even more, the presence of a rigging which is a gradient vector field prevents the null hypersurface to be compact (not necessarily totally umbilic), see [17,25].*

Or more generally, if there exists a rigging for a null hypersurface L and there is a gradient vector field which is not proportional to the rigged vector field at any point, then L is not compact. In fact, if ζ is a rigging for L and we decompose $\nabla f = X + a\xi + bN$, where $X \in \mathcal{S}$ and $a, b \in C^\infty(L)$, then we can easily check that the gradient of $f \circ i$ respect to the rigged metric is $\widetilde{\nabla}(f \circ i) = X + b\xi$. If L is compact, then there is a critical point $p \in L$ of $f \circ i$. Thus $X_p = 0, b(p) = 0$ and, therefore, $\nabla f_p = a(p)\xi_p$, contradiction.

We show several examples of the above theorem.

Example 2. *Consider the Minkowski space $\mathbb{L}^{n+1} = \left(\mathbb{R}^{n+1}, -dx_0^2 + \ldots + dx_n^2\right)$, the future null cone with vertex at the origin $C_0^+ = \{(x_0, \ldots, x_n) : -x_0^2 + \ldots + x_n^2 = 0, x_0 > 0\}$ and the rigging*

$\zeta = -\partial_{x_0}$. *The rigged vector field is* $\xi = \frac{1}{x_0}P$*, where* P *is the position vector field based at* $p = 0$ *and the null second fundamental form is* $B = -\frac{1}{x_0}g$.

If we take $p = (1, 1, \ldots, 0) \in C_0^+$*, the leaf through* p *of the screen distribution is a* $(n-1)$*-dimensional euclidean sphere of radius* 1 *and the integral curve of* ξ *with initial condition* p *is* $\gamma(t) = (t+1)p$*. Applying the above theorem, the Riemannian manifold* $(C_0^+, \widetilde{g})$ *is isometric to the warped product given by*

$$\left((-1,\infty) \times \mathbb{S}^{n-1}, dr^2 + (1+r)^2 g_{\mathbb{S}^{n+2}}\right),$$

which coincides with the usual metric on the null cone induced from the Euclidean metric in $\mathbb{R}^{n+1}$.

Example 3. *Consider the pseudosphere* $\mathbb{S}_1^n = \{x \in \mathbb{L}^{n+1} : -x_0^2 + \ldots + x_n^2 = 1\}$*, which is furnished with the induced metric from the Minkowski space* $\mathbb{L}^{n+1}$*, and decompose* $-\partial_{x_0} = \zeta + x_0 P$*, where* P *is the position vector field as above and* $\zeta_x \in P_x^{\perp} = T_x\mathbb{S}_1^n$ *for any* $x \in \mathbb{S}_1^n$*. The vector field* ζ *restricted to* $\mathbb{S}_1^n$ *is a timelike, closed, and conformal vector field and we suppose that it is past-directed.*

The future null cone of $\mathbb{S}_1^n$ *with vertex at* $p = (0, \ldots, 1) \in \mathbb{S}_1^n$ *is given by* $\mathcal{C}_p^+ = \mathbb{S}_1^n \cap C_p^+$*, where* C_p^+ *is the future null cone of* $\mathbb{L}^{n+1}$ *with vertex at* p*. Therefore,* $\mathcal{C}_p^+$ *is a hypersurface of* C_p^+ *that can be obtained intersecting* C_p^+ *and the hyperplane* $x_n = 1$*. If we consider the rigging* ζ*, then the rigged vector field is* $\frac{1}{x_0}P'$ *where* $P' = x_0\partial_{x_0} + x_1\partial_{x_1} + \ldots + x_{n-1}\partial x_{n-1}$*. The rigged metric on* $\mathcal{C}_p^+$ *coincides with the induced metric from the euclidean cone* $(C_p^+, \widetilde{g})$*. Thus,* $\mathcal{C}_p^+$ *is also a* $(n-1)$*-dimensional euclidean cone.*

Example 4. *Let* m *be a positive constant and consider* $Q = \{(u,v) \in \mathbb{R}^2 : uv > \frac{-2m}{e}\}$ *the Kruskal plane with metric* $2F(r(u,v))dudv$*, where* F *and* r *are certain functions. In the Kruskal spacetime* $Q \times_r \mathbb{S}^2$ *the hypersurfaces* $L_{u_0} = \{(u,v,x) \in Q \times \mathbb{S}^2 : u = u_0\}$ *are totally umbilic null hypersurfaces (totally geodesic if* $u_0 = 0$*). If we consider the closed rigging* $\zeta = \frac{1}{F(r)}\partial_u$*, then the rigged vector field is* ∂_v *and* $B = -\frac{r_v}{r}g$*. From Theorem 9,* $(L_{u_0}, \widetilde{g})$ *is isometric to the warped product* $\left(\frac{-2m}{u_0 e}, \infty\right) \times_{\frac{r(u_0,v)}{2m}} \mathbb{S}^2$ *if* $u_0 \neq 0$ *and to the direct product* $\mathbb{R} \times \mathbb{S}^2$ *if* $u_0 = 0$.

3.2. Completeness of the Rigged Metric

For further applications it is interesting to tackle the problem of the completeness of the rigged metric. Compact null hypersurfaces are of course the most relevant case. The non-compact case is not complete in general, but if we can identify some situation where completeness is fulfilled, then we have a good tool to apply standard Riemannian techniques. The details of this subsection can be seen in [30].

Let $M = I \times_f F$ be a generalized Robertson–Walker space. The rigging $\zeta = f\partial_t$ is both a gradient and conformal timelike vector field. Let L be a null hypersurface and h any primitive of $-f$. Take $\zeta = \nabla h = f\partial_t$ as a rigging for L. Its induced rigged vector field is $\xi = \widetilde{\nabla}(h \circ i)$ which has unit $\widetilde{g}$-norm. Recall from [31] the following important fact: A Riemannian manifold (M, g) is complete if, and only if, it admits a proper C^3 function with bounded gradient. So the following result follows.

Proposition 8. *If* L *is a null hypersurface in a generalized Robertson–Walker space* $M = I \times_f F$*, such that* $f \circ i$ *has a primitive which is a proper function on* L*, then* $(L, \widetilde{g})$ *is complete, where* $\widetilde{g}$ *is the rigged metric induced from the rigging* $f\partial_t$.

If F is compact, then the primitive $h : I \longrightarrow \mathbb{R}$ of f is a proper function if, and only if, M is null complete. In fact, $h' = -f < 0$ so it is a diffeomorphism onto its image, but it is surjective due to [32]. Considered as a function $h : I \times F \longrightarrow \mathbb{R}$, it is proper if, and only if, M is null complete. Finally if L is a closed subset then $h \circ i : L \longrightarrow \mathbb{R}$ is also proper. Hence we have.

Proposition 9. *Let $M = I \times_f L$ be a generalized Robertson–Walker space with compact Riemannian factor F. If M is null complete, then any topologically closed null hypersurface is $\widetilde{g}$-complete for the rigging $\zeta = f\partial_t$.*

Example 5. *If the fiber F of a generalized Robertson–Walker $M = I \times_f F$ is complete, then the null completeness of M depends exclusively on the behavior of the warping function, since in [32] it is shown that M is geodesically null complete if, and only if,*

$$\int_a^c f(t)dt = \int_c^b f(t)dt = \infty,$$

where $I = (a, b)$ and c is some fixed point in I.

For example, if F is complete, then any closed null hypersurface in $\mathbb{R} \times_{t^2+1} F$ is $\widetilde{g}$-complete for the rigging $\zeta = f\partial_t$.

The following theorem is based in an elementary observation. The associated rigged metric to the rigging $\zeta = \sqrt{2}\partial t$ is a direct product with one dimensional base.

Theorem 10. *Let $M = \mathbb{R} \times_f F$ be a generalized Robertson–Walker space with complete Riemannian factor (F, g_F). If L is a topologically closed null hypersurface, then the Riemannian structure $(L, \widetilde{g})$ induced by the rigging $\zeta = \sqrt{2}\partial_t$ is complete.*

There are other results that are applicable to more general Lorentzian manifolds with different kind of hypothesis. We show a couple of them in the following theorems.

Theorem 11. *Let (M, g) be a Lorentzian manifold and ζ a closed rigging for a connected noncompact null hypersurface. If ξ is complete and $\mathcal{S}$ has compact leaves then $(L, \widetilde{g})$ is complete.*

Theorem 12. *Let (M, g) be a Lorentzian manifold furnished with a proper function f whose gradient is timelike everywhere. For any topologically closed null hypersurface L, the rigging $\zeta = \nabla f$ makes $(L, \widetilde{g})$ complete.*

We illustrate the above results with some applications to the existence of closed geodesic in $(L, \widetilde{g})$. We need positive definiteness of the null second fundamental form, which has many consequences in the Riemannian case. A well-known theorem due to Hadamard states that if the second fundamental form of a compact immersed hypersurface M of a Euclidean space is positive definite, then M is embedded as the boundary of a convex body [33,34]. Recall that if a complete Riemannian manifold admits a convex function, any closed geodesic belongs to one of its level sets, (Proposition 2.1 [35]).

Proposition 10. *Let ζ be a closed rigging for a null hypersurface L in a simply connected Lorentzian manifold (M, g). If L is $\widetilde{g}$-complete and has screen definite second fundamental form, then $(L, \widetilde{g})$ contains no closed geodesics. In particular, if L is proper totally umbilic, then $(L, \widetilde{g})$ does not contain closed geodesics.*

This proposition remains true if M is not simply connected but the first De Rham cohomology group $H^1(L, \mathbb{R})$ is trivial or the rigged one form ω is exact so that the rigged vector field ξ is a gradient vector field.

The following application gives us a restriction on the topology of a proper totally umbilic null surface (in 3-dimensional Lorentzian manifold) which can admit a $\widetilde{g}$-complete Riemannian metric for a given rigging. It follows from the classification of complete surfaces without closed geodesic, see (Theorem 3.2 [36]).

Theorem 13. *Let (M, g) be a simply connected three dimensional Lorentzian manifold and L a null surface which is non totally geodesic at any point. If there is a closed rigging for L, such that $(L, \widetilde{g})$ is complete, then L is homeomorphic to the plane or the cylinder.*

In the following corollary, we take the rigging $\zeta = \sqrt{2}\partial_t$ to obtain a complete rigged metric, see Theorem 10.

Corollary 5. *Let $M = \mathbb{R} \times_f F$ be a 3-dimensional generalized Robertson–Walker space with complete Riemannian factor (F, g_0). If a null surface is topologically closed and non totally geodesic at any point, then it is homeomorphic to the plane or the cylinder.*

3.3. Null Cones

The rigging technique is specially useful in studying null cones. The main reason is the existence of a rigging with the remarkable property that its rigged vector field is g-geodesic and $\widetilde{g}$-geodesic simultaneously. The main references for this subsection are [17,37].

The definition of null cone itself depends on the authors, so we describe the notion of null cone that we are going to use. Roughly speaking, it is the image of the exponential map of the null cone at its tangent space.

Fixed a timelike vector $e \in T_pM$, its tangent null cone is

$$\widehat{C}_e = \{u \in T_pM : g(u, u) = 0, g(u, e) < 0\}.$$

If $\widehat{\theta}$ is the maximal definition domain of $\exp_p$, then we define the null cone of e as

$$\mathcal{C}_e = \{\exp_p(u) : u \in \widehat{\theta} \text{ is null and } g(u, e) < 0\} = \exp_p\left(\widehat{C}_e \cap \widehat{\theta}\right).$$

Observe that a null cone is not an embedded null hypersurface in general, due to the presence of null conjugate points or null crossing points, i.e., a point $x \in \mathcal{C}_e$, such that there are two distinct null vectors $u_1, u_2 \in \widehat{C}_e$ with $\exp_p(u_1) = \exp_p(u_2) = x$. However, it is clear that near the vertex it is always a embedded null hypersurface. We can take a maximal portion (in some sense and depending on the fixed vector e) of the null cone $\mathcal{C}_e$ which is an embedded null hypersurface as follows.

We define

$$\widehat{S}_{(0,t)} = \{v \in \widehat{C}_e : g(e, v) < t\},$$
$$S_{(0,t)} = \exp_p\left(\widehat{\theta} \cap \widehat{S}_{(0,t)}\right)$$

and take i_p the supremum of $t \in \mathbb{R}$, such that $\exp_p : \widehat{S}_{(0,t)} \to S_{(0,t)}$ is a diffeomorphism. Obviously, $S_{(0,i_p)}$ is an embedded hypersurface and the fact that it is a null hypersurface is a direct consequence of the Gauss lemma.

Null cones in Robertson–Walker spaces (and, therefore, in constant curvature spaces) are totally umbilic. In constant curvature it is well know that the converse holds, but we can find counterexamples in Robertson–Walker spaces. However, we can prove the following characterization [38].

Theorem 14. *Any totally umbilic null hypersurface in a Robertson–Walker space $I \times_f \mathbb{S}^{n-1}$ ($n > 3$) with*

$$\int_I \frac{1}{f(r)} dr > \pi$$

is an open set of a null cone. In particular, it cannot exist totally geodesic null hypersurfaces.

The inequality in the above theorem cannot be sharpened. For example, it is easy to construct a totally geodesic null hypersurface in the De Sitter space $\mathbb{S}^n_1 = \mathbb{R} \times_{cosh(t)} \mathbb{S}^{n-1}$, which is not contained in a null cone. For this, consider $\mathbb{S}^n_1$ as a subset of the Minkowski space $\mathbb{L}^{n+1}$ in the standard way and intersect it with a null plane of $\mathbb{L}^{n+1}$ through the origin. On the other hand, we can apply the theorem to two remarkable types of Robertson–Walker spaces. The first one is the closed Friedmann cosmological model and the second one the direct product $\mathbb{R} \times \mathbb{S}^{n-1}$.

In view of Theorem 14, it is natural to ask if we can find conditions on a null hypersurface to be inside a null cone for more general Lorentzian manifolds. This problem was settled for the first time in [39] where it is claimed that a totally umbilic null hypersurface in a Lorentzian manifold of constant curvature is contained in a null cone. The proof is based on their following claim: in a Lorentzian manifold any totally umbilic null hypersurfaces with zero null sectional curvature is contained in a null cone. However, this is wrong because null completeness is needed in an essential way. The example below shows a relevant counterexample.

Example 6. *Let $Q \times_r \mathbb{S}^2$ be the Kruskal spacetime [29]. Fixed $u_0 \in \mathbb{R}$, the hypersurface*

$$L_{u_0} = \{(u, v, x) \in Q \times \mathbb{S}^2 : u = u_0\}$$

is totally umbilic and null. Moreover, if Π is a null tangent plane to L_{u_0}, then it is spanned by ∂_v and $w \in T\mathbb{S}^2$, so

$$\mathcal{K}_{\partial_v}(\Pi) = -\frac{Hess_r(\partial_v, \partial_v)}{r} = 0,$$

but L_{u_0} is not contained in a null cone.

In any case, the answer to the above question is positive, but we have to assume a strong curvature hypothesis.

Theorem 15. *Let (M, g) be a geodesically null complete Lorentzian manifold with dimension $n > 3$. Take L a totally umbilic null hypersurface satisfying the following properties:*

1. *It has never vanishing null mean curvature;*
2. *$Ric(u, u) = 0$ for all null vector $u \in TL$;*
3. *It is strongly inextensible.*

Then, L is contained in a null cone.

An (embedded) null hypersurface L is strongly inextensible if it is inextensible in the category of immersed null hypersurfaces. This is a necessary topological condition to avoid naive examples.

The basic idea to prove Theorem 15 is to apply Theorem 9 to obtain a decomposition of the rigged metric. For this, we construct a geodesic rigged vector field and an integrable screen distribution, such that all null geodesics that start at a fixed leaf are defined in the same interval.

As we said in Section 2, in general it does not exist a geodesic rigged vector field, but we can construct it along a null geodesic, which is enough for our purposes.

Once this technical issues are solved, we obtain a decomposition of $(L, \widetilde{g})$ in a neighborhood along a null geodesic of the form $\left((0, b) \times S, -dt^2 + \lambda(t)^2 g_S\right)$, where $\lambda(t) = 1 - \frac{t}{b}$ and b is a constant. Since $\lim_{t \to b^-} \lambda(t) = 0$, the $\widetilde{g}$-distance between the curves (t, x) and (t, y) for fixed $x, y \in S$ is converging to zero, but in this decomposition the curves (t, x) are identified with null g-geodesics of the null hypersurface. Therefore, the null geodesics of the null hypersurface are converging to a common point, i.e., it is contained in a null cone. Details can be found in [37].

Now consider the following well known theorem in Lorentzian manifold.

Theorem 16. *Let (M, g) be a Lorentzian manifold and $\gamma : [0, a] \to M$ a null geodesic, such that $\gamma(a)$ is the first conjugate point to $\gamma(0)$ along γ. Let $c > 0$ be a constant.*

1. *If $c^2 \leq \mathcal{K}_{\gamma'}(\Pi)$ for all null plane containing γ', then $a \leq \frac{\pi}{c}$;*

2. *If $\mathcal{K}_{\gamma'}(\Pi) \leq c^2$ for all null plane containing γ', then $\frac{\pi}{c} \leq a$.*

Point 1 was proved in [13] and Point 2 was proved in [19] and after in [18]. It is a challenge to prove this theorem using the rigged metric and the classical Riemannian version of the theorem.

There are some previous issues. First, the rigged metric is in general not complete as we mentioned earlier in this paper. In any case, even if we can find a rigging whose rigged metric is complete, we have the problem that the vertex and its null conjugate point, the points of interest, do not belong to the null cone, which is the natural null hypersurface to set the problem [40]. Additionally, the geodesic γ in the theorem need not be a geodesic for the rigged metric.

We can see that the role of completeness in the Riemannian version of the theorem is not crucial for our purposes because it is enough the existence of the null geodesic segment containing a couple of candidates to be conjugate points at its ends. It is also easy to see that we can study the problem even if both candidates are not defined in the manifold. It is enough to take a parallelly propagated orthonormal basis and prove that the Jacobi equation, which is a second order ordinary differentiable system in matrix form, has finite limits when the parameter approaches the ends of the geodesic.

Let $\gamma : (0, a) \longrightarrow L$ be a geodesic segment in a null hypersurface. With some abuse of nomenclature, we will call $\gamma(a)$ a conjugate point of $\gamma(0)$ if a non-trivial Jacobi field J exists along γ with

$$lim_{t\to 0} J(t) = 0$$
$$lim_{t\to a} J(t) = 0.$$

This wider idea is applicable to $(L, \widetilde{g})$ which in general is a non complete Riemannian manifold.

So, the main preliminary problem is reduced to choose a suitable rigging vector field. Consider L_p the part of a null cone near the vertex p which is an embedding, that is $\exp_p : \widehat{C} \to L_p$ is a diffeomorphism for some open set $\widehat{C} \subset \widehat{C}_e$ which intersects any neighbourhood of the origin, and a null geodesic segment γ starting at p inside it without null conjugate points in L_p. The fact that all the points in the preimage of the geodesic segment are regular points of $\exp_p$ makes possible the construction of the above diffeomorphism.

We will construct a rigging such that γ is an integral curve of the rigged vector field which is both a $\widetilde{g}$-geodesic and a g-geodesic in a neighborhood of γ.

Fix a timelike vector $e \in T_pM$ and consider the function $h : L_p \to \mathbb{R}$ given by

$$h(\exp_p(v)) = -g(e, v),$$

for all $v \in \exp_p(\widehat{C}_e)$. Then $\zeta = \nabla h$ is a rigging for $\mathcal{C}_e$ with associated rigged vector field $\xi = \frac{1}{h}P$, being $P \in \mathfrak{X}(\mathcal{C}_e)$ the position vector field defined by $P_{\exp_p(v)} = \left(\exp_p\right)_{*_v}(v)$.

It is easy to see that the rigged vector field is g-geodesic and it holds $\xi_{\gamma(t)} = \gamma'(t)$. Since it is also a gradient, it is $\widetilde{g}$-geodesic too, Proposition 6. So γ is a geodesic for both the ambient and the rigged metric. This key fact, which is a nice evidence of the tuning of both geometries, allows us to use the following theorem applied to null cones.

Theorem 17. *Let (M, g) be a Lorentzian manifold, L a null hypersurface and ζ a closed rigging for L such that its rigged vector field ξ is g-geodesic. Take $\gamma : [0, 1] \to L$ an integral curve of ξ.*

1. *If J is a Jacobi field in (M, g) along γ with values in TL, then the projection of J onto the screen distribution $\mathcal{S}$ is a Jacobi field in $(L, \widetilde{g})$.*
2. *If V is a Jacobi vector field in $(L, \widetilde{g})$ along γ with $V(0) = V(a) = 0$, then there exists a Jacobi field J in (M, g) along γ with values in TL, such that $J(0) = J(a) = 0$.*

In particular, $\gamma(a)$ is a conjugate point to $\gamma(0)$ in (M,g) if, and only if, it is a conjugate point to $\gamma(0)$ in $(L,\widetilde{g})$ and both share the same multiplicity.

To use this theorem we need an adapted index lemma and Rauch comparison theorem for incomplete geodesics in a Riemannian manifold. We can do it following step by step the proof in [41] taking into account a suitable convergence approaching to the ends of the geodesic segment. We introduce the following definitions.

Definition 5. *Let $(L,\widetilde{g})$ be a Riemannian manifold and $\gamma : (0,a) \to L$ an arc length parametrized geodesic.*

- *If $\lim_{t\to 0} \widetilde{g}(\widetilde{R}_{X\gamma'}\gamma', Y)$ exists for all vector fields X, Y parallel along γ and orthogonal to γ', then we say that the tidal force operator is converging along γ;*
- *If for any Jacobi vector field $J : (0,a) \to TL$ along γ with $\lim_{t\to 0} |J(t)| = 0$ it holds $|J(t)| > 0$ for all $t \in (0,a)$, then we say that γ has not conjugate points in the interval $(0,a)$;*
- *If γ has not conjugate points in the interval $(0,a)$ and there exists a Jacobi vector field $J : (0,a) \to TL$ with $\lim_{t\to 0} |J(t)| = \lim_{t\to a} |J(t)| = 0$, we say that γ has a conjugate point in the limit.*

Lemma 3 (Adapted Index Lemma)**.** *Let $(L,\widetilde{g})$ be a Riemann manifold, $\gamma : (0,a) \to L$ an arc length parametrized geodesic without conjugate points in the interval $(0,a)$ and $J, V : (0,a) \to TL$ vector fields along γ with $\lim_{t\to 0} |J(t)| = \lim_{t\to 0} |V(t)| = 0$, $\lim_{t\to 0} |J'(t)|$ exists, $g(J,\gamma') = g(V,\gamma') = 0$ and $J(t_0) = V(t_0)$ for some $t_0 \in (0,a)$. If the tidal force operator is converging along γ and $I_{t_0}(V,V)$ exists, then $I_{t_0}(J,J) \le I_{t_0}(V,V)$.*

Theorem 18 (Adapted Rauch comparison theorem)**.** *Let $(L,\widetilde{g})$ and $(\overline{L},\overline{g})$ be two Riemannian manifolds and $\gamma : (0,a) \to L$, $\overline{\gamma} : [0,a] \to \overline{L}$ two arc length parametrized geodesics. Suppose that the tidal force operator is converging along γ and take $J : (0,a) \to TL$ and $\overline{J} : [0,a] \to T\overline{L}$ two Jacobi vector fields, such that $\lim_{t\to 0} |J(t)| = |\overline{J}(0)| = 0$, $\lim_{t\to 0} |J'(t)| = |\overline{J}'(0)|$ and $\widetilde{g}(J(t),\gamma'(t)) = \overline{g}(\overline{J}(t),\overline{\gamma}'(t)) = 0$ for all $t \in (0,a)$. Then, the following statements hold.*

- *If $\overline{\gamma}$ has not conjugate points to $\overline{\gamma}(0)$ and $K(span(\gamma',v)) \le \overline{K}(span(\overline{\gamma}',\overline{v}))$ for all $v \in \gamma'^{\perp}$ and all $\overline{v} \in \overline{\gamma}'^{\perp}$, then $|\overline{J}(t)| \le |J(t)|$ for all $t \in (0,a)$.*
- *If γ has not conjugate points in the interval $(0,a)$ and $K(span(\gamma',v)) \ge \overline{K}(span(\overline{\gamma}',\overline{v}))$, then $|\overline{J}(t)| \ge |J(t)|$.*

Now, we have all the machinery to give a new proof of the Theorem 16 following step by step the standard Riemannian arguments with the obvious modifications. We remark that the interest of this new proof is to make visible the powerful of the rigging technique once we chose a rigging vector field adapted to the problem. Details can be found in [17].

An application of this theorem and Theorem 17 is the following. Null conjugate points in Robertson–Walker spaces have maximum multiplicity [42]. These spaces are highly symmetric and it is not clear if the result is a consequence of them or there are some hidden conditions. The authors proved in [38] that it is also true for a null geodesic in a generalized Robertson–Walker space provided the geodesic is contained in a totally umbilic null cone. This suggests that it is a feature of totally umbilic null cones itself, which would be a nice geometric significance of this family of null hypersurfaces. The following theorem shows that it is the case [17].

Theorem 19. *Let (M,g) be a Lorentzian manifold and $\gamma : [0,a] \to M$ a null geodesic, such that $\gamma(a)$ is the first conjugate point to $\gamma(0)$ along γ. If the null cone with vertex at $\gamma(0)$ containing γ is totally umbilic, then $\gamma(a)$ has maximum multiplicity.*

3.4. Compact Null Hypersurfaces

As we mentioned above, compact null hypersurfaces is another distinguished family that deserves attention because any rigging vector field induces a complete rigged metric on it. Its study unveils interesting consequences, in particular several links with the properties of the ambient space. The main reference for this subsection is [43].

Theorem 20. *Let (M, g) be an orientable Lorentzian manifold of dimension* $\dim M \geq 4$ *which verifies the reverse null convergence condition. If there is a reference frame U which is geodesic, spatially conformal stationary and $\nabla(\operatorname{div} U) = \lambda U$, where λ is a non-negative function, then any compact totally umbilic null hypersurface is totally geodesic.*

Note that for a spatially conformal stationary reference frame U (see Definition 3), it holds

$$\operatorname{div} U = \frac{n-1}{2}\rho.$$

where $\dim M = n$.

We can improve the above theorem to any compact null hypersurface, totally umbilic or not, in the presence of the null convergence condition.

Theorem 21. *Let (M, g) be an orientable Lorentzian manifold which holds the null convergence condition. If there is a geodesic and spatially conformal stationary reference frame with conformal function that never vanishes, then any compact null hypersurface is totally geodesic.*

If U is a geodesic and spatially homothetic stationary reference frame, that is, the conformal function is constant, then it is obvious that $\operatorname{div} U$ has sign, so we have.

Corollary 6. *Let (M, g) be an orientable Lorentzian manifold which holds the null convergence condition. If there is a geodesic and spatially homothetic stationary reference frame, then any compact null hypersurface is totally geodesic.*

Since null hypersurfaces are invariant under conformal changes, it seems natural to impose some conditions on the conformal class of a Lorentzian metric to study them. Moreover, the property of being totally umbilic is also a conformal invariant, see [38]. We use this fact to prove the following.

Proposition 11. *Let (M, g) be a Lorentzian manifold with* $\dim M \geq 3$ *furnished with a timelike conformal vector field. If the null convergence condition holds on (M, g^*) for some g^* in the conformal class of g, then any compact null hypersurface in (M, g) is totally umbilic.*

The important null Raychaudhuri equation states that

$$Ric(\xi, \xi) = \xi(H) + \tau(\xi)H - \operatorname{trace}(A^{*2}).$$

Using it and the above proposition we can show that the null hypersurface is totally geodesic in (M, g^*), which implies that it is totally umbilic in (M, g). So we obtain the following corollary.

Corollary 7. *Let (M, g) be an orientable Lorentzian manifold with* $\dim M \geq 3$ *which obeys the reverse null convergence condition. If there is a conformal metric g^* to g which admits a timelike conformal vector field and holds the null convergence condition, then any compact null hypersurface in (M, g) is totally geodesic.*

Remark 1 shows that the presence of compact null hypersurfaces in Lorentzian manifolds is not general. On the other hand, Theorem 7 is a prototype result where some curvature hypothesis prevents the existence of strict totally umbilic and compact null hypersurfaces. In the following results we will see several obstructions to the existence of compact null hypersurfaces. We start with the following proposition [17].

Proposition 12. *Let M be a null complete Lorentzian manifold furnished with a timelike conformal vector field. If the null sectional curvature is positive for all degenerate plane, then it cannot exist any topologically closed embedded null hypersurface.*

The Lorentzian Berger sphere is the sphere $\mathbb{S}^{2n+1}$ furnished with the Lorentzian metric $g_L = g_0 - 2\Omega \otimes \Omega$, where g_0 is the standard metric in $\mathbb{S}^{2n+1}$ and Ω is the g_0-equivalent one form to the Hopf vector field. It is geodesically complete, the null sectional cuvature is positive for all null plane and the Hopf vector field is timelike and Killing [19,44]. Therefore, using the above proposition, it does not admit any topologically closed embedded null hypersurface.

Theorem 22. *Let $(M = M_1 \times M_2, g)$ be a Lorentzian manifold with M_1 Lorentzian and M_2 non compact. If there exists a timelike vector field in M_1, then M does not admit compact null hypersurfaces.*

The proof of the above result uses Remark 1 and the existence of a submersion on M_2, see [45]. It is powerful, since it can be applied to the important family of double twisted products. In fact, if $M = M_1 \times_{(f_1,f_2)} M_2$ is a time orientable doubly twisted product with (M_1, g_1) Lorentzian and (M_2, g_2) Riemannian and non compact, then M admits no compact null hypersurfaces.

This family includes examples of generalized Robertson–Walker spaces and standard static Lorentzian manifolds of type $\mathbb{S}^1 \times_{(f,1)} F$ with F non-compact. Observe that if we take $\mathbb{R}$ in this example as the first factor, then it is stably causal and we can apply directly Remark 1 to conclude that it does not admit compact null hypersurfaces. Another example included in the above family is the Lorentzian product $M = \mathbb{R}^3 \times \mathbb{R}$ with metric $g = -(dt + f dx)^2 + h dx^2 + dy^2 + dz^2$, being f and h smooth positive functions on $\mathbb{R}^3$. In particular, taking $f = e^x$ and $h = \frac{1}{2}e^{2x}$, it follows that the Gödel spacetime admits no compact null hypersurfaces.

The following result gives us an unexpected obstruction to the existence of compact null hypersurfaces in odd dimensional Lorentzian manifolds. Its proof is a nice argument using the Euler characteristic, see [46].

Theorem 23. *Let (M, g) be a time orientable odd dimensional Lorentzian manifold. If there is a spacelike gradient vector field with only two critical points, then it does not admit compact null hypersurfaces.*

A convex (resp. strictly convex) function on a Lorentzian manifold (M, g) is a smooth real-valued function whose Hessian is positive semidefinite (resp. positive definite). They have been used in [35,47] in a semi-Riemannian setting.

In the following results, we see that the presence of convex functions restricts the topological and geometrical properties of null hypersurfaces. This will have implication in the causal structure showing once more the relevance of the properties of null hypersurfaces in the ambient space through the rigging technique.

Proposition 13. *If a Lorentzian manifold admits a strictly convex function and a timelike conformal vector field, then it does not contain compact null hypersurfaces.*

If (M, g) admits no timelike conformal vector field but a geodesic spatially conformal stationary reference frame, then we obtain the following.

Proposition 14. *Let (M, g) be a Lorentzian manifold admitting a strictly convex function. If there is a geodesic spatially conformal stationary reference frame U, such that $\nabla(\operatorname{div} U) = \lambda U$, where λ is a non negative function on M, then (M, g) does not contain compact null hypersurfaces.*

Certain property on the curvature of two metrics in the same conformal class also gives us an obstruction to the existence of compact null hypersurfaces.

Proposition 15. *Let (M, g) be a Lorentzian manifold furnished with a timelike conformal vector field. If there are g_1, g_2 two metrics conformal to g, such that $Ric_1(u, u) < Ric_2(u, u)$ for all null vector $u \in TM$, then there are not compact null hypersurfaces in (M, g).*

The proof is based on the formula relating the Ricci curvature of two conformal metrics and the fact that the rigged vector field is geodesic for both metrics. Observe that being a null hypersurface is a conformal invariant, so the above proposition also implies that there are not compact null hypersurfaces in (M, g^*) for any conformal metric g^* to g.

The existence of a rigging vector field for a compact null hypersurfaces is very restrictive, because the existence of the induced rigged vector field forces the Euler characteristic to be zero. There is another situation in which the topology of the null hypersurface influences the existence of a rigging vector field for it. The reference for these ideas is [30].

Theorem 24. *A compact null hypersurface with trivial first De Rham cohomology group does not admit a closed rigging.*

As a corollary of this theorem, a Lorentzian manifold furnished with a closed timelike vector field does not admit compact null hypersurfaces with trivial first De Rham cohomology group. In particular, it does not admit simply connected compact null hypersurfaces either.

Given a rigging for a null hypersurface, we say that it is screen conformal if $C = \varphi B$ for some $\varphi \in C^\infty(L)$. Whereas the tensor B codifies important properties of the null hypersurface, the tensor C codifies properties of the screen distribution. The full geometric significance of the screen conformal condition is not well known yet. Observe that it implies that the restriction of C to the screen distribution is symmetric, so the screen distribution is integrable. There are important examples of null hypersurfaces which admit a screen conformal rigging vector field. Among them, we mention its existence in generalized Robertson–Walker spaces (in particular spaces of constant curvature), plane fronted waves, Kruskal space, etc.

Proposition 16. *Let ζ be a rigging for a compact null hypersurface L in a Lorentzian manifold of constant curvature. If the screen distribution is conformal and the first De Rham cohomology group is trivial, then L is totally geodesic.*

Using this, we find an obstruction to the existence of a screen conformal rigging in some class of compact null hypersurfaces.

Proposition 17. *A 4-dimensional Lorentzian manifold of constant curvature with a compact null hypersurface with finite fundamental group does not admits any screen conformal rigging.*

3.5. Causality

Now we study some relationships between causality theory and null hypersurfaces. The most immediate was given in Remark 1 of Section 3.5, where it is shown that in a stably causal Lorentzian manifold there are not compact null hypersurfaces. Here, we will see several implications of the properties of null hypersurfaces in causality theory. The main reference for this subsection is [43].

The causal hierarchy and its main properties can be found in [29,48–50] for example. In this subsection, all Lorentzian manifolds are supposed time oriented. We will use some causality conditions that are no so frequent in the literature, so we recall their definitions.

A point $p \in M$ is a future endpoint of a future-directed causal curve $\gamma : I \longrightarrow M$ if for every neighborhood O of p there exists a point $t_0 \in I$, such that $\gamma(t) \in O$ for all $t > t_0$.

A causal curve is future inextensible (respectively, past inextensible) if it has no future (respectively, past) endpoints.

A future inextensible causal curve $\gamma : I \longrightarrow M$ is totally future imprisoned in the compact set C if there is $t_0 \in I$ such that $\gamma(t) \in C$ for every $t > t_0$, i.e., it enters and remains in C. It is partially future imprisoned if for every $t_0 \in I$ there is $t > t_0$, such that $\gamma(t) \in C$, i.e., it continually returns to it. On the other hand, the curve escapes to infinity in the future if it is not partially future imprisoned in any compact set.

A spacetime is non-total future imprisoning if no future inextensible causal curve is totally future imprisoned in a compact set and it is non-partial future imprisoning if no future inextensible causal curve is partially future imprisoned in a compact set.

In [51], it is shown that a spacetime is non-total future imprisoning if, and only if, it is non-total past imprisoning, thus one can simply speak of the non-total imprisoning property.

Finally, (M, g) is future-distinguishing if for every $p, q \in M$ with $p \neq q$ it holds if $I^+(p) \neq I^+(q)$.

The non-total imprisoning condition is an obstruction to the existence of compact null submanifolds, not only hypersurfaces.

Proposition 18. *A non-total imprisoning Lorentzian manifold does not contain compact null submanifolds.*

The causal hierarchy and the above proposition tell us that distinguishing, strongly causal, stable causal, and globally hyperbolic Lorentzian manifolds cannot contain compact null submanifolds. However, there exists causal spacetimes which contain compact null hypersurfaces, as the following example shows.

Example 7. *Consider $\mathbb{R}^3$ described by coordinates (t, y, z) and identify (t, y, z) and $(t, y, z + 1)$, as well as (t, y, z) and $(t, y + 1, z + a)$, where a is an irrational number. The resulting manifold, which is diffeomorphic to $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$, with the metric*

$$g = -(\cosh t - 1)^2(dt^2 - dy^2) - dtdy + dz^2$$

is called Carter space. It is totally imprisoning and it is also causal, see ([52], p. 195), but $\{0\} \times \mathbb{S}^1 \times \mathbb{S}^1$ is a compact null hypersurface.

A spacetime is called disprisoning if for each inextensible geodesic $\gamma : (a, b) \longrightarrow M$ and any fixed $t_0 \in (a, b)$ the images of each of the two maps $\gamma_{|(a,t_0]}$ and $\gamma_{|[t_0,b)}$ fail to have compact closure. It is null disprisoning if this property is satisfied for each inextensible null geodesic. Now we see that this condition jointly with the existence of a timelike conformal vector field also prevents the existence of compact null hypersurfaces.

Proposition 19. *A null-disprisoning Lorentzian manifold admitting a timelike conformal vector field does not contain compact null hypersurfaces.*

In the proof of this proposition it is crucial the presence of the timelike conformal vector field, because it implies that the rigged vector field is geodesic.

Now we determine conditions to add to a chronological or causal spacetime to gain higher step in the causality hierarchy.

Theorem 25. *Let (M, g) be a time orientable and null complete 3-dimensional Lorentzian manifold satisfying the null convergence condition. Suppose that there exists a never vanishing nonspacelike vector field which is a gradient. If (M, g) is causal, then it is non total imprisoning.*

The proof uses rigging and dynamical system techniques, a result on the existence of a null line contained in a compact minimal invariant set given in [53] and a result in [12] to see that in fact the above set is a compact null hypersurface. This leads us to obtain a contradiction with the existence of a never vanishing nonspacelike vector field which is a gradient, see Remark 1.

We can modify the argument to avoid the use of the existence of compact null hypersurfaces. This allows us to improve the hypothesis from causal to chronological and the dimensional restriction.

Theorem 26. *Let (M, g) be a chronological null complete Lorentzian manifold of dimension greater than 2 which satisfies the null convergence condition. If there is a timelike conformal vector field and a geodesic spatially conformal stationary reference frame U, such that* $\operatorname{div} U$ *never vanishes, then (M, g) is non-total imprisoning.*

If we have a timelike conformal vector field, then its unitary is spatially conformal stationary. Therefore, we have the following corollary.

Corollary 8. *Let (M, g) be a null complete Lorentzian manifold of dimension greater than 2 satisfying the null convergence condition which admits a pregeodesic timelike conformal vector field with nowhere vanishing divergence. If (M, g) is chronological, then it is non total imprisoning.*

We can avoid the condition on the divergence of the timelike conformal vector field using a curvature hypothesis on the conformal class or the existence of a convex function.

Theorem 27. *Let (M, g) be a null complete Lorentzian manifold of dimension greater than 2 satisfying the null convergence condition and furnished with a timelike conformal vector field. Suppose that there exists a strictly convex function or some metric in the conformal class of g holds the strict reverse null convergence condition. If (M, g) is chronological, then it is non total imprisoning.*

It is well known that a compact spacetime admitting a timelike conformal vector field is totally vicious. This is not true if the vector field is only causal. Moreover, compact static spacetimes are causally geodesically connected [54]. We prove that a compact spacetime admitting a causal Killing vector field satisfying the null generic condition (see [50] for the definition) is totally vicious. A consequence is that if the universal covering is globally hyperbolic, then the spacetime is geodesically connected, see Theorem 29. To prove this statement, we need to previous results [55].

Lemma 4. *Let (M, g) be a Lorentzian manifold and ζ a timelike affine conformal Killing vector field (resp. a timelike projective vector field) which is a rigging for a compact null hypersurface, then the rigged vector field ξ is g-geodesic.*

Theorem 28. *If (M, g) is a compact Lorentzian manifold admitting a causal Killing vector field, then (M, g) is totally vicious or it contains a compact achronal Killing horizon.*

If, additionally, (M, g) admits a timelike projective vector field (resp. a timelike affine conformal Killing vector field), then the Killing horizon is extremal.

In any globally hyperbolic spacetime, any two causally related points p, q can be joined by a causal geodesic, with length equal to the time-separation between p and q. This is a classic result due to Avez and Seifert [56,57]. This result allows us to prove the second part of the following result.

Theorem 29. *Let (M, g) be a compact Lorentzian manifold satisfying the null generic condition. If there is a causal Killing vector field, then it is totally vicious.*

Moreover, if its universal Lorentzian covering is globally hyperbolic, then it is geodesically connected.

Finally, we see the influence of the presence of generalized time function and quasi-time functions in spacetimes [46,48,58].

Definition 6. *A function $f : M \longrightarrow \mathbb{R}$ is a generalized time function if $f(p) < f(q)$ provided that $p < q$. Observe that if f is continuous, then (M, g) is stably causal.*

A smooth function $f : M \to \mathbb{R}$ is a quasi-time function if its gradient is causal and past directed and every null segment of a null geodesic on which f is constant is injective.

Proposition 20. *A Lorentzian manifold admitting a lower semi-continuous (resp. upper semi-continuous) generalized time function does not contain compact null hypersurfaces.*

Theorem 30. *Let (M, g) be a null complete 3-dimensional Lorentzian manifold satisfying the null convergence condition. If it admits a quasi-time function with compact connected level sets non-diffeomorphic to a torus, then (M, g) is strongly causal.*

Theorem 31. *Let (M, g) be a null complete 3-dimensional Lorentzian manifold satisfying the null convergence condition. If it admits a quasi-time function with null gradient and non-compact connected level sets, then (M, g) is non total imprisoning.*

3.6. Codimension Two Spacelike Submanifolds through a Null Hypersurface

There are important examples of null hypersurfaces admitting an integrable screen distribution, which endows the null hypersurface with a spacelike foliation. This motivates the following question: under what conditions a codimension two spacelike submanifold contained in a null hypersurface is a leaf of an integrable screen distribution? The main reference for this subsection is [59], where this problem is tackled. In general, there is interest in the study of spacelike submanifolds contained in null hypersurfaces, see for example [60–66].

If the induced screen distribution in a null hypersurface L is integrable and S is a leaf, then we know that its second fundamental form and mean curvature vector field as codimension two submanifold of the ambient space are

$$\begin{aligned} \mathbb{I}_{\mathcal{S}}(X,Y) &= C(X,Y)\xi + B(X,Y)N, \\ \vec{H}_{\mathcal{S}} &= \Omega \cdot \xi + H \cdot N, \end{aligned} \tag{23}$$

where Ω is the screen mean curvature given by $\Omega = \operatorname{trace}_{\mathcal{S}} C$.

On the other hand, if Σ is a codimension two spacelike submanifold contained in L, then there is a unique null vector field η defined over Σ, such that $g(\xi, \eta) = 1$ and $T\Sigma^{\perp} = span\{\xi, \eta\}$. If we call $A_\eta : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the Weingarten endomorphism associated to η, $A_\eta(U) = -(\nabla_U \eta)^{T\Sigma}$, then the second fundamental form and the mean curvature vector field of Σ are given by

$$\begin{aligned} \mathbb{I}_{\Sigma}(U,V) &= g(A_\eta(U),V)\xi + B(U,V)\eta, \\ \vec{H}_{\Sigma} &= tr_{\Sigma} A_\eta \cdot \xi + H \cdot \eta. \end{aligned} \tag{24}$$

Now, we consider Σ and the leaf S as hypersurfaces of the Riemannian manifold $(L, \widetilde{g})$. In this case, ξ is a $\widetilde{g}$-unitary and normal vector field to S and we know that its second fundamental form and mean curvature as a hypersurface of $(L, \widetilde{g})$ are

$$\begin{aligned} \widetilde{\mathbb{I}}(X,Y) &= B(X,Y)\xi, \\ \widetilde{H}_{\mathcal{S}} &= H \cdot \xi. \end{aligned}$$

We need to compute the mean curvature $\widetilde{H}_\Sigma$ of Σ as a hypersurface of $(L, \widetilde{g})$. For this, we decompose η according to decompositions (2) and (3) as

$$\eta = X_0 + \alpha\xi + N,$$

where $X_0 \in \mathcal{S}$ and $\alpha = g(N, \eta)$. Since N and η are in the same cone, we have $\alpha \leq 0$. Moreover, $\alpha = -\frac{1}{2}\tan^2\theta$, where $\theta \in [0, \frac{\pi}{2})$ is the $\widetilde{g}$-angle between $T_x\Sigma$ and $\mathcal{S}$ and $E = \cos\theta(X_0 + \xi)$ is a vector field $\widetilde{g}$-unitary and normal to Σ.

In the following proposition we give an explicit formula for $\widetilde{H}_\Sigma$ assuming that the rigged one-form ω is closed, which in particular implies that the screend distribution is integrable.

Proposition 21. *Let L be a null hypersurface of a Lorentzian manifold and ζ a rigging vector field for it such that $d\omega = 0$. If Σ is a spacelike codimension two submanifold of M through L, then the mean curvature $\widetilde{H}_\Sigma$ of Σ respect to E holds*

$$\begin{aligned}\frac{\widetilde{H}_\Sigma}{\cos\theta} &= g(\vec{H}_\Sigma, N) - \Omega - B(X_0, X_0) + \frac{1}{\cos^2\theta}H \\ &\quad + \cos^2\theta\big(C(X_0, X_0) - g(\mathbb{I}_\Sigma(V_0, V_0), N) - \tau(X_0 + V_0)\big),\end{aligned}$$

where $V_0 = X_0 + 2\alpha\xi$.

We need to restrict ourselves to a small class of null hypersurfaces in order to simplify the terms involved in the above formula. For this, in the main results of this subsection we are going to consider null hypersurfaces which admit a distinguished rigging, that is, a rigging such that $\tau = 0$.

There are examples of null hypersurfaces which admit a screen conformal rigging vector field and distinguished rigging simultaneously. For example, they do exist in those Lorentzian manifolds cited in Section 3.4: generalized Robertson–Walker spaces, spaces of constant curvatures, plane fronted waves and Kruskal space. On the other hand, we have the following relations between screen conformal and distinguished riggings.

Lemma 5. *Let L be a null hypersurface and ζ a rigging vector field for it.*

1. *If ζ is screen conformal and distinguished, then $d\omega = 0$;*
2. *If the screen distribution is totally umbilic ($C = \frac{\Omega}{n-2}g$) and ζ is distinguished, then $d\omega = 0$;*
3. *If ζ is a conformal vector field and a screen conformal rigging, then it is distinguished. Moreover, if the conformal factor of ζ never vanishes, then L is totally umbilic.*

Now, we need to make precise the notion of ”being on one side of a leaf”. For this, recall that the rigged vector field ξ is always pregeodesic, so the null geodesics with initial velocity given by ξ are contained, at least locally, in the null hypersurface.

Definition 7. *Suppose that ζ is a rigging vector field for a null hypersurface L with integrable screen distribution and take S a leaf of the screen distribution. The signed distance function of S respect to ζ is*

$$d_S^\zeta = \Pi \circ \Phi^{-1},$$

where Φ is the diffeomorphism $\Phi : (-\varepsilon, \varepsilon) \times U \to V$ given by $\Phi(t, p) = \exp_p(t\xi_p)$, being $U \subset S$ and $V \subset L$ open neighborhoods, and Π is the projection onto the first factor.

Take $p_0 \in \Sigma$ and S the leaf of the screen distribution through p_0. If $d_S^\zeta \geq 0$ in a neighborhood of p_0 in Σ, then Σ is "on one side of S" at least locally. Moreover, Σ and S are tangent at the point p_0, since p_0 is a local minimum of d_S^ζ. In particular, it holds $E_{p_0} = \xi_{p_0}$.

Now, we have all the necessary ingredients to apply the classical Eschenburg maximum principle to Σ and a leaf of the screen considered as hypersurfaces of the Riemannian manifold $(L, \widetilde{g})$, [67]. This will provide us with some theorems ensuring that a codimension two spacelike submanifold through a null hypersurface coincides with a leaf of the screen distribution.

Theorem 32. *Let L be a null hypersurface of a Lorentzian manifold, ζ a rigging vector field for it and Σ a spacelike totally geodesic codimension two submanifold of M through L. Take a point $p_0 \in \Sigma$ and let S be the leaf of the screen distribution through p_0. Suppose that*

1. *ζ is distinguished and screen conformal;*
2. *$d_S^\zeta \geq 0$ in a neighborhood of p_0 in Σ;*
3. *$H(p) \geq 0$ for all p in a neighborhood of p_0 in S.*

Then Σ coincides with the leaf S in a neighborhood of p_0.

Theorem 33. *Let L be a null hypersurface of a Lorentzian manifold, ζ a rigging vector field for it and Σ a spacelike totally umbilic codimension two submanifold of M through L. Take a point $p_0 \in \Sigma$ and let S be the leaf of the screen distribution through p_0. Suppose that:*

1. *ζ is distinguished;*
2. *ζ is screen conformal with conformal factor φ;*
3. *$dH = c\omega$ for some non-positive function $c \in C^\infty(L)$;*
4. *$d_S^\zeta \geq 0$ in a neighborhood of p_0 in Σ;*
5. *$H(p_0) \leq 0$;*
6. *$g(\vec{H}_\Sigma, N) \leq \varphi H$ in a neighborhood of p_0 in Σ.*

Then Σ coincides with the leaf S in a neighborhood of p_0.

Observe that even if Σ is totally geodesic or totally umbilic, L does not need to be also totally geodesic or totally umbilic, since Equation (24) only holds along Σ.

Evidently, the conditions in the above theorems maybe are not satisfied if we change the rigging. However, if we only change the sign of the rigging vector field, then conditions 1–3 of the above theorem still hold, although the inequalities in conditions 4–6 change. On the other hand, since $\tau(\xi) = 0$, using the null Raychaudhuri equation we have that

$$dH(\xi) = \xi(H) = Ric(\xi, \xi) + tr((A^*)^2) \geq Ric(\xi, \xi).$$

Thus, if $Ric(\xi, \xi) \geq 0$ and H is not constant, condition 3 can not hold. This is why Theorem 33 can not be used, for example, in the case of a null cone in a constant curvature Lorentzian manifold.

Certainly, we need to assume many conditions in the above results, but there are examples where we can apply them.

Example 8. *Let (F, g_0) be a Riemannian manifold with dimension $n-1$ and define a generalized Robertson–Walker space*

$$(M, g) = \Big(I \times F, -dt^2 + \phi(t)^2 g_0\Big).$$

Suppose that (F, g_0) can be decomposed as a warped product with one-dimensional base, $(F, g_0) = (J \times K, ds^2 + \mu(s)^2 h_0)$, where $J \subset \mathbb{R}$ and (K, h_0) is a Riemannian manifold. The hypersurface L given by

$$L = \{(t, s, x) \in I \times J \times K : s = \int_{t_*}^{t} \frac{1}{\phi(r)} dr\}$$

for some fixed $t_* \in I$ is a totally umbilic null hypersurface, i.e., $B = \frac{H}{n-2} g$ [38]. Moreover, if we consider the rigging vector field $\zeta = \phi \partial_t$, then the null mean curvature is

$$H_{(t,s,x)} = \frac{n-2}{\phi(t)^2} \left(\phi'(t) + \frac{\mu'(s)}{\mu(s)} \right). \quad (25)$$

It is easy to see that ζ is a closed and conformal vector field, so $\nabla_U \zeta = \Phi' U$ for all $U \in \mathfrak{X}(M)$ and Proposition 3 implies that

$$C = \left(\frac{H\phi^2}{2(n-2)} - \phi' \right) g,$$
$$\Omega = \frac{H\phi^2}{2} - (n-2)\phi',$$
$$\tau = 0.$$

Therefore, the rigging vector field ζ is distinguished and if $H \neq 0$, then it is also screen conformal with factor

$$\varphi = \frac{\phi^2}{2} - \frac{(n-2)\phi'}{H}.$$

On the other hand, the leaf of the screen distribution through a point $p_0 = (t_0, s_0, x_0) \in L$ is given by $S = \{(t, s, x) \in I \times J \times K : t = t_0, s = s_0\}$, thus from Equation (25) we see that H is constant on the leaves and, therefore, $dH = c\omega$ for some $c \in C^\infty(L)$.

Since $\zeta = -\frac{1}{\phi}\partial_t - \frac{1}{\phi^2}\partial_s$, if we fix $p_0 \in L$ and S the leaf through p_0, the condition $d_S^\zeta(p) \geq 0$ is equivalent to $t(p) \leq t(p_0)$, where $t : M \to \mathbb{R}$ is the canonical projection onto the first factor. Moreover, the transverse vector field is $N = \frac{1}{2}(\phi\partial_t - \partial_s)$.

We particularize the above situation to the case of the Lorentzian manifold $(M, g) = \left(\mathbb{R} \times \mathbb{H}^{n-1}, -dt^2 + g_0\right)$. The hyperbolic space $\mathbb{H}^{n-1}$ can be decomposed as

$$\left(\mathbb{R} \times \mathbb{R}^{n-2}, ds^2 + e^{-2s} h_0\right),$$

being h_0 the Euclidean metric and so the null hypersurface L is given in this case by

$$L = \{(t, t, x) : t \in \mathbb{R}, x \in \mathbb{R}^{n-2}\},$$

which has constant null mean curvature $H = 2 - n$ respect to the rigging vector field $\zeta = \partial_t$. Therefore, conditions 1, 2, and 3 in Theorem 33 are fulfilled and we can apply it to obtain the following.

Suppose that Σ is a codimension two totally umbilic spacelike hypersurface in $\mathbb{R} \times \mathbb{H}^{n-1}$ contained in $L = \{(t, t, x) : t \in \mathbb{R}, x \in \mathbb{R}^{n-2}\}$. If there is a point $p_0 = (t_0, t_0, x_0) \in \Sigma$, such that $t(p) \leq t_0$ and $g(\vec{H}_\Sigma, \partial_t - \partial_s) \leq 2 - n$ for all p in a neighborhood of p_0 in Σ, then Σ is locally contained in $\{(t_0, t_0, x) : x \in \mathbb{R}^{n-2}\}$.

We can study the coincidence of a codimension two spacelike submanifold and a leaf of the screen distribution without using the rigged metric, but we need to assume that the null hypersurface has zero null mean curvature and it exists a rigging vector field which is a gradient. In this case, the screen distribution is integrable and the leaves are given by the intersection of the level hypersurfaces of the function and the null hypersurface.

Proposition 22. *Let L be a null hypersurface with zero null mean curvature and $f \in C^\infty(M)$ a function, such that $\zeta = \nabla f$ is a distinguished rigging vector field for L. If Σ is a codimension two spacelike submanifold through L and $p_0 \in \Sigma$ is a point, such that*

$$f(p_0) \leq f(p),$$
$$g(\vec{H}_\Sigma, \nabla f) + \triangle f + \xi g(\nabla f, \nabla f) \leq 0$$

for all p in a neighborhood of p_0 in Σ, then Σ coincides with a leaf of the screen distribution in a neighborhood of p_0.

Example 9. *A plane fronted wave is the Lorentzian manifold $M = M_0 \times \mathbb{R}^2$ endowed with the metric*

$$g = g_0 + 2dudv + \phi(x,u)du^2,$$

where (M_0, g_0) is a Riemannian manifold and $\phi : M_0 \times \mathbb{R} \to \mathbb{R}$ is some function. It holds that ∂_v is a parallel null vector field in M.

Call $u, v : M \to \mathbb{R}$ the canonical projections. We have that

$$L_{u_0} = \{p \in M : u(p) = u_0\}$$

is a totally geodesic null hypersurface for all $u_0 \in \mathbb{R}$. The vector field $\zeta = \nabla v = \partial_u - \Phi\partial_v$ is a rigging vector field for L_{u_0} and its rigged vector field is $\xi = \partial_v$. Moreover, $\tau = 0$ and the leaf of the screen distribution through a point $p_0 \in L_{u_0}$ is $S_{p_0} = \{p \in M : u(p) = u_0, v(p) = v(p_0)\}$.

Since $\triangle v = 0$ and $\partial_v(g(\nabla v, \nabla v)) = 0$, from the above proposition, if Σ is a codimension two spacelike submanifold contained in L_{u_0} and there is a point $p_0 \in \Sigma$, such that $v(p_0) \leq v(p)$ and $g(\vec{H}_\Sigma, \partial_u) \leq 0$ for all p in a neighborhood of p_0 in Σ, then Σ is locally contained in S_{p_0}.

Example 10. *Suppose that L is a null hypersurface with zero null mean curvature in a generalized Robertson–Walker space $(M, g) = \left(I \times F, -dt^2 + \phi(t)^2 g_0\right)$.*

We call $f : M \to \mathbb{R}$ the function given by $f(p) = -\int_c^{t(p)} \phi(s)ds$, being $c \in I$ a fixed point and $t : M \to \mathbb{R}$ the canonical projection. We know that $\zeta = \nabla f = \phi\partial_t$ is a distinguished rigging vector field for L. Since $\triangle f + \xi(g(\nabla f, \nabla f)) = (n-2)\phi'$, Proposition 22 implies that if Σ is a codimension two spacelike submanifold through L and $p_0 \in \Sigma$ holds $t(p) \leq t(p_0)$ and $g(\vec{H}_\Sigma, \partial_t) \leq -\frac{(n-2)\phi'}{\phi}$ in a neighborhood of p_0 in Σ, then Σ is contained in the slice $t = t(p_0)$ in a neighborhood of p_0.

3.7. Black Hole Horizons

A remarkable fact of black hole horizon theory is that there exists four laws that resembles the laws of thermodynamic. Physicists claim that this is not a mere coincidence but a deep property of nature. In this short section we illustrate how taking a suitable rigging we can easily prove the zeroth law of black hole thermodynamics. A review from a physical point of view is [68].

We say that a null hypersurface L is a Killing horizon if there is a Killing vector field $K \in \mathfrak{X}(M)$, such that K_x is null and tangent to L for each $x \in L$. Killing horizons appear as the event horizon of stationary black holes. For example, the horizon of the black hole in Kruskal space is a Killing horizon.

A Killing horizon is necessarily totally geodesic since the null second fundamental form $B(X,Y) = -g(\nabla_X K, Y)$ associated to K is symmetric and skew-symmetric for any $X, Y \in \mathcal{S}$. Since K is a null vector field over L, there is $\kappa \in C^\infty(L)$, such that $\nabla_K K = \kappa K$. This function κ is called the surface gravity in the case of stationary black holes and the zeroth law asserts that, under suitable conditions, it is constant.

To show this, we take a point $p \in L$ and a spacelike codimension two submanifold S with $p \in S \subset L$. We can pick a vector $\zeta_x \in T_x M$ with $g(\zeta_x, K_x) = 1$ for all $x \in S$. Since the flow Φ_t of K are isometries, there are $\varepsilon > 0$ and a open set θ with $p \in \theta \subset L$, such that $\psi : (-\varepsilon, \varepsilon) \times S \to \theta$ given by $\psi(t,x) = \Phi_t(x)$ is a diffeomorphism. Using it, we can construct a rigging ζ defined in θ with associated rigged vector feld $\xi = K$. Moreover, ζ is invariant by the flow of K, that is, $(\Phi_t)_{*_x}(\zeta) = \zeta_{\Phi_t(x)}$ for all $x \in S$.

Observe that $\kappa = -\tau(\xi)$, being τ the rotation one-form associated to ζ. Since ζ and ξ are invariant by the flow of K, which are formed by isometries, and τ is given by $\tau(U) = g(\nabla_U \zeta, \xi)$, it follows that τ is also invariant by the flow. This implies that $L_\xi \tau = 0$.

On the other hand, since L is totally geodesic, the Gauss–Codazzi Equations (14) and (16) reduce to

$$g(R_{UV}W,\xi) = 0,$$
$$g(R_{UV}\xi,N) = -d\tau(U,V)$$

for all $U,V \in \mathfrak{X}(L)$.

Now we suppose that it holds the null dominant energy condition, which can be stated as follows. If we call T the stress-energy tensor, given by $T = Ric - \frac{1}{2}Sg$, then we say that the null dominant energy condition holds if the metrically equivalent vector to $-T(u,\cdot)$ is causal and future directed for any null future directed vector u.

Under this condition we have that $d\tau(\xi,U) = 0$ for all $U \in \mathfrak{X}(L)$. In fact, using the above Gauss–Codazzi equations we have

$$Ric(\xi,U) = \sum_{i=3}^{n} g(R_{Ue_i}e_i,\xi) + g(R_{\xi U}\xi,N) = -d\tau(\xi,U)$$

and, therefore, $d\tau(\xi,U) = -T(\xi,U)$ for all $U \in \mathfrak{X}(L)$. Since ξ is future directed and null, the vector field W metrically equivalent to $-T(\xi,\cdot)$ is causal and future directed, thus $g(W,\xi) = -T(\xi,\xi) = -d\tau(\xi,\xi) = 0$. It can not be both timelike and ortogonal to ξ at any point, so it is null and orthogonal to ξ, which implies that W is proportional to ξ. Therefore

$$d\tau(\xi,U) = -T(\xi,U) = g(W,U) = 0$$

for all $U \in \mathfrak{X}(L)$.

Finally, using Cartan's formula

$$0 = \big(L_\xi\tau\big)(U) = d\big(i_\xi\tau\big)(U) + \big(i_\xi d\tau\big)(U) = U(\tau(\xi)) + d\tau(\xi,U) = U(\tau(\xi)),$$

which means that $\tau(\xi)$ is constant on L. So we have proved the following.

Proposition 23. *If a spacetime satisfies the null dominant energy condition, then the surface gravity is constant for any Killing horizon.*

4. Future Developments

We have seen that the rigging technique allows us to show a nice interplay between the geometry of a Lorentzian manifold and the properties of the family of null hypersurfaces. In each subsection we have selected a representative number of results where the rigging technique has been used, showing its potential, but as it can be seen, most results could be assigned to several subsections. Part of the philosophy is to take advantage of the rigged metric, which is Riemannian. For example, in Proposition 4 the fact that the null mean curvature is a divergence allows us to use Stoke's theorem. In other cases, we have had to choose a rigging adapted to the situation. An example is Theorem 17 where the chosen rigging allows us to compare the null conjugate points and its multiplicities both for the ambient metric and the rigged metric in a null cone. Another important example is that closed riggings unveil the structure of totally umbilic null hypersurfaces as a local twisted product, Theorem 9.

In this philosophy, it is essential to establish a relationship between the geometric data (connexion and curvature) of the ambient space and the rigged metric. This is done in Sections 2.1–2.3. The relations do not cover all the possible cases, but it is enough in many situations, such as those shown in this review. It would be desirable to complete all possible relations between these objects, perhaps under some hypotheses on the rigging vector field, such as being closed, conformal, etc.

Three kind of null hypersurfaces are of special interest because they appear naturally in the applications: null cones, black hole horizons, and compact null hypersurfaces.

Null cones are studied in Section 3.3 where as we said there does exist a rigging tuning conjugate points along null geodesics inside a suitable null cone containing it, for both geometries, the ambient and the rigged metric. We also show a problem first settled in [39]. Given a null hypersurface, what kind of geometric properties are sufficient to ensure that it is contained in a null cone. Totally umbilic null hypersurfaces in a Ricci flat ambient space are candidate to be inside a null cone. This is a challenge problem and it deserves to be studied in order to weaken the conditions, or to find another conditions. Theorem 14 is a nice example of other possibilities.

This kind of results illustrates the importance of totally umbilic and totally geodesic null hypersurfaces. This is studied in Section 3.1, but it is present in several other places as we have seen. An interesting observation is that totally geodesic null hypersurfaces have an interpretation that is the usual in the Riemannian case. On the other hand, totally umbilic hypothesis is different because the null second fundamental form has a distinguished direction, the rigged direction, which belongs to its kernel. So it would be interesting to unveil the full geometric significance of totally umbilic condition for null hypersurfaces, (we thanks J. M. M. Senovilla for pointing out this observation in a private conversation). A first clue is Theorem 19 where it is shown that in a totally umbilic null cone, the presence of a null conjugate point to the vertex has always maximum multiplicity, showing some degree of maximum symmetry. The more we can say today is suggested in Theorem 9 where totally umbilic null hypersurfaces can be furnished locally with a rigged metric that is a twisted product.

In this classification, totally geodesic null hypersurfaces are relevant because isolated black hole horizons are examples of them. It is interesting to find ways to study problems of physical interest with our technique. An illustration is Section 3.7 where we use a suitable rigging to prove a version of the zeroth law of black hole thermodynamic.

One of the most important results in the rigging technique is Theorem 9 as we have commented in several places. Twisted products with a one dimensional first factor are examples of metrics with a distinguished direction. We can think of other possibilities for rigging metrics with a privileged direction provided by the rigged vector field, for example contact or Sasakian structures. These structures are related with quantum theory through symplectic and Kähler manifolds, used in the geometric quantization program, and more recently in quantum gravity. If this can be implemented in the horizon of a black hole, it could be interesting to explore those possible relations.

Section 3.5 is dedicated to the influence of null hypersurfaces in the causal theory of the ambient space, which is of interest both in geometry and physics. The importance of this is that we are using properties of the family of null hypersurfaces to find cute information on causality theory, which is a theory of global nature of the ambient space. In Section 3.6 we study some special geometric properties of null hypersurfaces, such as those through the screen distribution. This kind of ideas could be related with the study of trapped surfaces and MOTS, of great importance in singularity theory.

In Section 3.4, we studied the family of compact null hypersurfaces. Those kind of hypersurfaces have a transversal character, because they can be used as a tool to study different situations including its influence in the global properties of the ambient space, such as some results in Section 3.5, but they are interesting themselves because they are important objects from the differential geometry point of view and exotic objects in Lorentzian manifolds.

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