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# A Relation-Theoretic Metrical Fixed Point Theorem for Rational Type Contraction Mapping with an Application

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**Abstract:** In this article, we discuss the relation theoretic aspect of rational type contractive mapping to obtain fixed point results in a complete metric space under arbitrary binary relation. Furthermore, we provide an application to find a solution to a non-linear integral equation.

**Keywords:** binary relation; $\Re$-completeness; rational contraction

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## 1. Introduction

In 1922, the first prosperous result was postulated by S. Banach [1] in the fixed point theory for contractive mapping. For its modesty, his work functioned as a schematic research tool in a different branch of mathematics. This theorem went in a different direction to verify its effectiveness. Such as

(i) Enlarging the ambient space;
(ii) Improving the underlying contraction condition;
(iii) Weakening the involved metrical notions.

Among the several extensions of the Banach contraction principle to various spaces, some are rectangular metric space, generalized metric space, partial metric space, b-metric space, partial b-metric space, symmetric space and quasi metric space. Partial metric space was introduced by Matthews [2] in 1994. Nowadays, there are many fixed point theories in Partial metric space.

Several researchers stated various contraction conditions [3–8] for the fixed point theorem. Inspired by Turinici's [9] work, Ran-Reurings in 2004 formulate the result that there will be a fixed point of self-mappings that is applied only for those points which are comparable to each other by an order relation in partial metric space. Later, the work was extended by J. J. Nieto and R. Rodríguez-López [10]. In 1975, Dass and Gupta [11], came up with a new contractive condition termed as a rational type contraction. Later, Canbrera et al. [12] used the result of Dass and Gupta [11] in 2013 to obtain the fixed point results in partial ordered metric space.

Alternatively, Alam and Imdad [13] established a profound generalization of the Banach contraction principle with an amorphous binary relation. With this structure, various relation-theoretic results were proposed in different aspects of the binary relation or contractive condition.

There are too many applications of fixed point theory in the field of ordinary differential equations, systems of matrix equation, integral equations, game theory, economics, optimization models and numerical models in statistics. Moreover, for the multivalued maps in the equilibrium in the duopoly markets and in aquatic ecosystem there are also too many applications. In an ordinary differential equation, the application provided by J. J. Nieto and R. Rodríguez-López [10] and the system of matrix equations by Ran and

Reurings [14], the fixed point for iteration to find optimal solution in statistics [15], for the stability problem in Intuitionistics Fuzzy Banach Space [16], and many more such as [17].

This article intends to establish some fixed point theorems under contractive mapping over a complete metric space. Ultimately, an example is provided to establish the result for our assumptions. Furthermore, we provide an application [18] in a non-linear integral equation to obtain a fixed point.

## 2. Preliminaries

In this section, we present some basic definitions which will be required in proving our main results. We denote $\mathbb{N} \cup \{0\}$ as $\mathbb{N}_0$ throughout the paper.

**Definition 1** ([11]). *Let $(W, d)$ be a complete metric space and $T$ a self-mapping on $W$. Then, $T$ is said to be a rational type contraction if there exist $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 + \delta_2 < 1$, satisfying*

$$d(T\mu, T\nu) \leq \delta_1 \frac{d(\nu, T\nu)[1 + d(\mu, T\mu)]}{[1 + d(\mu, \nu)]} + \delta_2 d(\mu, \nu) \quad for \;\; all \;\; \mu, \nu \in W.$$

**Definition 2** ([19]). *Let $W$ be a nonempty set. A subset $\Re$ of $W^2$ is called a binary relation on $W$. The subsets $W^2$ and $\emptyset$ of $W^2$ are in trivial relation.*

**Definition 3** ([13]). *Consider a binary relation $\Re$ on a nonempty set $W$. For $\mu, \nu \in W$, one may say that $\mu$ and $\nu$ are $\Re$-comparative if either $(\mu, \nu) \in \Re$ or $(\nu, \mu) \in \Re$. We symbolize it with $[\mu, \nu] \in \Re$.*

**Definition 4** ([19–24]). *On a nonempty set $W$, a binary relation $\Re$ is termed as*

- *(i) Reflexive if $(\mu, \mu) \in \Re \; \forall \; \mu \in W$;*
- *(ii) Symmetric if $(\mu, \nu) \in \Re$ then $(\nu, \mu) \in \Re$;*
- *(iii) Anti-symmetric if $(\mu, \nu) \in \Re$ and $(\nu, \mu) \in \Re$ then $\mu = \nu$;*
- *(iv) Transitive if $(\mu, \nu) \in \Re$ and $(\nu, \kappa) \in \Re$ then $(\mu, \kappa) \in \Re$;*
- *(v) A partial order if $\Re$ is reflexive, anti-symmetric and transitive.*

**Definition 5** ([19]). *Let $W$ be a nonempty set and $\Re$ a binary relation on $W$.*

- *(i) The dual relation, transpose or inverse of $\Re$, signified by $\Re^{-1}$ is interpreted by,*

$$\Re^{-1} = \{(\mu, \nu) \in W^2 : (\nu, \mu) \in \Re\}.$$

- *(ii) Symmetric closure $\Re^s$ of $\Re$, is defined to be the set $\Re \cup \Re^{-1}$ (i.e., $\Re^s = \Re \cup \Re^{-1}$).*

**Proposition 1** ([13]). *For a binary relation $\Re$ defined on a nonempty set $W$,*

$$(\mu, \nu) \in \Re^s \implies [\mu, \nu] \in \Re.$$

**Definition 6** ([13]). *Consider a nonempty set $W$ and let $\Re$ be a binary relation on $W$. A sequence $\{\mu_n\} \subset W$ is called $\Re$-preserving if*

$$(\mu_n, \mu_{n+1}) \in \Re \quad \forall \, n \, \in \, \mathbb{N}_0.$$

**Definition 7** ([13]). *For a nonempty set $W$ with a self-mapping $T$ on it. Any binary relation $\Re$ on $W$ is $T$-closed if $\forall \; \mu, \nu \in W$,*

$$(\mu, \nu) \in \Re \implies (T\mu, T\nu) \in \Re.$$

**Definition 8** ([25]). *Let $(W, d)$ be a metric space and $\Re$ a binary relation on $W$. Then, $(W, d)$ is $\Re$-complete if every $\Re$-preserving Cauchy sequence in $W$ converges.*

It is obvious that every complete metric space is $\Re$-complete with respect to a binary relation $\Re$ but not conversely. For instance, Suppose $W = (-2, 2]$ together with the usual metric $d$. Notice that $(W, d)$ is not complete. Now endow $W$ with the following relation:

$$\Re = \{(\mu, \nu) \in W^2 : \mu, \nu \geq 0\}.$$

Then, $(W, d)$ is a $\Re$-complete metric space.

**Definition 9** ([22])**.** *Let $W$ be a nonempty set endowed with a binary relation $\Re$. A subset $D$ of $W$ is called $\Re$-directed if for each $\mu, \nu \in D$, there exists $\kappa \in W$ such that $(\mu, \kappa) \in \Re$ and $(\nu, \kappa) \in \Re$.*

**Definition 10** ([25])**.** *Let $(W, d)$ be a metric space endowed with a binary relation $\Re$ with $\mu \in W$. Then $T : W \to W$ is called $\Re$-continuous at $\mu$ if for any $\Re$-preserving sequence $\{\mu_n\}$ with $\mu_n \xrightarrow{d} \mu$, we obtain $T(\mu_n) \xrightarrow{d} T(\mu)$. Furthermore, $T$ is called $\Re$-continuous if it is $\Re$-continuous at each point of $W$.*

**Definition 11** ([13])**.** *Let $(W, d)$ be a metric space. A binary relation $\Re$ on $W$ is termed as d-self-closed if whenever $\{\mu_n\}$ is an $\Re$-preserving sequence and*

$$\mu_n \xrightarrow{d} \mu,$$

*then there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ with $[\mu_{n_k}, \mu] \in \Re \; \forall \; k \in \mathbb{N}_0$.*

**Proposition 2** ([13])**.** *If $(W, d)$ is a metric space, $\Re$ is a binary relation on $W$,$T$ is a self-mapping on $W$ and $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 + \delta_2 < 1$ then the following conditions are equivalent*

*(i)* $d(T\mu, T\nu) \leq \delta_1 \frac{d(\nu, T\nu)[1 + d(\mu, T\mu)]}{[1 + d(\mu, \nu)]} + \delta_2 d(\mu, \nu) \quad \forall \; \mu, \nu \in W \;$ *with* $\; (\mu, \nu) \in \Re$,

*(ii)* $d(T\mu, T\nu) \leq \delta_1 \frac{d(\nu, T\nu)[1 + d(\mu, T\mu)]}{[1 + d(\mu, \nu)]} + \delta_2 d(\mu, \nu) \quad \forall \; \mu, \nu \in W \;$ *with* $\; [\mu, \nu] \in \Re$.

*The proof is followed by the symmetrycity of the metric $d$.*

## 3. Main Result

In this fragment, we will introduce the fixed point theorem under rational contraction in the relation theoretic sense.

**Theorem 1.** *Consider $(W, d)$ as a metric space together with a binary relation $\Re$ and a self-mapping $T$ on it. Assume that the following conditions hold:*

*(i)* *$(W, d)$ is $\Re$-complete;*
*(ii)* *$W(T; \Re)$ is non-empty;*
*(iii)* *$\Re$ is $T$-closed;*
*(iv)* *Either $T$ is continuous or $\Re$ is d-self closed;*
*(v)* *There exist $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 + \delta_2 < 1$ such that*

$$d(T\mu, T\nu) \leq \delta_1 \frac{d(\nu, T\nu)[1 + d(\mu, T\mu)]}{1 + d(\mu, \nu)} + \delta_2 d(\mu, \nu) \quad for \quad \mu, \nu \in W \quad with \;\; (\mu, \nu) \in \Re.$$

*Then $T$ has a fixed point.*

**Proof.** From the condition $(ii)$, we always have a $\mu_0 \in W$ such that $(\mu_0, T\mu_0) \in \Re$, then define a Picard sequence of iterates $\mu_{n+1} = T\mu_n$. If $T\mu_0 = \mu_0$ then nothing to prove.
If $T\mu_0 \neq \mu_0$ then by condition $\Re$ is $T$-closed we obtain

$$(T\mu_0, T^2\mu_0), ((T^2\mu_0, T^3\mu_0)), ((T^3\mu_0, T^4\mu_0)) \cdots ((T^n\mu_0, T^{n+1}\mu_0)) \cdots \in \Re.$$

So we have $(\mu_n, \mu_{n+1}) \in \Re \quad \forall n \in \mathbb{N}_0$ *i.e.*, $\{\mu_n\}$ is a $\Re$ preserving sequence.
If $\mu_{n_0+1} = T\mu_{n_0} = \mu_{n_0}$ then $\mu_{n_0}$ is a fixed point of $T$; then the proof is complete.
If $\mu_{n+1} \neq \mu_n$ for $n \geq 1$ then by condition $(v)$ for $(\mu_n, \mu_{n+1}) \in \Re$, we have

$$\begin{aligned}
d(\mu_n, \mu_{n+1}) = d(T\mu_{n-1}, T\mu_n) &\leq \delta_1 \frac{d(\mu_n, T\mu_n)[1 + d(\mu_{n-1}, T\mu_{n-1})]}{[1 + d(\mu_{n-1}, \mu_n)]} + \delta_2 d(\mu_{n-1}, \mu_n) \\
&= \delta_1 \frac{d(\mu_n, \mu_{n+1})[1 + d(\mu_{n-1}, \mu_n)]}{[1 + d(\mu_{n-1}, \mu_n)]} + \delta_2 d(\mu_{n-1}, \mu_n) \\
&= \delta_1 d(\mu_n, \mu_{n+1}) + \delta_2 d(\mu_{n-1}, \mu_n) \\
(1 - \delta_1) d(\mu_n, \mu_{n+1}) &\leq \delta_2 d(\mu_{n-1}, \mu_n) \\
d(\mu_n, \mu_{n+1}) &\leq \frac{\delta_2}{1 - \delta_1} d(\mu_{n-1}, \mu_n).
\end{aligned}$$

Then, by an induction process, we will obtain

$$d(\mu_n, \mu_{n+1}) \leq \left( \frac{\delta_2}{1 - \delta_1} \right)^n d(\mu_0, \mu_1) \quad \text{for any } \ n \in \mathbb{N}_0. \tag{1}$$

Denote $\gamma = \frac{\delta_2}{1-\delta_1} < 1$, then Equation (1) can be rewritten as

$$d(\mu_n, \mu_{n+1}) \leq (\gamma)^n d(\mu_0, \mu_1) \quad \text{for any} \quad n \in \mathbb{N}_0.$$

Next, for $\{\mu_n\}$ to be a Cauchy sequence, let $m > n$ then

$$\begin{aligned}
d(\mu_n, \mu_m) &\leq d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_{n+2}) + \ldots\ldots. + d(\mu_{m-1}, \mu_m) \\
&\leq \gamma^n d(\mu_0, \mu_1) + \gamma^{n+1} d(\mu_0, \mu_1) + \ldots\ldots. + \gamma^{m-1} d(\mu_0, \mu_1) \\
&\leq (\gamma^n + \gamma^{n+1} + \ldots\ldots + \gamma^{m-1}) d(\mu_0, \mu_1) \\
&\leq \gamma^n (1 + \gamma + \gamma^2 + \ldots\ldots + \gamma^{m-n-1}) d(\mu_0, \mu_1) \\
&\leq \gamma^n \left( \frac{1 - \gamma^{m-n}}{1 - \gamma} \right) d(\mu_0, \mu_1).
\end{aligned}$$

For $m, n \to \infty$ and as $\gamma < 1$, then we obtain $\lim\limits_{n,m \to \infty} d(\mu_n, \mu_m) = 0$. So, we have proved that $\{\mu_n\}$ is a Cauchy sequence.
Since the space $(W, d)$ is a $\Re$-complete metric space, then there always exists $\mu \in W$ such that $\mu_n \to \mu$.
Then by continuity of $T$, we have

$$T\mu = T(\lim_{n \to \infty} \mu_n) = \lim_{n \to \infty} T\mu_n = \lim_{n \to \infty} \mu_{n+1} = \mu.$$

So, $\mu$ is a fixed point of $T$.
If otherwise, $\Re$ is $d$-self closed then for the $\Re$-preserving sequence $\{\mu_n\} \xrightarrow{d} \mu$ there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ with $[\mu_{n_k}, \mu] \in \Re \quad \forall\, k \in \mathbb{N}_0$.
Then, by condition $(v)$, $[\mu_{n_k}, \mu] \in \Re$ and Proposition 2 and $\mu_{n_k} \to \mu$

$$\begin{aligned}
d(\mu_{n_k+1}, T\mu) = d(T\mu_{n_k}, T\mu) &\leq \delta_1 \frac{d(\mu, T\mu)[1 + d(\mu_{n_k}, T\mu_{n_k})]}{1 + d(\mu_{n_k}, \mu)} + \delta_2 d(\mu_{n_k}, \mu) \\
&\leq \delta_1 \frac{d(\mu, T\mu)[1 + d(\mu_{n_k}, \mu_{n_k+1})]}{1 + d(\mu_{n_k}, \mu)} + \delta_2 d(\mu_{n_k}, \mu).
\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\begin{aligned} d(\mu, T\mu) &\leq \delta_1 d(\mu, T\mu) \\ (1-\delta_1) d(\mu, T\mu) &\leq 0. \end{aligned}$$

Since $0 \leq \delta_1 < 1$ then, the only possibility is $d(\mu, T\mu) = 0$. Hence, $\mu = T\mu$. Then, $\mu$ is a fixed point of $T$. □

**Theorem 2.** *If in addition to Theorem 1 we have the condition:*
*$(vi)$ T(W) is $\Re^s$-directed.*
*Then T has a unique fixed point.*

**Proof.** Let us suppose that $\mu, \nu$ are two fixed points, i.e., $T\mu = \mu$ and $T\nu = \nu$ then we have the two cases,
Case I: if $(\mu, \nu) \in \Re$ then

$$\begin{aligned} d(\mu, \nu) = d(T\mu, T\nu) &\leq \delta_1 \frac{d(\nu, T\nu)[1 + d(\mu, T\mu)]}{[1 + d(\mu, \nu)]} + \delta_2 d(\mu, \nu) \\ d(\mu, \nu) &\leq \delta_2 d(\mu, \nu) \end{aligned}$$

then $(1-\delta_2) d(\mu, \nu) \leq 0$ implies that $d(\mu, \nu) = 0$ as $\delta_2 > 0$.
Case II: if $(\mu, \nu) \notin \Re$ then by $T(W)$ is $\Re^s$-directed then there exists $\kappa \in W$ such that $(\mu, \kappa) \in \Re$ and $(\kappa, \nu) \in \Re$. Since $\Re$ is $T$-closed $T^n\kappa$ will be related to $T^n\mu$, *i.e.*, $(T^n\kappa, T^n\mu = \mu) \in \Re$ for any $n \in \mathbb{N}_0$. Then, by contractive condition $(v)$ of Theorem 1, for any $n \in \mathbb{N}_0$, we have

$$\begin{aligned} d(T^n\kappa, \mu) = d(T^n\kappa, T^n\mu) &\leq \delta_1 \frac{d(T^{n-1}\mu, T^n\mu)[1 + d(T^{n-1}\kappa, T^n\kappa)]}{1 + d(T^{n-1}\kappa, T^{n-1}\mu)} + \delta_2 d(T^{n-1}\kappa, T^{n-1}\mu) \\ &= \delta_1 \frac{d(\mu, \mu)[1 + d(T^{n-1}\kappa, T^n\kappa)]}{1 + d(T^{n-1}\kappa, \mu)} + \delta_2 d(T^{n-1}\kappa, \mu) \\ &= \delta_2 d(T^{n-1}\kappa, \mu). \end{aligned}$$

Then by mathematical induction, we obtain

$$d(T^n\kappa, \mu) \leq (\delta_2)^n d(\kappa, \mu).$$

Since $\delta_2 < 1$ then $\lim_{n\to\infty} d(T^n\kappa, \mu) = 0$, which provides us $\lim_{n\to\infty} T^n\kappa = \mu$.
In a similar fashion we also obtain $\lim_{n\to\infty} T^n\kappa = \nu$.
Then, by the unity of limits we obtain $\mu = \nu$.
So our supposition that $\mu$ and $\nu$ are two different fixed points is wrong. Hence, the mapping $T$ has a unique fixed point. □

**Corollary 1.** *If we substitute $\delta_2 = 0$ into Theorems 1 and 2, we have the following fixed point theorem.*
*Consider $(W, d)$ as a metric space together with a binary relation $\Re$ and a self mapping $T$ on it. Assume that the following conditions holds:*

*(i′)* *$W(T; \Re)$ is non-empty;*
*(ii′)* *$\Re$ is T-closed;*
*(iii′)* *$(W, d)$ is $\Re$-complete;*
*(iv′)* *Either $T$ is $\Re$-continuous or $\Re$ is d-self closed;*
*(v′)* *There exist $\delta_1 \in [0, 1)$ such that*

$$d(T\mu, T\nu) \leq \delta_1 \frac{d(\nu, T\nu)[1 + d(\mu, T\mu)]}{[1 + d(\mu, \nu)]} \quad for \;\; (\mu, \nu) \in \Re \; and \;\; \mu, \nu \in W,$$

*(vi′)* $T(W)$ *is* $\Re^s$*-directed.*

*Then, T has a unique fixed point.*

**Remark 1.** *If we put* $\delta_1 = 0$ *into Theorems 1 and 2, then under the setting of* $\Re = \preceq$ *as the partial order, we obtain Theorems* (2.1), (2.2) *and* (2.3) *of [26].*

**Remark 2.** *If we substitute* $\delta_1 = 0$ *and* $0 < \delta_2 < \frac{1}{3}$ *into Theorems 1 and 2, then the condition* $(v)$ *reduces to the Kannan contraction [27]* $(\delta_3 \in (0, \frac{1}{2}))$

$$d(T\mu, T\nu) \leq \delta_3[d(\mu, T\mu) + d(\nu, T\nu)] \ \ for \ \ \mu, \nu \in W \ \ with \ (\mu, \nu) \in \Re.$$

**Proof.** As $\delta_1 = 0$ and $0 \leq \delta_2 < \frac{1}{3}$, then the condition $(v)$ of Theorem 1 reduces to the form

$$\begin{aligned} d(T\mu, T\nu) \leq \delta_2 d(\mu, \nu) &\leq \delta_2\big[d(\mu, T\mu) + d(T\mu, T\nu) + d(T\nu, \nu)\big] \\ &\leq \delta_2\big[d(\mu, T\mu) + d(\nu, T\nu)\big] + \delta_2 d(T\mu, T\nu) \\ (1-\delta_2)d(T\mu, T\nu) &\leq \big[d(\mu, T\mu) + d(\nu, T\nu)\big] \\ d(T\mu, T\nu) &\leq \frac{\delta_2}{1-\delta_2}\big[d(\mu, T\mu) + d(\nu, T\nu)\big] \\ d(T\mu, T\nu) &\leq \delta_3\big[d(\mu, T\mu) + d(\nu, T\nu)\big] \ \ for \ \ \delta_3 = \frac{\delta_2}{1-\delta_2} < \frac{1}{2}. \end{aligned}$$

□

**Remark 3.** *If diameter* $(W) \leq 1$ *and* $2\delta_1 + \delta_2 < 1$, *then conditions* $(v)$ *of Theorem 1 reduces to the Riech [28] type conditions:*

$$d(T\mu, T\nu) \leq \delta_1 d(\mu, T\mu) + \delta_1 d(\nu, T\nu) + \delta_2 d(\mu, \nu).$$

**Proof.** For any $\mu, \nu \in W$ with $(\mu, \nu) \in \Re$,

$$\begin{aligned} d(T\mu, T\nu) &\leq \delta_1 \frac{d(\nu, T\nu)[1 + d(\mu, T\mu)]}{[1 + d(\mu, \nu)]} + \delta_2 d(\mu, \nu) \\ &\leq \delta_1 d(\nu, T\nu) + \delta_1 d(\nu, T\nu) d(\mu, T\mu) + \delta_2 d(\mu, \nu). \end{aligned}$$

As the diameter of $W$ is less than equal to one then $d(\nu, T\nu) < 1$. Then, we have

$$d(T\mu, T\nu) \leq \delta_1 d(\nu, T\nu) + \delta_1 d(\mu, T\mu) + \delta_2 d(\mu, \nu).$$

□

Finally, we produce an illustrative example to substantiate the utility of our result, which does not satisfy the hypotheses of the existing results [1,11–13,18], but satisfies the hypotheses of our result, and hence has a fixed point.

**Example 1.** *Consider the metric space* $W = (-1, 1]$ *with the usual metric d and a binary relation* $\Re = \{(\mu, \nu) \in W^2 : \nu > \mu \geq 0\}$ *together with a mapping* $T : W \to W$ *defined by*

$$T(x) = \begin{cases} \frac{1}{2}, \ if & -1 < \mu < 0, \\ \frac{\mu}{4}, \ if & 0 \leq \mu \leq 1. \end{cases}$$

*It is clear that $\Re$ is T-closed and T is not a continuous function.*
*Now, for $(\mu, \nu) \in \Re$*

$$\begin{aligned}
d(T\mu, T\nu) &= |T\mu - T\nu| \\
&= |\frac{\mu}{4} - \frac{\nu}{4}| \\
&= \frac{1}{4}|\mu - \nu| \\
&\leq \frac{1}{2}\frac{|\mu - \nu|}{[1 + |\mu - \nu|]} \\
&< \frac{1}{2}\frac{1}{[1 + |\mu - \nu|]} \times \frac{8}{5}|\frac{3\nu}{4}|\left[1 + |\frac{3\mu}{4}|\right] \\
&< \frac{1}{2} \times \frac{8}{5}\frac{|\frac{3\nu}{4}|[1 + |\frac{3\mu}{4}|]}{1 + |\mu - \nu|} \\
&< \frac{4}{5}\frac{|\nu - \frac{\nu}{4}|[1 + |\mu - \frac{\mu}{4}|]}{1 + |\mu - \nu|} + \frac{1}{10}|\mu - \nu| \\
&= \delta_1\frac{d(\nu, T\nu)[1 + d(\mu, T\mu)]}{[1 + d(\mu, \nu)]} + \delta_2 d(\mu, \nu) \ \textit{for} \ \delta_1 = \frac{4}{5} < 1 \ \textit{and} \ \delta_2 = \frac{1}{10} < 1.
\end{aligned}$$

*So, $\delta_1 + \delta_2 = \frac{4}{5} + \frac{1}{10} = \frac{9}{10} < 1$.*

Then $T$ has fixed point $\mu = 0$.

Notice that condition $(v)$ of Theorem 1 does not hold for the whole space (for example, take $\mu = 1$ and $\nu = 0$). Therefore, this example cannot be solved by the existing results, which establishes the importance of our result.

## 4. Application to Non-Linear Integral Equations

Consider $W = C[a, b]$ the class of all continuous functions from $[a, b]$ to $[a, b]$ with metric

$$d(\mu, \nu) = \sup_{\tau \in [a,b]} \{|\mu(\tau) - \nu(\tau)|\}$$

then $(W, d)$ is a complete metric space.

**Theorem 3.** *Consider the non-linear integral equation*

$$\mu(\tau) = f(\tau) + \int_a^b Q(\tau, r, \mu(r))dr \tag{2}$$

*where $\tau \in [a, b]$, $f : [a, b] \to \mathbb{R}$ and $Q : [a, b]^2 \times \mathbb{R} \to \mathbb{R}$. Suppose the following conditions holds:*

*(i) $f$ is continuous and $Q(\tau, r, \mu(r))$ is integrable w.r.t $r$ on $[a, b]$;*

*(ii) $T\mu \in C[a, b]$ for all $\mu \in C[a, b]$ where*

$$T\mu(\tau) = f(\tau) + \textstyle\int_a^b Q(\tau, r, \mu(r))dr,$$

*(iii) For all $r, \tau \in [a, b]$ and $\mu, \nu \in C[a, b]$ with $(\mu, \nu) \in \Re$*

$$Q(\tau, r, \nu(r)) - Q(\tau, r, \mu(r)) < \eta(t, r)|\mu(r) - \nu(r)|$$

*where $\eta : [a, b]^2 \to \mathbb{R}^+$ is a continuous function satisfying*

$$\sup_{\tau \in [a,b]} \left( \textstyle\int_a^b \eta(\tau, r)dr \right) < 1,$$

*(iv) There exist $\mu_0(\tau) \leq f(\tau) + \int_a^b Q(\tau, r, \mu_0(r))dr$ for all $\tau \in [a, b]$.*

*Then, the non-linear integral Equation (2) has a unique solution $\mu \in C[a, b]$.*

**Proof.** For the proof, let us define a binary relation on $W$

$$\Re = \left\{ (\mu,\nu) \in W^2 \quad \text{if} \quad \mu(\tau) < \nu(\tau) \;\; \text{for all} \;\; \tau \in [a,b] \right\}.$$

By assumption $(iv)$ we have $\mu_0 \in C[a,b]$ such that

$$\begin{aligned} \mu_0(\tau) &\leq f(\tau) + \int_a^b Q(\tau, r, \mu_0(r))dr \\ &\leq T\mu_0(\tau) \end{aligned}$$

this implies that $\mu_0 \Re T\mu_0$, then $W(T;\Re)$ is non-empty.
Now, to prove that the relation $\Re$ is $T$-closed, choose $\mu,\nu \in C[a,b]$ such that $\mu\Re\nu$,then

$$\begin{aligned} \mu(r) &< \nu(r) \\ |\mu(r) - \nu(r)| &> 0 \\ \eta(\tau,r)|\mu(r) - \nu(r)| &> 0 \end{aligned}$$

then by condition (iii) of Theorem 3, we have

$$\begin{aligned} Q(\tau,r,\mu(r)) &< Q(\tau,r,\nu(r)) \\ \int_a^b Q(\tau,r,\mu(r))dr &< \int_a^b Q(\tau,r,\nu(r))dr \\ f(\tau) + \int_a^b Q(\tau,r,\mu(r))dr &< f(\tau) + \int_a^b Q(\tau,r,\nu(r))dr \\ T\mu(r) &< T\nu(r) \end{aligned}$$

then $(T\mu, T\nu) \in \Re$. Hence, $\Re$ is $T$-closed.
Now for $(\mu,\nu) \in \Re$ means that $\mu(\tau) < \nu(\tau)$

$$\begin{aligned} d(T\mu,T\nu) = |T\mu - T\nu| &= \left| f(\tau) + \int_a^b Q(\tau,r,\mu(r))dr - f(\tau) - \int_a^b Q(\tau,r,\nu(r))dr \right| \\ &= \left| \int_a^b Q(\tau,r,\mu(r))dr - \int_a^b Q(\tau,r,\nu(r))dr \right| \\ &< \int_a^b \eta(\tau,r)|\mu(r) - \nu(r)|dr \quad \text{from condition (iii)} \\ &< \sup_{\tau\in[a,b]} \left( \int_a^b \eta(\tau,r)dr \right) |\mu(r) - \nu(r)| \\ &< \delta_2|\mu(r) - \nu(r)| \quad for \;\; \delta_2 = \sup_{\tau\in[a,b]} \left( \int_a^b \eta(\tau,r)dr \right) < 1 \\ &< \frac{\delta_1|\nu(r) - T\nu(r)|[1 + |\mu(r) - T\mu(r)|]}{[1 + |\mu(r) - \nu(r)|]} + \delta_2|\mu(r) - \nu(r)| \; for \; \delta_1 = 1 - \delta_2 \\ &< \frac{\delta_1 d(\nu, T\nu)[1 + d(\mu, T\mu)]}{[1 + d(\mu,\nu)]} + \delta_2 d(\mu,\nu). \end{aligned}$$

So, the contractive condition also satisfied.
For $\Re$ to be $d$-self closed, consider $\{\mu_n\}$ a $\Re$-preserving Cauchy sequence converging to $\mu \in C[a,b]$. As $\{\mu_n\}$ is $\Re$ preserving, we have

$$\mu_0(\tau) \leq \mu_1(\tau) \leq \mu_2(\tau) \leq \mu_3(\tau) \leq \ldots\ldots\ldots\ldots \leq \mu_n(\tau) \leq \mu_{n+1}(\tau) \leq \ldots\ldots\ldots \leq \mu(\tau) \quad \tau \in [a,b]$$

then we have $\mu_n \Re\, \mu \quad \forall\, n \in \mathbb{N}$. Therefore, $\Re$ is $d$-self closed.

Now, let us assume that $\kappa(\tau) = max\{\mu(\tau), \nu(\tau)\}$ then $\kappa(\tau) \in C[a,b]$
$\implies \quad \mu(\tau) \leq \kappa(\tau) \;\; and \;\; \nu(\tau) \leq \kappa(\tau)$. So, $\Re^s$- directed.
Hence, we observe that the integral Equation (2) satisfies all the conditions of Theorem 1 and Theorem 2 under the given assumption of Theorem 3, which amounts to saying that the integral Equation (2) has a unique solution. □

Now to show the guarantees, the existence of the function $Q(\tau, r, \mu(r))$ satisfies all the assumptions of the above application.

**Example 2.** *Consider $W = C[0, \frac{\pi}{4}]$ together with the metric*

$$d(\mu, \nu) = \sup_{\tau \in [0, \frac{\pi}{4}]} \{|\mu(\tau) - \nu(\tau)|\}.$$

*Define a binary relation*

$$\Re = \left\{ \mu \Re \nu \;\; if \;\; \mu(\tau) < \nu(\tau) \;\; for\ all \;\; \tau \in \left[0, \frac{\pi}{4}\right] \right\}.$$

*Consider the non-linear integral equation as*

$$\mu(\tau) = 3\tau - 4 \int_0^{\frac{\pi}{4}} (\tau r) sin(\mu(r)) dr \;\; for \;\; \mu \in C\left[0, \frac{\pi}{4}\right],$$

*and*

$$T\mu(\tau) = 3\tau - 4 \int_0^{\frac{\pi}{4}} (\tau r) sin(\mu(r)) dr \;\; for \;\; \mu \in C\left[0, \frac{\pi}{4}\right].$$

**Proof.** Since $f(\tau) = 3\tau$, which is continuous on $\left[0, \frac{\pi}{4}\right]$ and $Q(\tau, r, \mu(r)) = 4(\tau r) sin(\mu(r))$ is integrable w.r.t $r$ on $\left[0, \frac{\pi}{4}\right]$.
Now for every $\tau \in \left[0, \frac{\pi}{4}\right]$ and the sequence $\{\tau_n\} \subset \left[0, \frac{\pi}{4}\right]$ with $\lim_{n \to \infty} \tau_n = \tau$.
Then, for any $\mu \in C\left[0, \frac{\pi}{4}\right]$,

$$\begin{aligned}
\left|T\mu(\tau_n) - T\mu(\tau)\right| &= \left|f(\tau_n) - f(\tau) - 4 \int_0^{\frac{\pi}{4}} r(\tau_n - \tau) sin(\mu(r)) dr\right| \\
&\leq \left|f(\tau_n) - f(\tau)\right| + 4 \int_0^{\frac{\pi}{4}} r\left|\tau_n - \tau\right| \left|sin(\mu(r))\right| dr \\
&\leq \left|3\tau_n - 3\tau\right| + 4\left|\tau_n - \tau\right| \int_0^{\frac{\pi}{4}} r dr \\
&\leq 3\left|\tau_n - \tau\right| + \frac{\pi^2}{8}\left|\tau_n - \tau\right|.
\end{aligned}$$

Taking as a limit $n \to \infty$

$$\left|T\mu(\tau_n) - T\mu(\tau)\right| = 0.$$

which implies that $T\mu(\tau_n) = T\mu(\tau)$. Hence, $T\mu \in C\left[0, \frac{\pi}{4}\right]$ for all $\mu \in C\left[0, \frac{\pi}{4}\right]$.
Now for all $r, \tau \in \left[0, \frac{\pi}{4}\right]$ and $\mu, \nu \in C\left[0, \frac{\pi}{4}\right]$ with $\mu \Re \nu$ we have

$$\begin{aligned}
Q(\tau, r, \nu(r)) - Q(\tau, r, \mu(r)) &\leq 4 \times \tau r \left|sin(\nu(r)) - sin(\nu(r))\right| \\
&\leq \eta(\tau, r)\left|\mu(r) - \nu(r)\right| \quad \text{for} \;\; \eta(\tau, r) = 4\tau r.
\end{aligned}$$

Consequently, $\eta(\tau, r) = 4\tau r$ is a continuous function from $\left[0, \frac{\pi}{4}\right] \times \left[0, \frac{\pi}{4}\right] \to [0, \infty)$ and

$$\begin{aligned}
\sup_{\tau \in [0, \frac{\pi}{4}]} \left( \int_0^{\frac{\pi}{4}} \eta(\tau, r) dr \right) &= \sup_{\tau \in [0, \frac{\pi}{4}]} \int_0^{\frac{\pi}{4}} 4\tau r \, dr \\
&= \sup_{\tau \in [0, \frac{\pi}{4}]} 4\tau \left[ \frac{r^2}{2} \right]_0^{\frac{\pi}{4}} \\
&= \sup_{\tau \in [0, \frac{\pi}{4}]} 4\tau \left( \frac{\pi^2}{32} \right) \\
&= \frac{\pi^2}{8} \sup_{\tau \in [0, \frac{\pi}{4}]} \tau \\
&= \frac{\pi^2}{8} \times \frac{\pi}{4} \\
&= \frac{\pi^3}{32} < 1.
\end{aligned}$$

Now choosing $\mu_0(\tau) = 2\tau$ for $\tau \in [0, \frac{\pi}{4}]$, we have $T\mu_0(\tau) = \mu_0(\tau) = 2\tau$.
Hence $\mu_0(\tau) = 2\tau$ is a fixed point of $T$. □

## 5. Conclusions

In this article, we have established the relation theoretical fixed point results for the rational type contraction. One may observe that, for the uniqueness of the fixed point, the $\Re^s$-directed condition can be replaced by other conditions. Here, we also included some contractions that can be obtained on restriction to the rational contraction. Our results deduce some well known fixed point results if the binary relation is universal. The example we provided is unique in that it will satisfy all the relational elements but fails for many elements outside of the relation. Moreover, we provide an abstract version of an application to a non-linear integral equation. Lastly, we include an example that guarantees the existence of such a non-linear integral equation.

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