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The Darboux Transformation and N-Soliton Solutions of Gerdjikov–Ivanov Equation on a Time–Space Scale

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Abstract: The Gerdjikov–Ivanov (GI) equation is one type of derivative nonlinear Schrödinger equation used widely in quantum field theory, nonlinear optics, weakly nonlinear dispersion water waves and other fields. In this paper, the coupled GI equation on a time–space scale is deduced from Lax pairs and the zero curvature equation on a time–space scale, which can be reduced to the classical and the semi-discrete GI equation by considering different time–space scales. Furthermore, the Darboux transformation (DT) of the GI equation on a time–space scale is constructed via a gauge transformation. Finally, N-soliton solutions of the GI equation are given through applying its DT, which are expressed by the Cayley exponential function. At the same time, one-solition solutions are obtained on three different time–space scales ($\mathbb{X} = \mathbb{R}$, $\mathbb{X} = \mathbb{C}$ and $\mathbb{X} = \mathbb{K}p$).

Keywords: Gerdjikov-Ivanov equation; time–space scales; Darboux transformation; N-soliton solution

MSC: 35Q51; 35K05; 34N05

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1. Introduction

There are some practical problems that cannot be solved accurately by using only continuous or discrete analysis. In order to unify continuous and discrete analysis, a time scale was initiated by Stefan Hilger in 1988, which is an arbitrary nonempty closed subset of the real numbers [1–3]. In recent years, extensive research about time scales has been conducted, particularly in stability, oscillation and initial-boundary value problems [4–8]. In addition, time scale dynamic equations have wide application prospects in many areas, such as population dynamic models [9], epidemic models [10,11] and models of the financial consumption process [12,13].

Toda's lattice, Hirota's network and nonlinear Schrödinger dynamic equations were derived on a time–space scale by extending an Ablowitz–Ladik hierarchy of integrable dynamic systems on a time–space scale [14]. This extension facilitates a variety of modeling applications of Ablowitz–Ladik hierarchies, including optics and chaos in dispersion numerical schemes [15]. The formulas for solutions of boundary value problem of Burgers equation and heat equation were derived on a time–space scale by using the Cole–Hopf transformation. These formulas may be used to study the wave motion on a time–space scale. Sine-Gordon equation was obtained on a time–space scale and its solution expressed by the Cayley exponential function was given [16–18]. However, the development of time–space scales is relatively slow in nonlinear dynamical systems compared to other fields.

There are important applications regarding the derivative nonlinear Schrödinger (DNLS) equation in many fields [19]. In particular, in situations where higher order nonlinear effects need to be restored, a family of DNLS equations was investigated [20]. There are three famous DNLS equations, which are the DNLS I equation [21,22], DNLS

II equation [23,24] and DNLS III equation [25]. The forms of these three equations are as follows

$$iq_t + q_{xx} + i\left(q^2 q^*\right)_x = 0,$$

$$iq_t + q_{xx} + iqq^* q_x = 0,$$

$$iq_t + q_{xx} + \frac{1}{2}q^3 q^{*2} - iq^2 q_x^* = 0,$$

where $q*$ represents the complex conjugate of q. They can be transformed into each other by a gauge transformation [26]. Specifically, the last equation is also known as the Gerdjikov–Ivanov (GI) equation, which was discovered by Gerdjikov and Ivanov [27]. In recent years, several useful methods have been proposed for obtaining solutions of the GI equation, such as the Darboux transformation (DT) [28,29], algebra-geometric solution [30–33], Wronskian type solution [29,34] and Hamiltonian structures [35,36].

The advantage of DT is that new solutions can be obtained successively through iteration. The explicit soliton-like solution of the GI equation was obtained by its DT [26]. The explicit N-fold DT with multiparameters for the GI equation was constructed with the help of a gauge transformation [28]. The dark soliton, bright soliton, breather solution and periodic solution are given explicitly from different seed solutions. In this paper, the coupled GI equation on a time–space scale is deduced by the Lax matrix equation extended on a time–space scale. This extension will provide a wider range of nonlinear integrable dynamic models and promote solutions to practical problems.

This paper is organized as follows. In Section 2, the coupled GI equation on a time–space scale is obtained, which can be reduced to the classical and the semi-discrete GI equation. In Section 3, N-fold DT and N-soliton solutions of the GI equation on a time–space scale are constructed with the help of a gauge transformation. In particular, one-soliton solutions of the GI equation on three different time–space scales are obtained from seed solution. The last section is our conclusions.

2. GI Equation on a Time–Space Scale

For constructing the GI equation on a time–space scale, jump operators, graininess functions and the $\nabla-$derivative are introduced as follows [1–3].

Definition 1. *For $(t, x) \in \mathbb{T} \times \mathbb{X}$, backward jump operators are defined as*

$$\sigma : \mathbb{T} \to \mathbb{T},\ \rho : \mathbb{X} \to \mathbb{X},$$

$$\sigma(t) = \sup\{s \in \mathbb{T} : s < t\},\ \rho(x) = \sup\{y \in \mathbb{X} : y < x\}. \tag{1}$$

For $x \in \mathbb{X}$, the forward jump operator $\beta(x) : \mathbb{X} \to \mathbb{X}$ is defined as $\beta(x) = \rho^{-1}(x) = \inf\{y \in \mathbb{X} : y > x\}$.

Definition 2. *The $\nabla-$derivative associated with t (time) and x (space) variables is defined as*

$$\nabla_t f(t, x) = \lim_{p \to \mu(t)} \frac{f(t, x) - f^\sigma(t, x)}{p}, \tag{2}$$

$$\nabla_x f(t, x) = \lim_{q \to \nu(x)} \frac{f(t, x) - f^\rho(t, x)}{q}, \tag{3}$$

where the graininess functions $\mu : \mathbb{T} \to [0, +\infty)$, $\nu : \mathbb{X} \to [0, +\infty)$ are defined as

$$\mu(t) = t - \sigma(t),\ \nu(x) = x - \rho(x). \tag{4}$$

Note that,

$$f^\sigma(t, x) := f(\sigma(t), x) = f(t, x) - \mu(t)\nabla_t f(t, x), \tag{5}$$

$$f^\rho(t, x) := f(t, \rho(x)) = f(t, x) - \nu(x)\nabla_x f(t, x). \tag{6}$$

Definition 3. *The Cayley exponential function on a time scale is defined by*

$$e_\alpha(x,x_0) := \exp\left(\int_{t_0}^{x} \zeta_{\mu(s)}(\alpha(s))\Delta s\right),\ e_\alpha(x) := e_\alpha(x,0),$$

where $\alpha = \alpha(x)$ is a given rd-continuous regressive function and

$$\zeta_h(z) := \frac{1}{h}\log\frac{1+\frac{1}{2}zh}{1-\frac{1}{2}zh},\ h>0,\ \zeta_0(z) := z.$$

When $\mathbb{X} = \mathbb{R}$ and $\mathbb{X} = h\mathbb{Z}$, the Cayley exponential function becomes

$$e_\alpha(x) = e^{\int_0^x \alpha(s)ds} \text{and}$$
$$e_\alpha(x) = \left(\frac{1+\frac{1}{2}\alpha h}{1-\frac{1}{2}\alpha h}\right)^{\frac{x}{h}},$$

respectively.

Lemma 1. *Take $\mathbb{T}\times\mathbb{X} = \mathbb{R}\times\mathbb{R}$. The backward jump operators*

$$\sigma(t) = \sup(-\infty,t) = t,\ \rho(x) = \sup(-\infty,x) = x, \tag{7}$$

and the graininess functions

$$\mu(t) = t-\sigma(t) = 0,\ \nu(x) = x-\rho(x) = 0. \tag{8}$$

Lemma 2. *Take $\mathbb{T}\times\mathbb{X} = \mathbb{R}\times\mathbb{Z}$. The backward jump operators*

$$\sigma(t) = \sup(-\infty,t) = t,\ \rho(x) = \sup\{x-1,x-2,\cdots\} = x-1, \tag{9}$$

and the graininess functions

$$\mu(t) = t-\sigma(t) = 0,\ \nu(x) = x-\rho(x) = 1. \tag{10}$$

Lemma 3. *When $\mathbb{X} = \mathbb{R}$, $\mathbb{X} = \hbar\mathbb{Z}$ and $\mathbb{X} = \mathbb{K}_p$, the $\nabla-$derivative becomes*

$$\nabla_x f(x) = f_x(x),$$
$$\nabla_x f(x) = \frac{f(x)-f(x-\hbar)}{\hbar}\text{and}$$
$$\nabla_x f(x) = \frac{f(x)-f(p^{-1}x)}{(1-p^{-1})x},$$

respectively.

In what follows, based on Lax pairs of DNLS equation from the generalized Kaup–Newell spectrum problem [32], a ∇-dynamical system is introduced

$$\begin{cases}\nabla_x\psi(t,x) = U(t,x)\psi(t,x),\\ \nabla_t\psi(t,x) = V(t,x)\psi(t,x),\end{cases} \tag{11}$$

where

$$\begin{cases}U = \begin{pmatrix}-i\lambda^2-\frac{1}{2}iqr & \lambda q\\ \lambda r & i\lambda^2+\frac{1}{2}iqr\end{pmatrix},\\ V = \begin{pmatrix}A(t,x) & B(t,x)\\ C(t,x) & -A(t,x)\end{pmatrix},\end{cases} \tag{12}$$

with $\psi = \begin{pmatrix} \psi_1(t,x) \\ \psi_2(t,x) \end{pmatrix}$, q and r are potential functions, and λ is a spectral parameter.

According to the compatibility condition $\nabla_{xt}\psi = \nabla_{tx}\psi$ and $\nabla-$derivative product rules [15], the zero curvature equation on a time–space scale is obtained

$$\nabla_t U - \nabla_x V + U^\sigma V - V^\rho U = 0. \tag{13}$$

Then, substituting Equation (12) into Equation (13), we find

$$\begin{cases} -i(C+C^\rho)\lambda^2 - (r^\sigma A + rA^\rho + \nabla_t r)\lambda - \frac{1}{2}i(qr)^\sigma C - \frac{1}{2}iqrC^\rho + \nabla_x C = 0, \\ -i(B+B^\rho)\lambda^2 - (q^\sigma A + qA^\rho + \nabla_t q)\lambda - \frac{1}{2}i(qr)^\sigma B - \frac{1}{2}iqrB^\rho - \nabla_x B = 0, \\ -i(A-A^\rho)\lambda^2 + (q^\sigma C - rB^\rho)\lambda - \frac{1}{2}i(qr)^\sigma A + \frac{1}{2}iqrA^\rho - \frac{1}{2}i\nabla_t(qr) - \nabla_x A = 0, \\ -i(A-A^\rho)\lambda^2 + \lambda(r^\sigma B - qC^\rho) - \frac{1}{2}i(qr)^\sigma A + \frac{1}{2}iqrA^\rho - \frac{1}{2}i\nabla_t(qr) + \nabla_x A = 0. \end{cases} \tag{14}$$

Take $A,\ B$ and C as quaternary polynomials of λ,

$$A = \sum_{j=0}^{4} a_j\lambda^j,\ B = \sum_{j=0}^{4} b_j\lambda^j,\ C = \sum_{j=0}^{4} c_j\lambda^j. \tag{15}$$

Then, by substituting Equation (15) into Equation (14), these relations are obtained

$$\begin{cases} a_4 = -2i,\ a_1 = a_3 = b_0 = b_2 = b_4 = c_0 = c_2 = c_4 = 0, \\ b_3 = -b_3^\rho + 2(q^\sigma + q),\ c_3 = -c_3^\rho + 2(r^\sigma + r) = 0, \\ b_1 = -b_1^\rho + iqa_2^\rho + iq^\sigma a_2 - \frac{1}{2}qrb_3^\rho - \frac{1}{2}(qr)^\sigma b_3 + i\nabla_x b_3, \\ c_1 = -c_1^\rho + ira_2^\rho + ir^\sigma a_2 - \frac{1}{2}qrc_3^\rho - \frac{1}{2}(qr)^\sigma c_3 - i\nabla_x c_3, \\ a_2 = \nabla_x^{-1}(\frac{1}{2}q^\sigma c_1 - \frac{1}{2}r^\sigma b_1 - \frac{1}{2}rb_1^\rho + \frac{1}{2}qc_1^\rho), \\ a_0 = \nabla_x^{-1}(-\frac{1}{2}iqr^\sigma a_0^\rho + \frac{1}{4}qrr^\sigma b_1^\rho - \frac{1}{2}ir^\sigma b_1^x - \frac{1}{2}iq^\sigma r^\sigma a_0 + \frac{1}{4}(qr)^\sigma b_1 r^\sigma \\ \quad + \frac{1}{2}iqra_0^\rho - \frac{1}{4}q^2rc_1^\rho + \frac{1}{2}ir^\sigma qa_0 - \frac{1}{4}(qr)^\sigma qc_1 - \frac{1}{2}ic_{1x}q), \end{cases} \tag{16}$$

and evolution equations on a time–space scale are obtained

$$\nabla_t q = qa_0^\rho + q^\sigma a_0 + \frac{1}{2}iqrb_1^\rho + \frac{1}{2}i(qr)^\sigma b_1 + \nabla_x b_1, \tag{17}$$

$$\nabla_t r = -ra_0^\rho - r^\sigma a_0 - \frac{1}{2}iqrc_1^\rho - \frac{1}{2}i(qr)^\sigma c_1 + \nabla_x c_1. \tag{18}$$

According to Equations (5) and (6), Equation (16) is reduced to

$$b_3 = 2(2 - \nu(x)\nabla_x)^{-1}(q + q^\sigma), \tag{19}$$

$$c_3 = 2(2 - \nu(x)\nabla_x)^{-1}(r + r^\sigma), \tag{20}$$

$$b_1 = 2i(2 - \nu(x)\nabla_x)^{-1}m_1 a_2 + \frac{1}{2}(2 - \nu(x)\nabla_x)^{-1}m_4(q + q^\sigma), \tag{21}$$

$$c_1 = 2i(2 - \nu(x)\nabla_x)^{-1}m_2 a_2 + \frac{1}{2}(2 - \nu(x)\nabla_x)^{-1}m_3(r + r^\sigma), \tag{22}$$

$$\nabla_x a_0 = \frac{1}{2}im_5 a_0 + \frac{1}{2}i(r^\sigma m_4 m_1 - qm_4 m_2)a_2 + \frac{1}{8}r^\sigma m_4^2(q + q^\sigma) - \frac{1}{8}qm_4 m_3(r + r^\sigma), \tag{23}$$

$$\nabla_x a_2 = \frac{1}{2}i(m_1m_2 - m_2m_1)(2 - \nu(x)\nabla_x)a_2 + \frac{1}{2}m_1m_3(r + r^\sigma) + \frac{1}{2}m_2m_4(q + q^\sigma), \tag{24}$$

with

$$\begin{aligned}
m_1 &= [q^\sigma + q(1 - \nu(x)\nabla_x)](2 - \nu(x)\nabla_x)^{-1},\\
m_2 &= [r^\sigma + r(1 - \nu(x)\nabla_x)](2 - \nu(x)\nabla_x)^{-1},\\
m_3 &= [(qr)^\sigma + (qr)(1 - \nu(x)\nabla_x) + 2i\nabla_x](2 - \nu(x)\nabla_x)^{-1},\\
m_4 &= [(qr)^\sigma + (qr)(1 - \nu(x)\nabla_x) - 2i\nabla_x](2 - \nu(x)\nabla_x)^{-1},\\
m_5 &= (qr - qr^\sigma)(1 - \nu(x)\nabla_x) + qr^\sigma - q^\sigma r^\sigma.
\end{aligned}$$

Then, the coupled GI equation on a time–space scale is obtained

$$\begin{cases}
\nabla_t q = q(1 - \nu(x)\nabla_x)a_0 + q^\sigma a_0 + \frac{1}{2}iqr(1 - \nu(x)\nabla_x)b_1 + \frac{1}{2}i(qr)^\sigma b_1 + \nabla_x b_1,\\
\nabla_t r = -r(1 - \nu(x)\nabla_x)a_0 - r^\sigma a_0 - \frac{1}{2}iqr(1 - \nu(x)\nabla_x)c_1 - \frac{1}{2}i(qr)^\sigma c_1 + \nabla_x c_1,
\end{cases} \tag{25}$$

where a_0, b_1, c_1 are defined by Equations (21)–(23), respectively.

In the following, two special kinds of equations are given as follows.

Case I: Taking $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{R}$, we find $\mu(t) = 0, \nu(x) = 0$. Equations (21)–(23) are reduced to

$$\begin{aligned}
b_1 &= iq_x,\\
c_1 &= -ir_x,\\
a_0 &= \frac{1}{2}(rq_x - qr_x) + \frac{1}{4}iq^2r^2.
\end{aligned}$$

Then, Equation (25) is reduced to the coupled GI equation

$$\begin{cases}
iq_t + q_{xx} + iq^2r_x + \frac{1}{2}q^3r^2 = 0,\\
ir_t - r_{xx} + ir^2q_x - \frac{1}{2}q^2r^3 = 0.
\end{cases} \tag{26}$$

When $r = -q^*$, the classical GI equation is obtained

$$iq_t + q_{xx} + \frac{1}{2}q^3q^{*2} - iq^2q^*_x = 0. \tag{27}$$

Case II: Taking $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{Z}$, we find $\mu(t) = 0,\ \nu(x) = 1$.

$$\begin{aligned}
f^\sigma(x,t) &= f(x,t),\\
f^\rho(x,t) &= Ef(x,t) = f(x,t) - (1 - E)f(x,t),
\end{aligned} \tag{28}$$

where E is the shift operator. Then, Equations (19)–(24) are reduced to

$$b_3 = 4(1 + E)^{-1}q, \tag{29}$$

$$c_3 = 4(1 + E)^{-1}r, \tag{30}$$

$$a_2 = (1 - E)^{-1}\left(qr^2 + rm_7q\right), \tag{31}$$

$$b_1 = 2i(1 + E)^{-1}q(1 - E)^{-1}\left(qr^2 + rm_7q\right) + (1 + E)^{-1}m_7q, \tag{32}$$

$$c_1 = 2i(1 + E)^{-1}r(1 - E)^{-1}\left(qr^2 + rm_7q\right) + (1 + E)^{-1}m_6q, \tag{33}$$

$$
\begin{aligned}
a_0 =& \frac{1}{2}i(1-E)^{-1}(rm_7q - qm_7r)(1-E)^{-1}\left(qr^2 + rm_7q\right) \\
&+ \frac{1}{4}(1-E)^{-1}\left(rm_7^2q - qm_7m_6r\right),
\end{aligned}
\tag{34}
$$

with

$$
\begin{aligned}
m_6 &= qr + 2i(1-E)(1+E)^{-1}, \\
m_7 &= qr - 2i(1-E)(1+E)^{-1}.
\end{aligned}
$$

Therefore, the semi-discrete coupled GI equation is obtained

$$
\begin{aligned}
q_t &= q(1+E)a_0 + \frac{1}{2}iqr(1+E)b_1 + (1-E)b_1, \\
r_t &= -r(1+E)a_0 - \frac{1}{2}iqr(1+E)c_1 + (1-E)c_1,
\end{aligned}
\tag{35}
$$

where a_0, b_1, and c_1 are defined by Equations (32)–(34), respectively.

3. DT of GI Equation on a Time–Space Scale

In this section, we construct a DT for GI equation and give its N-soliton solutions on a time–space scale.

3.1. Construction of DT on a Time–Space Scale

First, it can be shown by long calculations that Equation (12) is transformed to

$$
\begin{cases}
U = -i\lambda^2\sigma_3 + \lambda Q - \frac{1}{2}iQ^2\sigma_3, \\
V = -2i\sigma_3\lambda^4 + B_3\lambda^3 + a_2\sigma_3\lambda^2 + B_1\lambda + a_0\sigma_3,
\end{cases}
\tag{36}
$$

with σ_3 is a Pauli matrix where $Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 & b_1 \\ c_1 & 0 \end{pmatrix}$, $B_3 = \begin{pmatrix} 0 & b_3 \\ c_3 & 0 \end{pmatrix}$, $a_j(j = 0, 2), b_j, c_j(j = 1, 3)$ are defined by Equations (19)–(24), respectively.

Then, the ∇-dynamical system Equation (11) is transformed into

$$
\begin{cases}
\nabla_x\psi[1] = U[1]\psi[1], \\
\nabla_t\psi[1] = V[1]\psi[1],
\end{cases}
\tag{37}
$$

under a gauge transformation

$$
\psi[1] = T[1]\psi. \tag{38}
$$

Substituting Equation (38) into Equation (37), we find

$$
U[1]T[1] = \nabla_x T[1] + T[1]^\rho U, \tag{39}
$$

$$
V[1]T[1] = \nabla_t T[1] + T[1]^\sigma V, \tag{40}
$$

where

$$
\begin{cases}
U[1] = -i\lambda^2\sigma_3 + \lambda Q[1] - \frac{1}{2}iQ[1]^2\sigma_3, \\
V[1] = -2i\sigma_3\lambda^4 + B_3[1]\lambda^3 + a_2[1]\sigma_3\lambda^2 + B_1[1]\lambda + a_0[1]\sigma_3,
\end{cases}
\tag{41}
$$

with $Q[1] = \begin{pmatrix} 0 & q[1] \\ -q[1]^* & 0 \end{pmatrix}$, $B_1[1] = \begin{pmatrix} 0 & b_1[1] \\ c_1[1] & 0 \end{pmatrix}$, $B_3[1] = \begin{pmatrix} 0 & b_3[1] \\ c_3[1] & 0 \end{pmatrix}$.

Assume

$$
T[1] = T_0 + T_1\lambda \tag{42}
$$

where $T_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T_1 = \begin{pmatrix} a_{11} & b_{12} \\ c_{21} & d_{22} \end{pmatrix}$.

Substituting Equation (42) into Equation (39) and comparing the coefficients in the terms of the same powers $\lambda^j (j = 0, \cdots, 5)$ on both sides of equation, we find

$$\begin{cases} c_{21} = b_{12} = 0, \\ a_{11} = d_{22} = 1, \\ q[1] = q + ib + ib^{\rho}, \\ q[1]^* = q^* + ic + ic^{\rho}. \end{cases} \tag{43}$$

Setting $S = -T_0 = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$, we obtain

$$T[1] = \lambda I - S, \tag{44}$$

$$q[1] = q - is_{12} - is_{12}^{\rho}. \tag{45}$$

Substituting Equation (44) into Equation (39), we obtain

$$\nabla_x S = \frac{1}{2} iS^{\rho} Q^2 \sigma_3 + \frac{1}{2} iSQ^2 \sigma_3 + QS^2 - S^{\rho} QS + iS^{\rho} S^2 \sigma_3 + iS^3 \sigma_3. \tag{46}$$

Assume

$$S = H \Lambda H^{-1} \tag{47}$$

with $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}$ is an eigenvalue matrix, $H = \begin{pmatrix} \psi_1 & \psi_2^* \\ \psi_2 & -\psi_1^* \end{pmatrix}$ is a fundamental solution matrix and satisfies

$$\begin{cases} \nabla_x H = -i\sigma_3 H \Lambda^2 + QH\Lambda - \frac{1}{2} iQ^2 \sigma_3 H, \\ \nabla_t H = -2i\sigma_3 H \Lambda^4 + B_3 \Lambda^3 + a_2 \sigma_3 H \Lambda^2 + B_1 H \Lambda + a_0 \sigma_3 H. \end{cases} \tag{48}$$

It is easy to obtain

$$\begin{aligned} \nabla_x S &= \nabla_x \Big(H \Lambda H^{-1} \Big) \\ &= \frac{1}{2} iS^{\rho} Q^2 \sigma_3 + \frac{1}{2} iSQ^2 \sigma_3 + QS^2 - S^{\rho} QS + iS^{\rho} S^2 \sigma_3 + iS^3 \sigma_3, \end{aligned} \tag{49}$$

which means that Equation (47) yields Equation (46). From $T[1]_t + T[1]^{\sigma} V = V[1] T[1]$, we find

$$\begin{aligned} &- \nabla_t S + (\lambda I - S^{\sigma}) \Big(-2i\sigma_3 \lambda^4 + B_3 \lambda^3 + a_2 \sigma_3 \lambda^2 + B_1 \lambda + a_0 \sigma_3 \Big) \\ =& \Big(-2i\sigma_3 \lambda^4 + B_3[1] \lambda^3 + a_2[1] \sigma_3 \lambda^2 + B_1[1] \lambda + a_0[1] \sigma_3 \Big) (\lambda I - S). \end{aligned} \tag{50}$$

Comparing the coefficients in terms of the same powers $\lambda^j (j = 0, \cdots, 5)$ on both sides of Equation (50), we obtain

$$\begin{aligned} &\lambda^0 : -\nabla_t S - a_0 S^{\sigma} \sigma_3 = -a_0[1] \sigma_3 S, \\ &\lambda^1 : a_0 \sigma_3 - S^{\sigma} B_1 = -B_1[1] S + a_0[1] \sigma_3, \\ &\lambda^2 : B_1 - a_2 S^{\sigma} \sigma_3 = B_1[1] - a_2[1] \sigma_3 S, \\ &\lambda^3 : a_2 \sigma_3 - S^{\sigma} B_3 = a_2[1] \sigma_3 - B_3[1] S, \\ &\lambda^4 : B_3 + 2iS^{\sigma} \sigma_3 = B_3[1] + 2i\sigma_3 S, \\ &\lambda^5 : -2i\sigma_3 = -2i\sigma_3. \end{aligned} \tag{51}$$

Then, the gauge transformations Equations (44) and (45) are proven to be DT of the GI equation on a time–space scale.

3.2. Soliton Solutions of the GI Equation on a Time–Space Scale

Soliton solutions of the GI equation on a time–space scale are constructed by applying its DT. First, Equation (11) is transformed to

$$\begin{cases} \nabla_x \psi[0] = U[0]\psi[0] = \begin{pmatrix} -i\lambda^2 + \frac{1}{2}iq[0]q[0]^* & \lambda q[0] \\ -\lambda q[0]^* & i\lambda^2 - \frac{1}{2}iq[0]q[0]^* \end{pmatrix}\psi[0], \\ \nabla_t \psi[0] = V[0]\psi[0] = \begin{pmatrix} -2i\lambda^4 + a_2[0]\lambda^2 + a_0[0] & b_3[0]\lambda^3 + b_1[0]\lambda \\ c_3[0]\lambda^3 + c_1[0]\lambda & 2i\lambda^4 - a_2[0]\lambda^2 - a_0[0] \end{pmatrix}\psi[0], \end{cases} \tag{52}$$

where $\psi[0]=\begin{pmatrix} \psi_1[0] \\ \psi_2[0] \end{pmatrix}$.

Let us set the spectral parameter $\lambda = \lambda_1$. A one-fold DT of the GI equation on a time–space scale is constructed

$$\begin{aligned} \psi[1] &= T[1]\psi[0] \\ &= (\lambda I - S[0])\psi[0] \\ &= \begin{pmatrix} \lambda - s_{11}[0] & -s_{12}[0] \\ -s_{21}[0] & \lambda - s_{22}[0] \end{pmatrix}\psi[0], \\ q[1] &= q[0] - is_{12}[0] - is_{12}[0]^{\rho} \\ &= q[0] - i\frac{(\lambda_1 - \lambda_1^*)\psi_1[0]\psi_2^*[0]}{\Delta_0} - i\frac{(\lambda_1 - \lambda_1^*)\psi_1^{\rho}[0]\psi_2^{*\rho}[0]}{\Delta_0^{\rho}}, \end{aligned} \tag{53}$$

where

$$S = \frac{1}{\Delta_0}\begin{pmatrix} -\lambda_1|\psi_1[0]|^2 + \lambda_1^*|\psi_2[0]|^2 & (\lambda_1^* - \lambda_1)\psi_1[0]\psi_2^*[0] \\ (-\lambda_1^* - \lambda_1)\psi_1^*[0]\psi_2[0] & -\lambda_1^*|\psi_1[0]|^2 - \lambda_1|\psi_2[0]|^2 \end{pmatrix}, \tag{54}$$

with $\Delta_0 = -|\psi_1[0]|^2 - |\psi_2[0]|^2$.

Under the DT (53), the ∇-dynamical system (52) is transformed into

$$\begin{cases} \nabla_x \psi[1] = U[1]\psi[1] = \begin{pmatrix} -i\lambda^2 + \frac{1}{2}iq[1]q[1]^* & \lambda q[1] \\ -\lambda q[1]^* & i\lambda^2 - \frac{1}{2}iq[1]q[1]^* \end{pmatrix}\psi[1], \\ \nabla_t \psi[1] = V[1]\psi[1] = \begin{pmatrix} -2i\lambda^4 + a_2[1]\lambda^2 + a_0[1] & b_3[1]\lambda^3 + b_1[1]\lambda \\ c_3[1]\lambda^3 + c_1[1]\lambda & 2i\lambda^4 - a_2[1]\lambda^2 - a_0[1] \end{pmatrix}\psi[1]. \end{cases} \tag{55}$$

In what follows, taking the "seed solution" $q[0] = 0$, we obtain eigenvectors $\psi[0]$ of Equation (52) with $\lambda = \lambda_1$

$$\psi[0]=\begin{pmatrix} \psi_1[0] \\ \psi_2[0] \end{pmatrix}=\begin{pmatrix} e_{-i\lambda_1^2}(x,0)e_{-2i\lambda_1^4}(t,0) \\ e_{i\lambda_1^2}(x,0)e_{2i\lambda_1^4}(t,0) \end{pmatrix}, \tag{56}$$

$$\psi^{\rho}[0]=\begin{pmatrix} \psi_1^{\rho}[0] \\ \psi_2^{\rho}[0] \end{pmatrix}=\begin{pmatrix} [1 - i\lambda_1^2\nu(x)]e_{-i\lambda_1^2}(x,0)e_{-2i\lambda_1^4}(t,0) \\ [1 + i\lambda_1^2\nu(x)]e_{i\lambda_1^2}(x,0)e_{2i\lambda_1^4}(t,0) \end{pmatrix}, \tag{57}$$

where $e_{\pm i\lambda_1^2}(x,0)$ and $e_{\pm 2i\lambda_1^4}(t,0)$ are Cayley exponential functions [18]. Then, a one-soliton solution of the GI equation on a time–space scale is obtained

$$q[1] = \frac{i(\lambda_1 - \lambda_1^*)E_3}{E_1 + E_2} + \frac{i(\lambda_1 - \lambda_1^*)(1 - i\lambda_1^{*2}\nu(x))E_3}{(1 + i\lambda_1^{*2}\nu(x))E_1 + (1 - i\lambda_1^{*2}\nu(x))E_2}, \tag{58}$$

where

$$E_1 = -e_{-i\left(\lambda_1^2-\lambda_1^{*2}\right)}(x,0)e_{-2i\left(\lambda_1^4-\lambda_1^{*4}\right)}(t,0),$$
$$E_2 = -e_{i\left(\lambda_1^2-\lambda_1^{*2}\right)}(x,0)e_{2i\left(\lambda_1^4-\lambda_1^{*4}\right)}(t,0),$$
$$E_3 = -e_{-i\left(\lambda_1^2+\lambda_1^{*2}\right)}(x,0)e_{-2i\left(\lambda_1^4+\lambda_1^{*4}\right)}(t,0).$$

Similarly, we take the spectral parameter $\lambda = \lambda_2$. A two-fold DT of the GI equation on a time–space scale is constructed

$$\begin{aligned}
\psi[2] &= T[2]\psi[1] \\
&= (\lambda I - S[1])\psi[1] \\
&= \begin{pmatrix} \lambda - s_{11}[1] & -s_{12}[1] \\ -s_{21}[1] & \lambda - s_{22}[1] \end{pmatrix}\psi[1] \\
&= T[2]T[1]\psi[0], \\
q[2] &= q[1] - is_{12}[1] - is_{12}[1]^{\rho} \\
&= q[1] - i\frac{(\lambda_2 - \lambda_2^*)\psi_1[1]\psi_2^*[1]}{\Delta_1} - i\frac{(\lambda_2 - \lambda_2^*)\psi_1^{\rho}[1]\psi_2^{*\rho}[1]}{\Delta_1^{\rho}},
\end{aligned} \tag{59}$$

where

$$S[1] = \frac{1}{\Delta_1}\begin{pmatrix} -\lambda_2|\psi_1[1]|^2 + \lambda_2^*|\psi_2[1]|^2 & (\lambda_2^* - \lambda_2)\psi_1[1]\psi_2[1]^* \\ (-\lambda_2^* - \lambda_2)\psi_1[1]^*\psi_2[1] & -\lambda_2^*|\psi_1[1]|^2 - \lambda_2|\psi_2[1]|^2 \end{pmatrix}, \tag{60}$$

with $\Delta_1 = -|\psi_1[1]|^2 - |\psi_2[1]|^2$.

When the spectral parameter $\lambda = \lambda_N$, N-fold DT is constructed as follows

$$\begin{aligned}
\psi[N] &= T[N]\psi[N-1] \\
&= (\lambda I - S[N-1])\psi[N-1] \\
&= \begin{pmatrix} \lambda - s_{11}[N-1] & -s_{12}[N-1] \\ -s_{21}[N-1] & \lambda - s_{22}[N-1] \end{pmatrix}\psi[N-1] \\
&= T[N]\cdots T[3]T[2]T[1]\psi[0], \\
q[N] &= q[N-1] - is_{12}[N-1] - is_{12}^{\rho}[N-1] \\
&= q[0] + i\sum_{j=1}^{N}\frac{\left(\lambda_j - \lambda_j^*\right)\psi_1[j-1]\psi_2^*[j-1]}{|\psi_1[j-1]|^2 + |\psi_2[j-1]|^2} + i\sum_{j=1}^{N}\frac{\left(\lambda_j - \lambda_j^*\right)\psi_1^{\rho}[j-1]\psi_2^{*\rho}[j-1]}{\left|\psi_1^{\rho}[j-1]\right|^2 + \left|\psi_2^{\rho}[j-1]\right|^2}.
\end{aligned} \tag{61}$$

An N-soliton solution of the GI equation on a time–space scale is obtained

$$q[N] = i\sum_{j=1}^{N}\frac{\left(\lambda_j - \lambda_j^*\right)\psi_1[j-1]\psi_2^*[j-1]}{|\psi_1[j-1]|^2 + |\psi_2[j-1]|^2} + i\sum_{j=1}^{N}\frac{\left(\lambda_j - \lambda_j^*\right)\psi_1^{\rho}[j-1]\psi_2^{*\rho}[j-1]}{\left|\psi_1^{\rho}[j-1]\right|^2 + \left|\psi_2^{\rho}[j-1]\right|^2}. \tag{62}$$

In what follows, N-fold DT and N-soliton solutions of the GI equation on three special time–space scales are obtained as follows.

Case I: Taking $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{R}$, we obtain an N-fold DT of the classical GI equation

$$\begin{aligned} \psi[N] &= T[N]\psi[N-1] \\ &= (\lambda I - S[N-1])\psi[N-1] \\ &= \begin{pmatrix} \lambda - s_{11}[N-1] & -s_{12}[N-1] \\ -s_{21}[N-1] & \lambda - s_{22}[N-1] \end{pmatrix}\psi[N-1] \\ &= T[N]\cdots T[3]T[2]T[1]\psi[0], \\ q[N] &= q[N-1] - 2is_{12}[N-1] \\ &= q[0] + 2i\sum_{j=1}^{N} \frac{\left(\lambda_j - \lambda_j^*\right)\psi_1[j-1]\psi_2^*[j-1]}{|\psi_1[j-1]|^2 + |\psi_2[j-1]|^2}. \end{aligned} \tag{63}$$

When $N = 1$, $q[0]$=0 and the spectral parameter $\lambda_1 = \alpha_1 + i\eta_1$, we obtain a one-soliton solution of Equation (27)

$$q[1] = -2\eta_1 e^{2iY_1}\,\mathrm{sech}(2X_1), \tag{64}$$

where

$$\begin{aligned} X_1 &= 4\alpha_1\eta_1 x + 16\left(\alpha_1^3\eta_1 - \alpha_1\eta_1^3\right)t, \\ Y_1 &= -2\left(\alpha_1^2 - \eta_1^2\right)x - 4\left(\alpha_1^4 - 6\alpha_1^2\eta_1^2\right)t. \end{aligned}$$

The profile of the one-soliton in Figure 1.

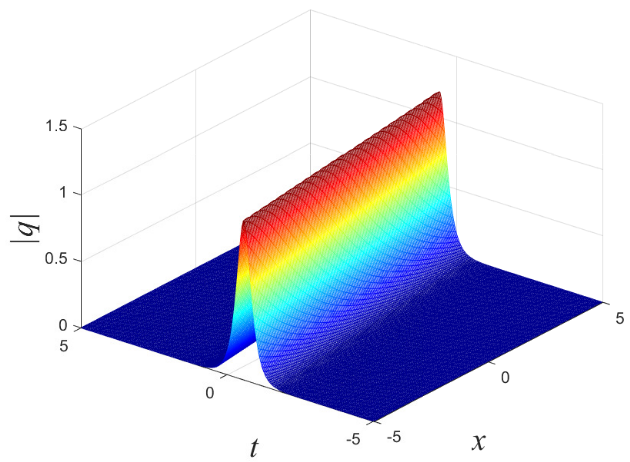

Figure 1. One-soliton solution (64) with $\alpha_1 = 0.7$, $\eta_1 = 0.6$.

When $N = 2$, $q[0]$=0 and the spectral parameter $\lambda_2 = \alpha_2 + i\eta_2$, we obtain a two-soliton solution of Equation (27)

$$\begin{aligned} q[2] =& -2\eta_1 e^{2iY_1}\,\mathrm{sech}(2X_1) - 4\eta_2 \frac{M_1 M_2 e^{-2iY_2} - \alpha_1 M_1\,\mathrm{sech}(2X_1)e^{2X_2 - 2iY_1}}{|M_1|^2 e^{2X_2} + |M_2|^2 e^{-2X_2} + M_4 + M_5 + M_6} \\ &+ 4\eta_2 \frac{i\eta_1 M_2\,\mathrm{sech}(2X_1)e^{-2X_2 - 2iY_1} - M_3}{|M_1|^2 e^{2X_2} + |M_2|^2 e^{-2X_2} + M_4 + M_5 + M_6}, \end{aligned} \tag{65}$$

where

$$
\begin{aligned}
M_1 &= \alpha_2 - \alpha_1 \tanh(2X_1) + (\eta_2 - \eta_1)i, \\
M_2 &= \alpha_2 - \alpha_1 + (\eta_2 - \eta_1 \tanh(2X_1))i, \\
M_3 &= i\alpha_1\eta_1 \operatorname{sech}^2(2X_1)e^{-4iY_1+2iY_2}, \\
M_4 &= 2i \operatorname{sech}(2X_1)\sinh(2iY_1 - 2iY_2)(\eta_1\alpha_2 + \alpha_1\eta_2), \\
M_5 &= 2\operatorname{sech}(2X_1)\cosh(2iY_1 - 2iY_2)\left(\eta_1\eta_2 - \eta_1^2 + \alpha_1\alpha_2 - \alpha_1^2\right), \\
M_6 &= \operatorname{sech}^2(2X_1)\left(\eta_1^2 e^{-2X_2} + \alpha_1^2 e^{2X_2}\right), \\
X_2 &= 4\alpha_2\eta_2 x + 16\left(\alpha_2^3\eta_2 - \alpha_2\eta_2^3\right)t, \\
Y_2 &= -2\left(\alpha_2^2 - \eta_2^2\right)x - 4\left(\alpha_2^2 - 6\alpha_2^2\eta_2^2\right)t.
\end{aligned}
$$

Case II: Taking $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{C}$, we find

$$
\begin{aligned}
&\mu(t) = 0, \\
&\nu(x) = \begin{cases} \dfrac{1}{3^{m+1}}, \ x \in \mathbb{L}, \\ 0, \ x \in \mathbb{C}\backslash\mathbb{L}, \end{cases}
\end{aligned} \tag{66}
$$

where $\mathbb{C}$ is a Cantor set. $\mathbb{L}$ contains left discrete elements of $\mathbb{C}$,

$$
\mathbb{L} = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : m \in N, \ a_k \in \{0,2\}, 1 \le k \le m \right\}.
$$

Then, an N-fold DT of the GI equation is constructed

$$
\begin{aligned}
\psi[N] &= T[N]\psi[N-1] \\
&= (\lambda I - S[N-1])\psi[N-1] \\
&= \begin{pmatrix} \lambda - s_{11}[N-1] & -s_{12}[N-1] \\ -s_{21}[N-1] & \lambda - s_{22}[N-1] \end{pmatrix}\psi[N-1] \\
&= T[N]\cdots T[3]T[2]T[1]\psi[0], \\
q[N] &= \begin{cases} q[0] + i\sum\limits_{j=1}^{N} \frac{\left(\lambda_j - \lambda_j^*\right)\psi_1[j-1]\psi_2^*[j-1]}{|\psi_1[j-1]|^2 + |\psi_2[j-1]|^2} + i\sum\limits_{j=1}^{N} \frac{\left(\lambda_j - \lambda_j^*\right)\psi_1^\rho[j-1]\psi_2^{*\rho}[j-1]}{\left|\psi_1^\rho[j-1]\right|^2 + \left|\psi_2^\rho[j-1]\right|^2}, \ x \in \mathbb{L}, \ t \in \mathbb{R}, \\ q[0] + 2i\sum\limits_{j=1}^{N} \frac{\left(\lambda_j - \lambda_j^*\right)\psi_1[j-1]\psi_2^*[j-1]}{|\psi_1[j-1]|^2 + |\psi_2[j-1]|^2}, \ x \in \mathbb{C}\backslash\mathbb{L}, \ t \in \mathbb{R}. \end{cases}
\end{aligned} \tag{67}
$$

According to Definition 3, we have

$$
e_{\pm 2i\lambda_1^4}(x,0) = \left[\frac{1 \pm \frac{i\lambda_1^4}{3^{m+1}}}{1 \mp \frac{i\lambda_1^4}{3^{m+1}}}\right]^{\frac{x}{3^{m+1}}}, \ e_{\pm i\lambda_1^2}(x,0) = \left[\frac{1 \pm \frac{i\lambda_1^2}{2\times 3^{m+1}}}{1 \mp \frac{i\lambda_1^2}{2\times 3^{m+1}}}\right]^{\frac{x}{3^{m+1}}}.
$$

When $N = 1$, $q[0] = 0$ and the spectral parameter $\lambda_1 = \alpha_1 + i\eta_1$, a one-soliton solution is obtained

$$
q[1] = \begin{cases} \dfrac{1}{N_1} - \dfrac{\left(3^{m+1} - i\alpha_1^2\right)^2 M_7 - \left(i\eta_1^2 - 2\alpha_1\eta_1\right)^2 M_7}{\left(i\alpha_1^2 - i\eta_1^2\right)N_2}, \ x \in \mathbb{L}, \ t \in \mathbb{R}, \\ -2\eta_1 e^{2iY_1}\operatorname{sech}(2X_1), \ x \in \mathbb{C}\backslash\mathbb{L}, \ t \in \mathbb{R}, \end{cases} \tag{68}
$$

where

$$N_1 = E_1|_{\lambda_1=\alpha_1+i\eta_1} + E_2|_{\lambda_1=\alpha_1+i\eta_1},$$
$$N_2 = E_1|_{\lambda_1=\alpha_1+i\eta_1} - E_2|_{\lambda_1=\alpha_1+i\eta_1},$$
$$M_7 = 2\eta_1\left[1+\frac{i\alpha_1^2}{3^{m+1}}\left(2i\eta_1^2-2i\alpha_1^2\right)\right]^{3^{m+1}x} e^{\left(24i\alpha_1^2\eta_1^2-4i\alpha_1^4\right)t}.$$

Case III: Taking $\mathbb{T}\times\mathbb{X}=\mathbb{R}\times\mathbb{K}_p$, we find

$$\mu(t)=0, \quad \nu(x)=\begin{cases}(1-p^{-1})x, \ x=p^k\in p^{\mathbb{Z}},\\ 0, \ x=0,\end{cases} \tag{69}$$

where $p>1$, $p^{\mathbb{Z}}=\left\{p^k : k\in\mathbb{Z}\right\}$ and $\mathbb{K}_p=p^{\mathbb{Z}}\bigcup\{0\}$.

Then, an N-fold DT is constructed

$$\begin{aligned}\psi[N] &= T[N]\psi[N-1]\\ &=(\lambda I - S[N-1])\psi[N-1]\\ &=\begin{pmatrix}\lambda - s_{11}[N-1] & -s_{12}[N-1]\\ -s_{21}[N-1] & \lambda - s_{22}[N-1]\end{pmatrix}\psi[N-1]\\ &=T[N]\cdots T[3]T[2]T[1]\psi[0],\end{aligned}$$

$$q[N]=\begin{cases} q[0]+i\sum\limits_{j=1}^{N}\frac{\left(\lambda_j-\lambda_j^*\right)\psi_1[j-1]\psi_2^*[j-1]}{|\psi_1[j-1]|^2+|\psi_2[j-1]|^2}+i\sum\limits_{j=1}^{N}\frac{\left(\lambda_j-\lambda_j^*\right)\psi_1^{\rho}[j-1]\psi_2^{*\rho}[j-1]}{\left|\psi_1^{\rho}[j-1]\right|^2+\left|\psi_2^{\rho}[j-1]\right|^2}, \ x\in p^{\mathbb{Z}}, \ t\in\mathbb{R},\\ q[0]+2i\sum\limits_{j=1}^{N}\frac{\left(\lambda_j-\lambda_j^*\right)\psi_1[j-1]\psi_2^*[j-1]}{|\psi_1[j-1]|^2+|\psi_2[j-1]|^2}, \ x=0, \ t\in\mathbb{R}.\end{cases} \tag{70}$$

According to

$$\int_a^b f(x)\nabla x = \left(1-p^{-1}\right)\sum_{x=a}^{b} xf(x),$$
$$\int_a^b f(\rho(x))\nabla x = (p-1)\sum_{x=a}^{b} xf(x),$$

we have

$$e_{\pm i\lambda_1^2}(x,0)=e^{\left(1-p^{-1}\right)\sum\limits_{x=0}^{p^k}\pm i\lambda_1^2},$$
$$e_{\pm i\lambda_1^2}(\rho(x),0)=e^{(p-1)\sum\limits_{x=0}^{p^k}\pm i\lambda_1^2}.$$

When $N=1$, $q[0]$=0 and the spectral parameter $\lambda_1=\alpha_1+i\eta_1$, a one-soliton solution is obtained

$$q[1]=\begin{cases}-\eta_1 e^{\left(16\alpha_1\eta_1^3-16\alpha_1^3\eta_1-4i\alpha_1^4+24i\alpha_1^2\eta_1^2\right)t}M_8, \ x\in p^{\mathbb{Z}}, \ t\in\mathbb{R},\\ -2\eta_1 e^{2iY_1}\operatorname{sech}(2X_1), \ x=0, \ t\in\mathbb{R},\end{cases} \tag{71}$$

where

$$\begin{aligned}M_8 =& e^{\left(1-p^{-1}\right)\sum\limits_{x=0}^{p^k}\left(-2i\alpha_1^2+2i\eta_1^2\right)x}\operatorname{sech}\left(1-p^{-1}\right)\sum_{x=0}^{p^k}4\alpha_1\eta_1 x\\ &+e^{(p-1)\sum\limits_{k=0}^{p^k}\left(-2i\alpha_1^2+2i\eta_1^2\right)x}\operatorname{sech}(p-1)\sum_{x=0}^{p^k}4\alpha_1\eta_1 x.\end{aligned}$$

4. Conclusions

In this paper, the coupled GI equation on a time–space scale was obtained by extending the Lax matrix equation on a time–space scale, which can be reduced to the classical GI equation. In particular, the semi-discrete GI equation was given by providing parallel computations for the discrete and continuous case. The standard DT of the GI equation was extended on a time–space scale. On this basis, its N-soliton solutions on a time–space scale were obtained, which were expressed using Cayley exponential functions.

The extension provides a wider range of nonlinear integrable dynamic models and promotes the study of nonlinear dynamic systems. By taking the "seed solution" $q = 0$ and $\lambda = \alpha + i\beta$, one-solition solutions of the GI equation were obtained on three different time–space scales ($\mathbb{X} = \mathbb{R}$, $\mathbb{X} = \mathbb{C}$ and $\mathbb{X} = \mathbb{K}p$). In one case, the exact solution (64) and its dynamic figure were obtained when $x \in \mathbb{R}$. In the other cases, when $x \in \mathbb{C}\backslash\mathbb{L}$ and $x = 0$, exact solutions (68) and (71) were obtained and were similar to Equation (64). However, when $x \in \mathbb{L}$ and $x \in p^{\mathbb{Z}}$, the structures of solutions (68) and (71) were more complicated and their values were different from those of Equation (64) at those discontinuity points.

Due to the limitations of the computer, it was difficult to obtain their dynamic figures at this stage. Furthermore, there is another well-known equation, the Eckhaus equation, which possesses a very similar structure. The Eckhaus equation is also integrable and has soliton-like solutions expressed in terms of the hyperbolic functions [37,38]. Therefore, we will find the most effective way to reduce structures of solutions (68) and (71) on $\mathbb{C}$ and $\mathbb{K}p$, and study the Eckhaus equation on a time–space scale in our future work.

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