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Article

Nabla Fractional Derivative and Fractional Integral on Time Scales

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Abstract: In this paper, we introduce the nabla fractional derivative and fractional integral on time scales in the Riemann–Liouville sense. We also introduce the nabla fractional derivative in Grünwald–Letnikov sense. Some of the basic properties and theorems related to nabla fractional calculus are discussed.

Keywords: fractional calculus; Grünwald–Letnikov derivative; Riemann–Liouville derivative; time scales calculus; nabla derivative; nabla integral

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1. Introduction

Fractional calculus is a very significant branch in Mathematics whose applications are very useful for engineering students and researchers in both pure and applied field. The concept of "fractional calculus" was developed in the early of 17th century when L'Hopital asked Leibnitz the value of $\frac{1}{2}$th order derivative. After that many mathematicians showed their interest on this topic. Initially the theory was developed mainly as a purely theoretical area. However, in the last decade it has been used in various fields such as mechanics, physics, chemistry, control theory, and many more, for instance one can see [1–4].

The analysis of time scales calculus is a fairly new topic for researchers. Stefen Hilger and his Ph.D. supervisor, Bernd Aulbach, initiated the topic in the year 1988. After that, Hilger published two more paper on this topic [5,6]. The theory was highly raised after the publication of two book on time scales by Martin Bohner and Allan Peterson [7,8]. It combines the traditional areas of continuous and discrete analysis into one theory, which has various applications in discrete and continuous hybrid phenomena, quantum calculus and in various problems of economics [9].

The inception of the idea of combining the time scales calculus and fractional calculus occurred in the Ph.D. dissertation of N.R.O. Bastos in 2012, where the delta (Hilger), and nabla derivative on time scales were discussed in fractional calculus using the tool of Laplace transform on some specific real and discrete time scales [10,11]. After the inception of the topic, a number of papers were published see [12–19]. Recently, D. F. M. Torres introduced a generalized definition of Hilger derivative and integrals in a pure sense of Riemann–Liouville (RL) derivative [20,21]. Many research works have been completed in a conformable delta and nabla fractional derivative and integrals [22–24].

2. Motivation of the Article

On the basis of above work, here we are motivated to study the nabla derivative and integral using a Grünwald–Letnikov (GL) fractional derivative approach and then we arrive to the Riemann–Liouville sense. We introduce nabla fractional derivative and integral in unified approach of discrete and continuous time scales. Then, we generalize the definition of nabla fractional derivative and integral in arbitrary time scales and develop certain properties of nabla fractional derivative and fractional integral.

The paper is organized as follows. In Section 3, we review briefly the essentials of time scales, as well as some basic definitions of nabla fractional derivative and integral which helps the readers to recognize easily our main findings. We assume that the readers are familiar with the basic view of time scales calculus and we refer the reader to go through [7,8]. The paper also assess the Riemann–Liouville and Grünwald–Letnikov fractional derivative and integral. Our main findings are given in Section 4 with some preliminaries definition and then we present fractional integral and fractional derivative in an arbitrary time scale $\mathbb{T}$. After that we prove certain important characteristics of fractional derivative and integral. We end with Section 5 of conclusions.

3. Preliminaries and Auxiliary Results

Definition 1 ([7]). *A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$, with the subspace topology inherited from the stranded topology of $\mathbb{R}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$ for $t \in \mathbb{T}$ and forward jump operator $\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$. If $\rho(t) < t$ then t is said to be a left scattered and if $\rho(t) = t$, then we say t is a left dense point of $\mathbb{T}$, if $\sigma(t) > t$ and $\sigma(t) = t$, then we say t is right scattered and right dense, respectively. Again, if $\mathbb{T}$ has a right scattered minimum a, then let $\mathbb{T}^\kappa = \mathbb{T} - \{a\}$, or else set $\mathbb{T}^\kappa = \mathbb{T}$. Here we consider the backward graininess $\nu : \mathbb{T}^\kappa \to [0, \infty]$, which is defined by $\nu(t) = t - \rho(t)$.*

Definition 2 ([8]). *A function $h : \mathbb{T} \to \mathbb{R}$ is said to be a nabla differentiable at $t \in \mathbb{T}$, if for any $\varepsilon > 0$ there exists a neighborhood V of t, such that*

$$|h(\rho(t) - h(u) - h_\nabla(t)(\rho(t) - u)| \leq \varepsilon|\rho(t) - u|$$

for all $u \in V$. If $h_\nabla(t)$ exists for all $t \in \mathbb{T}^\kappa$ then it is called nabla derivative of h.

Theorem 1 ([8]). *Let us consider a function $h : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. Then we have*

(i) If h is continuous at a left-scattered t, then h is nabla differentiable at t with

$$h_\nabla(t) = \frac{h(t) - h(\rho(t))}{\nu(t)};$$

(ii) If t is left dense, then h is nabla differentiable at t if and only if the limit

$$\lim_{u \to t} \frac{h(t) - h(u)}{t - u}$$

exists as a finite number. In this case

$$h_\nabla(t) = \lim_{u \to t} \frac{h(t) - h(u)}{t - u}.$$

Definition 3 ([8,11]). *(Higher order nabla derivative): Assume a function $h : \mathbb{T} \to \mathbb{R}$, we first define the second order derivative $h_{\nabla\nabla}$ provided h_∇ is differentiable on $\mathbb{T}^{K^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $h_{\nabla\nabla} = (h_\nabla)_\nabla : \mathbb{T}^{K^2} \to \mathbb{R}$. Similarly, proceeding up to n^{th} order, here we obtain $h_{\nabla^n} : \mathbb{T}^{K^n} \to \mathbb{R}$, where $\mathbb{T}^{K^n}$ is a time scales which is obtained by removing n right scattered left end point.*

Definition 4 ([3])**.** *The Riemann–Liouville fractional differentiation of random order α is defined in the following manner:*

$$^{RL}D_x^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} h(t)dt$$

for $Re(\alpha) > 0$.

Riemann–Liouville derivative of order $\alpha \in \mathbb{R}$ is given by

$$^{RL}D_x^{\alpha} = D^n \left\{ ^{RL}D_x^{\alpha-n} h(t) \right\}$$

for $\alpha < n < \alpha + 1$.

Definition 5 ([2])**.** *Let $\alpha > 0$. The Grünwald–Letnikov derivative of fractional order α of a function h is defined by*

$$^{GL}D^{\alpha} h(t) = \frac{1}{h^{\alpha}} \sum_{i=0}^{\infty} \binom{\alpha}{i} h(t-ih), t \in \mathbb{R}.$$

Referring Definition 5 as the Grünwald–Letnikov fractional derivative is quite common in literature (see [25]). Moreover, once a starting point t_0 has been assigned, for practical reason then the following (see [26]) is often preferred, since it can be applied to function not defined (or simply not known) in $(-\infty, t_0)$.

Definition 6 ([22])**.** *A function $h : \mathbb{T} \to \mathbb{R}$ is called ld-continuous if h is continuous at left dense point in $\mathbb{T}$ and right sided limit exists (finite) at right dense point in $\mathbb{T}$.*

Definition 7 ([27])**.** *(Change of order of integration) If we have any function $h(x,y,z)$ which is integrable with respect to y and z, then the change of order of integration is given by the following formula:*

$$\int_{x_0}^{x} \int_{x_0}^{y} h(x,y,z)dzdy = \int_{x_0}^{x} \int_{z}^{x} h(x,y,z)dydz.$$

Definition 8 ([11])**.** *Let $B \subset \mathbb{R}$. A subset $J \subset \mathbb{T}$ is called a time scales interval if $J = B \cap \mathbb{T}$. A function $h : J \to \mathbb{T}$ is said to be left-dense absolutely continuous, if for all $\varepsilon > 0$ there exists $\delta > 0$, such that $\sum_{k=1}^{m} |h(y_k) - h(x_k)| < \varepsilon$, whenever a disjoint finite collection of sub-time scales intervals $(x_k, y_k) \cap \mathbb{T} \subset J$ for $1 \leq k \leq n$ satisfies $\sum_{k=1}^{m} |y_k - x_k| < \delta$. One denotes $h \in AC_{\nabla}$, if $h_{\nabla}^{(n-1)} \in AC$ then one denotes $h \in AC_{\nabla}^{(n)}$.*

4. Nabla Fractional Derivative and Nabla Fractional Integral

Definition 9. *For any time scale $\mathbb{T}$, a function $h : \mathbb{T} \to \mathbb{R}$ is said to be nabla fractional differentiable of order μ at $t \in \mathbb{T}^{\kappa}$, where $0 < \mu \leq 1$, if for any $\varepsilon > 0$, there exists a neighborhood V of t, such that*

$$|h(\rho(t)) - h(u) - h_{\nabla}^{(\mu)}(t)(\rho(t) - u)^{\mu})| \leq \varepsilon |(\rho(t) - u)^{\mu}| \quad (1)$$

for all $u \in V$. If for all $t \in \mathbb{T}^{\kappa}$, h holds the Equation (1), then we call $h_{\nabla}^{(\mu)}(t)$ the nabla fractional derivative of order μ.

Theorem 2. *Nabla fractional derivative is not well defined in $\mathbb{T}$, but in $\mathbb{T}^{\kappa}$.*

Proof. Let $h_{\nabla}^{(\mu)}(t)$ be defined at a point t on a time scale $\mathbb{T}$, and assume that $t \notin \mathbb{T}^{\kappa}$. Then, $t \in \mathbb{T} \setminus \mathbb{T}^{\kappa}$. From Definition 1, t must be unique which is equal to a, later, for any $\varepsilon > 0$ there exists a neighborhood $V = \{t\}$ of t, we obtain for $u \in V$

$$h(\rho(t)) = h(u) = h(\rho(a)) = h(a).$$

Thus for $\zeta \in \mathbb{R}$ and $\mu \in (0,1]$ we have

$$|h(\rho(t)) - h(u) - \zeta[(\rho(t) - u)^{\mu}]| = |h(a) - h(a) - \zeta(a-a)^{\mu}| \leq \varepsilon|(\rho(t) - u)^{\mu}|. \quad (2)$$

Here, Equation (2) is true for each $\zeta \in \mathbb{R}$, which means for each ζ is the nabla derivative of h of order μ if $t \notin \mathbb{T}^{\kappa}$, which cannot be true, so $h_{\nabla}^{(\mu)}$ is well defined only on $\mathbb{T}^{\kappa}$. □

Theorem 3. *For any time scale $\mathbb{T}$, let $h : \mathbb{T}^{\kappa} \to \mathbb{R}$. Then, for $\mu \in (0,1]$ we have the following:*

(i) If t is left dense and h is nabla differentiable of order μ at t, then h is continuous at t;

(ii) If h is continuous at t and t is left scattered, then h is nabla differentiable at t of order μ with

$$h_{\nabla}^{(\mu)}(t) = \frac{h(t) - h(\rho(t))}{\nu(t)^{\mu}};$$

(iii) If t is left dense, then h is differentiable at t if and only if the limit

$$\lim_{u \to t} \frac{h(t) - h(u)}{(t-u)^{\mu}}$$

exists as a finite number. In this case

$$h_{\nabla}^{(\mu)}(t) = \lim_{u \to t} \frac{h(t) - h(u)}{(t-u)^{\mu}};$$

(iv) If h is nabla differentiable of order μ at t, then

$$h(\rho(t)) = h(t) - h_{\nabla}^{(\mu)}(t)(\nu(t))^{\mu}.$$

Proof. (i) Given that h is nabla fractional differentiable at t, then for $\varepsilon > 0$ there exists a neighborhood V of t, such that

$$|h(\rho(t)) - h(u) - h_{\nabla}^{(\mu)}(t)(\rho(t) - u)^{\mu}| \leq \varepsilon|(\rho(t) - u)^{\mu}|$$

for $u \in V$. Therefore,

$$\begin{aligned} |h(t) - h(u)| \leq & |h(\rho(t)) - h(u) - h_{\nabla}^{(\mu)}(t)(\rho(t) - u)^{\mu}| + |h(\rho(t)) - h(t) - h_{\nabla}^{(\mu)}(t)(\rho(t) - t)^{\mu}| \\ & + |h_{\nabla}^{(\mu)}(t)||(\rho(t) - u)^{\mu} + (\rho(t) - t)^{\mu}| \end{aligned}$$

for all $u \in V \cap (t - \varepsilon, t + \varepsilon)$ and since t is a left dense point, so

$$\begin{aligned} |h(t) - h(u)| &\leq |h(\rho(t)) - h(u) - h_{\nabla}^{(\mu)}(t)(\rho(t) - u)^{\mu}| + |h_{\nabla}^{(\mu)}(t)||(t-u)^{\mu}| \\ &\leq \varepsilon|t-u|^{\mu} + |h_{\nabla}^{(\mu)}(t)||t-u|^{\mu} \\ &\leq \varepsilon^{\mu}\left[\varepsilon + |h_{\nabla}^{(\mu)}(t)|\right]. \end{aligned}$$

It follows the continuity of h at t.

(ii) Given that h is continuous and t is left scattered, by continuity

$$\lim_{u \to t} \frac{h(\rho(t)) - h(u)}{(\rho(t) - u)^{\mu}} = \frac{h(\rho(t)) - h(t)}{(\rho(t) - t)^{\mu}} = \frac{h(t) - h(\rho(t))}{(t - \rho(t))^{\mu}}.$$

Hence, there exists a neighborhood V of t, such that

$$\left|\frac{h(\rho(t)) - h(u)}{(\rho(t) - u)^{\mu}} - \frac{h(t) - h(\rho(t))}{(t - \rho(t))^{\mu}}\right| \leq \varepsilon.$$

For all $u \in V$, it follows that

$$\left|h(\rho(t)) - h(u) - \frac{h(tt - h(\rho(t))}{(t - \rho(t))^{\mu}}(\rho(t) - u)^{\mu}\right| \leq \varepsilon|(\rho(t) - u)^{\mu}|.$$

From Definition 9, we obtain our result:

$$h_{\nabla}^{(\mu)}(t) = \frac{h(t) - h(\rho(t))}{\nu(t)^{\mu}}.$$

(iii) Given that t is left dense, then we obtain $\rho(t) = t$, so there exists a neighborhood V of t, such that

$$|h(t) - h(u) - h_{\nabla}^{(\mu)}(t)(t - u)^{\mu}| \leq \varepsilon|(t - u)^{\mu}|$$

for all $u \in V$. It follows that

$$\left|\frac{h(t) - h(u)}{(t - u)^{\mu}} - h_{\nabla}^{(\mu)}(t)\right| \leq \varepsilon.$$

So, we obtain

$$h_{\nabla}^{(\mu)}(t) = \lim_{u \to t} \frac{h(t) - h(u)}{(t - u)^{\mu}}.$$

Now, assume that $\lim\limits_{u \to t} \frac{h(t)-h(u)}{(t-u)^{\mu}}$ exists as a finite number, say $\mathcal{L}$, and t is left dense. Then, for any $\varepsilon > 0$,

$$\left|\frac{h(t) - h(u)}{(t - u)^{\mu}} - \mathcal{L}\right| = \left|\frac{h(\rho(t) - h(u)}{(\rho(t) - u)^{\mu}} - \mathcal{L}\right| \leq \varepsilon.$$

Therefore,

$$|h(\rho(t)) - h(u) - \mathcal{L}(\rho(t) - u)^{\alpha}| \leq \varepsilon|(\rho(t) - u)^{\mu}|$$

from which we conclude that h is fractional differentiable of order μ at t and $h_{\nabla}^{(\mu)}(t) = \mathcal{L}$.
(iv) For all $t \in \mathbb{T}^{\kappa}$, there exist two possibilities of t.
Case 1: If t is left dense, then $\rho(t) = t$, and we have $\nu(t)^{\mu} = 0$,

$$h(\rho(t)) = h(t) - h_{\nabla}^{(\mu)}(t)(\nu(t))^{\mu},$$

so left hand side $\Longrightarrow$ right hand side.
Case 2: If t is left scattered, then $\rho(t) < t$, and by using Theorem 3 (ii), we obtain

$$h(\rho(t)) = h(t) - h_{\nabla}^{(\mu)}(t)(\nu(t))^{\mu}.$$

The proof is complete. □

Proposition 1. *Let $h : \mathbb{T}^{\kappa} \to \mathbb{R}$. If $h(t) = k$ for all $t \in \mathbb{T}^{\kappa}$, and $k \in \mathbb{R}$, then $h_{\nabla}^{(\mu)}(t) = 0$ for all $\mu \in \mathbb{R}$.*

Proof. From the results (ii) and (iii) of Theorem 3, we have: if t is left-scattered, then

$$h_{\nabla}^{(\mu)}(t) = \frac{h(t) - h(\rho(t))}{(\nu(t))^{\mu}} = \frac{(k - k)}{(\nu(t))^{\mu}} = 0,$$

if t is left dense, then

$$h_{\nabla}^{(\mu)}(t) = \lim_{u \to t} \frac{h(t) - h(u)}{(t-u)^{\mu}} = \lim_{u \to t} \frac{(k-k)}{(t-u)^{\mu}} = 0,$$

which completes the proof. □

Proposition 2. *For $h : \mathbb{T}^{\kappa} \to \mathbb{R}$, if $h(t) = t$, then for all $t \in \mathbb{T}^{\kappa}$*

$$h_{\nabla}^{(\mu)}(t) = \begin{cases} \nu(t)^{1-\mu} & \text{if } \mu \neq 1, \\ 1 & \text{if } \mu = 1. \end{cases}$$

Proof. Let $h_{\nabla}^{(\mu)}$ exist at $t \in \mathbb{T}^{\kappa}$. Then, from Theorem 3 (iv), we have

$$h(\rho(t)) = h(t) - h_{\nabla}^{(\mu)}(t)(\nu(t))^{\mu},$$

that is, $\nu(t) = (\nu(t))^{\mu} h_{\nabla}^{(\mu)}(t)$. If $\nu(t) \neq 0$, then we obtain our desired result which is $h_{\nabla}^{(\mu)}(t) = (\nu(t))^{1-\mu}$ for $\mu \neq 1$. If $\mu = 1$ then we obtain $h_{\nabla}^{(\mu)}(t) = (\nu(t))^{1-1} = (\nu(t))^{0} = 1$, hence the proof is complete. □

Theorem 4. *Let $h, g : \mathbb{T} \to \mathbb{R}$ be two nabla differentiable functions of order $\mu \in (0,1]$ at $t \in \mathbb{T}^{\kappa}$. Then, the following holds:*

(i) The sum $(\lambda_1 h + \lambda_2 g) : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t of order μ,

$$(\lambda_1 h + \lambda_2 g)_{\nabla}^{(\mu)}(t) = \lambda_1 h_{\nabla}^{(\mu)}(t) + \lambda_2 g_{\nabla}^{(\mu)}(t),$$

where λ_1 and λ_2 are any two arbitrary constants;

(ii) The product $hg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable of order μ at t,

$$(hg)_{\nabla}^{(\mu)} = h_{\nabla}^{(\mu)}(t) g(t) + h(\rho(t)) g_{\nabla}^{(\mu)}(t) = h(t) g_{\nabla}^{(\mu)}(t) + h_{\nabla}^{(\mu)}(t) g(\rho(t));$$

(iii) If $h(t)h(\rho(t)) \neq 0$ then $\frac{1}{h}$ is nabla differentiable at t of order μ,

$$\left(\frac{1}{h}\right)_{\nabla}^{(\mu)}(t) = -\frac{h_{\nabla}^{(\mu)}(t)}{h(t)h(\rho(t))};$$

(iv) If $g(t)g(\rho(t)) \neq 0$, then $\frac{h}{g}$ is nabla differentiable at t of order μ,

$$\left(\frac{h}{g}\right)_{\nabla}^{(\mu)}(t) = \frac{h_{\nabla}^{(\mu)}(t) g(t) - h(t) g_{\nabla}^{(\mu)}(t)}{g(t)g(\rho(t))}.$$

Proof. (i) Let $\mu \in (0,1]$. Given that h and g are nabla differentiable at $t \in \mathbb{T}^{\kappa}$ of order μ, for any $\varepsilon > 0$ there exist neighborhoods V_1 and V_2 of t, thus for all $u \in V_1$

$$|\lambda_1 h(\rho(t)) - \lambda_1 h(u) - \lambda_1 h_{\nabla}^{(\mu)}(t)((\rho(t) - u)^{\mu}| \leq \frac{\varepsilon}{2} |(\rho(t) - u)^{\mu}|, \tag{3}$$

also for all $u \in V_2$

$$|\lambda_2 g(\rho(t)) - \lambda_2 g(u) - \lambda_2 g_{\nabla}^{(\mu)}(t)((\rho(t) - u)^{\mu}| \leq \frac{\varepsilon}{2} |(\rho(t) - u)^{\mu}|. \tag{4}$$

Let $u \in V = V_1 \cap V_2$. Then, we obtain

$$\begin{aligned} &|(\lambda_1 h + \lambda_2 g)(\rho(t)) - (\lambda_1 h + \lambda_2 g)(u) - [\lambda_1 h_{\nabla}^{(\mu)}(t) + \lambda_2 g_{\nabla}^{(\mu)}(t)]((\rho(t) - u)^{\mu}| \\ &\leq |[\lambda_1 h(\rho(t)) - \lambda_1 h(u) - \lambda_1 h_{\nabla}^{(\mu)}(t)((\rho(t) - u)^{\mu}]| \\ &+ |[\lambda_2 g(\rho(t)) - \lambda_2 g(u) - \lambda_2 g_{\nabla}^{(\mu)}(t)((\rho(t) - u)^{\mu}]|. \end{aligned} \tag{5}$$

By using the Equations (3) and (4), we obtain that

$$\begin{aligned} &|(\lambda_1 h + \lambda_2 g)(\rho(t)) - (\lambda_1 h + \lambda_2 g)(u) - [\lambda_1 h_{\nabla}^{(\mu)}(t) + \lambda_2 g_{\nabla}^{(\mu)}(t)]((\rho(t) - u)^{\mu}| \\ &< \frac{\varepsilon}{2}|(\rho(t) - u)^{\mu}| + \frac{\varepsilon}{2}|(\rho(t) - u)^{\mu}| \\ &= \varepsilon|(\rho(t) - u)^{\mu}|. \end{aligned}$$

From Theorem 3, it holds that $(\lambda_1 h + \lambda_2 g)_{\nabla}^{(\mu)}$ is a nabla differentiable at $t \in \mathbb{T}^{\kappa}$ of order μ.
(ii) If t is left dense, i.e., $\rho(t) = t$ for $t \in \mathbb{T}^{\kappa}$, then

$$\begin{aligned} (hg)_{\nabla}^{(\mu)}(t) &= \lim_{u \to t} \frac{(hg)(t) - (hg)(t)}{(t - u)^{(\mu)}} \\ &= \lim_{u \to t} \frac{h(t) - h(u)}{(t - u)^{\mu}} g(t) + \lim_{u \to t} \frac{g(t) - g(u)}{(t - u)^{\mu}} h(u) \\ &= h_{\nabla}^{(\mu)}(t) g(t) + g_{\nabla}^{(\mu)}(t) h(t) \\ &= h_{\nabla}^{(\mu)}(t) g(t) + g_{\nabla}^{(\mu)}(t) h(\rho(t)). \end{aligned}$$

Additionally, if $\rho(t) < t$, then

$$\begin{aligned} (hg)_{\nabla}^{(\mu)}(t) &= \frac{(hg)(t) - (hg)(\rho(t))}{(\nu(t))^{\mu}} \\ &= \frac{h(t) - h(\rho(t))}{(\nu(t))^{\mu}} g(t) + \frac{g(t) - g(\rho(t))}{(\nu(t))^{\mu}} f(\rho(t)) \\ &= h_{\nabla}^{(\mu)}(t) g(t) + h(\rho(t)) g_{\nabla}^{(\mu)}(t). \end{aligned}$$

Other part of the proof is very similar to this.
(iii) Using the above result and Proposition 1, we obtain

$$\left(h \cdot \frac{1}{h}\right)_{\nabla}^{(\mu)}(t) = (1)_{\nabla}^{(\mu)}(t) = 0$$

and hence, by (ii),

$$\left(\frac{1}{h}\right)_{\nabla}^{(\mu)}(t) h(\rho(t)) + h_{\nabla}^{(\mu)}(t) \frac{1}{h(t)} = 0.$$

Since $h(t)h(\rho(t)) \neq 0$, so we obtain

$$\left(\frac{1}{h}\right)_{\nabla}^{(\mu)}(t) = -\frac{h_{\nabla}^{(\mu)}(t)}{h(t)h(\rho(t))}.$$

(iv) Using the result of Theorem 4 (ii) and (iii) we obtain the following:

$$\begin{aligned}\left(\frac{h}{g}\right)^{(\mu)}_{\nabla}(t) &= \left(h\cdot\frac{1}{g}\right)^{(\mu)}_{\nabla}(t)\\ &= h(t)\left(\frac{1}{g}\right)^{(\mu)}_{\nabla}(t) + h^{(\mu)}_{\nabla}(t)\frac{1}{g(\rho(t))}\\ &= -h(t)\frac{g^{(\mu)}_{\nabla}(t)}{g(t)g(\rho(t))} + h^{(\mu)}_{\nabla}(t)\frac{1}{g(\rho(t))}\\ &= \frac{h^{(\mu)}_{\nabla}(t)g(t) - h(v)g^{(\mu)}_{\nabla}(t)}{g(t)g(\rho(v))}.\end{aligned}$$

This completes the proof. □

Theorem 5. *Let k be a constant, $n \in \mathbb{N}$. Then, for $0 < \mu \le 1$, we obtain the following:*

(i) *If $h(t) = (t-k)^n$, then*

$$h^{\mu}_{\nabla}(t) = (\nu(t))^{1-\mu}\sum_{j=0}^{n-1}(\rho(t)-k)^j(t-k)^{n-1-j};$$

(ii) *If $g(t) = \frac{1}{(t-k)^n}$, then*

$$g^{(\mu)}_{\nabla}(t) = -(\nu(t))^{1-\mu}\sum_{j=0}^{n-1}\frac{1}{(\rho(t)-k)^{n-j}(t-k)^{j+1}}$$

provided $(t-k)(\rho(t)-k) \neq 0$.

Proof. (i) Here we prove this result by using the method of induction. If $n = 1$, then $h(t) = t-k$ hence $h^{(\mu)}_{\nabla}(t) = (\nu(t))^{1-\mu}$ is true from Propositions 4 and 2. We assume that

$$h^{\mu}_{\nabla}(t) = (\nu(v))^{1-\mu}\sum_{j=0}^{n-1}(\rho(t)-k)^j(t-k)^{n-1-j}$$

holds for $h(t) = (t-k)^n$. We shall prove the result is true for

$$H(t) = (t-k)^{n+1} = (t-k)h(t).$$

By using Theorem 4 (ii), we obtain

$$\begin{aligned}h^{(\mu)}_{\nabla}(t) &= (t-k)^{(\mu)}_{\nabla}h(\rho(t)) + h^{(\mu)}_{\nabla}(t)(t-k) = (\nu(t))^{1-\mu}h(\rho(t)) + h^{(\mu)}_{\nabla}(t)(t-k)\\ &= (\nu(t))^{1-\mu}(\rho(t)-k)^n + (\nu(t))^{1-\mu}(t-k)\sum_{j=0}^{n-1}(\rho(t)-k)^j(t-k)^{n-1-j}\\ &= (\nu(t))^{1-\mu}\left[(\rho(t)-k)^n + \sum_{j=0}^{n-1}(\rho(t)-k)^j(t-k)^{n-j}\right]\\ &= (\nu(t))^{1-\mu}\sum_{j=0}^{n}(\rho(t)-k)^j(t-k)^{n-j}.\end{aligned}$$

(ii) Let $g(t) = \frac{1}{(t-k)^n} = \frac{1}{h(t)}$. Using Theorem 4 (iii), we obtain

$$\begin{aligned} g_{\nabla}^{(\mu)}(t) &= \left(\frac{1}{h(t)}\right)_{\nabla}^{(\mu)}(t) \\ &= -\frac{h_{\nabla}^{(\mu)}(t)}{h(t)h(\rho(t))} \\ &= -(\nu(t))^{1-\mu} \frac{\sum_{j=0}^{n-1}(\rho(t)-k)^j (t-k)^{n-1-j}}{(t-k)^n(\rho(t)-k)^n} \\ &= -(\nu(t))^{1-\mu} \sum_{j=0}^{n-1} \frac{1}{(t-k)^{j+1}(\rho(t)-k)^{n-j}}, \end{aligned}$$

provided $(t-k)(\rho(t)-k) \neq 0$. □

Example 1. *Let $\mu \in (0,1]$.*

(i) *If $g(t) = t^2$, then from Theorem 5, we obtain*

$$\begin{aligned} g_{\nabla}^{(\mu)}(t) &= (\nu(t))^{1-\mu}\left[\sum_{j=0}^{1}(\rho(t))^j (t)^{1-j}\right] \\ &= (\nu(t))^{1-\mu}\left[(\rho(t))^0 (t)^1 + (\rho(t))^1 (t)^0\right] \\ &= (\nu(t))^{1-\mu}(t+\rho(t)). \end{aligned}$$

By using Theorem 5, we obtain the following results:

(ii) *If $g(t) = t^3$, then $g_{\nabla}^{(\mu)}(t) = (\nu(t))^{1-\mu}[t^2 + t\rho(t) + (\rho(t))^2]$.*

(iii) *If $g(t) = \frac{1}{t}$, then $g_{\nabla}^{(\mu)}(t) = -\frac{(\nu(t))^{1-\mu}}{t(\rho(t))}$.*

(iv) *If $g(t) = \frac{1}{t^2}$, then $g_{\nabla}^{(\mu)}(t) = -\frac{\nu(t)^{1-\mu}}{t(\rho(t))^2 + t^2\rho(t)}$.*

Corollary 1. *Nabla fractional derivative in some specific time scales $\mathbb{T}$.*

(i) *If we consider the real time scale $\mathbb{T} = \mathbb{R}$, then all the elements of $\mathbb{T}$ are dense. So, by using Theorem 3 (iii), we have that*

$$h_{\nabla}^{(\mu)}(t) = \lim_{u \to t} \frac{h(t) - h(u)}{(t-u)^{\mu}}$$

exists, if $\mu = 1$, then we have $h_{\nabla}^{(\mu)} = h'(t)$, which is similar to the ordinary derivative.

(ii) *If $\mathbb{T} = \mathbb{Z}$, for $t \in \mathbb{T}$ one has $\rho(t) = t-1$ and then $\nu(t) = t-(t-1) = 1$. Now, by using Theorem 3 (ii), we obtain*

$$\begin{aligned} h_{\nabla}^{(\mu)}(t) &= \frac{h(t) - h(\rho(t))}{(t-\rho(t))^{\mu}} \\ &= \frac{h(t) - h(t-1)}{(t-(t-1))^{\mu}}. \end{aligned}$$

If $\alpha = 1$ we have that $h_{\nabla}^{(\mu)}(t) = h(t) - h(t-1) = \nabla h(t)$, which is similar as the usual backward operator;

(iii) *Let $\mathbb{T} = h\mathbb{Z}$, where $h > 0$. Then we obtain*

$$\rho(t) = \sup\{u \in \mathbb{T} : u < t\} = \sup\{t - nh : \textit{for } n \in \mathbb{N}\} = t - h$$

and then the function $\nu(t) = t - \rho(t) = t - (t-h) = h$*, which is constant.*
For $g : h\mathbb{Z} \to \mathbb{R}$*, we have from Theorem 1 that*

$$g_{\nabla}(t) = g_{\nabla}^{(1)}(t) = \frac{g(t - h(\rho(t))}{\rho(t) - t} = \frac{g(t-h) - g(t)}{(t-h-t)} = \frac{g(t) - g(t-h)}{h}.$$

From Definition 3, the second order nabla derivative is

$$g_{\nabla\nabla}(t) = g_{\nabla}^{(2)}(t) = \frac{g_{\nabla}(t) - g_{\nabla}(t-h)}{h} = \frac{g(t) - 2g(t-h) + g(t-2h)}{h^2}.$$

In general, the m^{th} *derivative for* $t \in h\mathbb{Z}$ *and* $m \in \mathbb{N}$*,*

$$g_{\nabla}^{(m)}(t) = \frac{1}{h^m}\sum_{r=0}^{m}(-1)^r\binom{m}{j}g(t-rh),$$

where the binomial coefficient $\binom{m}{r}$ *is defined as follows:*

$$\begin{aligned}\binom{m}{r} &= \frac{m(m-1)(m-2)\cdots(m-r+1)}{r!} \\ &= \begin{cases} \frac{m!}{r!(m-r)!}, & r = 0,1,2,\cdots \\ 0, & r > m. \end{cases}\end{aligned} \tag{6}$$

Since the binomial coefficient vanish when $r > m$*, so no contribution in the summation is given from the presence of terms with* $r > m$*, the upper limit of the formula can be raised to any value greater than m and hence, the finite summation in this formula can be replaced with the infinite series, i.e.,*

$$g_{\nabla}^{(m)}(t) = \frac{1}{h^m}\sum_{r=0}^{\infty}(-1)^r\binom{m}{j}g(t-rh).$$

Letting h tend to zero, then all points of the time scale become dense, and the time scale becomes the continuous time scale. If the value of m is replaced by an arbitrary real number $\mu \in \mathbb{R}$*,* $\mu > 0$*, and changing the factorial function with a Euler gamma function using the recurrence relation* $(n-1)! = \Gamma(n)$*, then without losing the generality, if we replace m by any arbitrary real number* $\mu \in \mathbb{R}$*, then the nabla fractional derivative, from Definition 3 and Theorem 3, is*

$${}^{\mathbb{T}}D_t^{\mu} = g_{\nabla}^{(\mu)}(t) = \lim_{h\to 0}\frac{1}{h^{\mu}}\sum_{r=0}^{\infty}(-1)^r\binom{\mu}{r}g(t-rh). \tag{7}$$

Moreover, once a starting point a assign as $nh = t - a$ *for* $t > a$*, such that*

$${}_a^{\mathbb{T}}D_t^{\mu}g(t) = \lim_{h\to 0}\frac{1}{h^{\mu}}\sum_{r=0}^{n}(-1)^r\binom{\mu}{r}g(t-rh). \tag{8}$$

Since for any continuous function $g(t)$ *Grünwald–Letnikov derivative and Riemann–Liouville derivative coincide with positive non integer order derivative, so we have*

$${}_a^{\mathbb{T}}D_t^{\mu}g(t) = {}_a^{RL}D_t^{\mu}g(t),$$

where ${}_a^{RL}D_t^{\mu}g(t)$ *denote the Riemann–Liouville fractional derivative defined on time scales, which is most useful in the study of fractional calculus.*
If $\mu < 0$*, then we have*

$$\begin{bmatrix}\mu \\ r\end{bmatrix} = \frac{(\mu)(\mu+1)(\mu+2)(\mu+3)\cdots(\mu+r-1)}{r!}, \tag{9}$$

i.e., when $\mu = -\mu$, then from Equations (6) and (9) we obtain

$$\binom{-\mu}{r} = \frac{(-\mu-1)(-\mu-2)(-\mu-3)\cdots(-\mu-r+1)}{r!} = (-1)^r \begin{bmatrix}\mu\\ r\end{bmatrix}$$

or we can write $(-1)^r\binom{\mu}{r} = \begin{bmatrix}\mu\\ r\end{bmatrix}$. For any ld-continuous function and for $\mu = -\mu$, then from Equation (8) we obtain

$$\begin{aligned} {}_a^{\mathbb{T}}D_t^{-\mu} g(t) &= \lim_{h\to 0} h^{\mu} \sum_{r=0}^{n} \begin{bmatrix}\mu\\ r\end{bmatrix} g(t-rh) \\ &= \lim_{h\to 0} \sum_{r=0}^{n} \frac{1}{r^{\mu-1}} \begin{bmatrix}\mu\\ r\end{bmatrix} h(rh)^{\mu-1} g(t-rh) \\ &= \frac{1}{\Gamma(\mu)} \lim_{h\to 0} \sum_{h\to 0}^{n} \frac{\Gamma(\mu)}{r^{\mu-1}} \begin{bmatrix}\mu\\ r\end{bmatrix} h(rh)^{\mu-1} g(t-rh) \\ &= \frac{1}{\Gamma(\mu)} \lim_{n\to\infty} \sum_{r=0}^{n} \frac{\Gamma(\mu)}{r^{\mu-1}} \begin{bmatrix}\mu\\ r\end{bmatrix} \left(\frac{t-a}{n}\right)\left(r\frac{t-a}{n}\right)^{\mu-1} g\left(t - r\frac{t-a}{n}\right). \end{aligned} \tag{10}$$

Let us take

$$\beta_r = \frac{\Gamma(\mu)}{r^{\mu-1}} \begin{bmatrix}\mu\\ r\end{bmatrix}, \eta_{n,r} = \left(\frac{t-a}{n}\right)\left(r\frac{t-a}{n}\right)^{\mu-1} g\left(t - r\frac{t-a}{n}\right).$$

Now,

$$\lim_{r\to\infty} \beta_r = \lim_{r\to\infty} \frac{\Gamma(\mu)}{r^{\mu-1}} \begin{bmatrix}\mu\\ r\end{bmatrix} = 1. \tag{11}$$

$$\begin{aligned} \lim_{n\to\infty} \sum_{r=0}^{n} \eta_{n,r} &= \lim_{n\to\infty} \sum_{r=0}^{n} \left(\frac{t-a}{n}\right)\left(r\frac{t-a}{n}\right)^{\mu-1} g\left(t - r\frac{t-a}{n}\right) \\ &= \lim_{n\to\infty} \sum_{r=0}^{n} h(rh)^{\mu-1} g(t-rh) \\ &= \int_a^t (t-s)^{\mu-1} g(s) \nabla s. \end{aligned} \tag{12}$$

Here, we obtain a condition (see [28]) that, if

$$\lim_{r\to\infty} \beta_r = A \text{ and } \lim_{n\to\infty} \sum_{r=0}^{n} \eta_{n,r} = B, \text{ then } \lim_{n\to\infty} \sum_{r=0}^{n} \eta_{n,r}\beta_r = AB. \tag{13}$$

By using the Equations (10)–(13), we obtain

$${}_a^{\mathbb{T}}D_t^{-\mu} g(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1} g(s) \nabla s, \tag{14}$$

which represents the nabla integral of any arbitrary order μ in a Riemann–Liouville sense.

Remark 1. *The definition of nabla fractional integral defined in Equation (14) is not the natural one for arbitrary time scales $\mathbb{T}$. For showing this we take an example. If $g(t) = t^2$, then from Example 1, $g_{\nabla}^{\mu}(t) = t + \rho(t)$, for $\mu = 1$. If the time scale is the continuous time scale $\mathbb{T} = \mathbb{R}$, then $\rho(t) = t$ and, hence, from Corollary 1, we find that $g_{\nabla}^{(\mu)}(t) = g'(v) = 2t$. But if we take the discrete time scale, we obtain $\rho(t) = t-1$, and the nabla derivative on $\mathbb{T} = \mathbb{Z}$ means the backward difference of t^2, i.e., $\nabla(t^2) = g(t) - g(t-1) = 2t-1$. Again, since every ld-continuous function is nabla integrable, so in this case we can claim that $\int_0^t g_{\nabla}^{(\mu)} = t^2$. In $\mathbb{T} = \mathbb{R}$ also means* *that* $\int_a^b f(s)\nabla s = \int_a^b f(s)ds$, *so* $\int_0^t (s+\rho(s))\nabla s = \int_0^t (2s)ds = t^2$, *but in* $\mathbb{T} = \mathbb{Z}$ *we obtain* $\int_a^b f(s)\nabla s = \sum_a^b f(s)$ *for* $a < b$, *so* $\int_0^t (s+\rho(s))\nabla s = \sum_0^t (2s-1) = t^2$ *for* $t > 0$. *For the appearance of* $\rho(s)$, *we claim that the correct definition of nabla fractional integral on an arbitrary time scale* $\mathbb{T}$ *must be*

$$ {}_a^{\mathbb{T}}\mathbb{I}_t^{\mu} g(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-\rho(s))^{\mu-1} g(s)\nabla s, \tag{15}$$

which is the generalization of the nabla fractional integral defined on the Equation (14), in a Riemann–Liouville sense.

Proposition 3. *The nabla fractional integral for any function g defined on* $[a,b]$ *satisfies*

$$ {}_a^{\mathbb{T}}\mathbb{I}_t^{\mu} \circ {}_a^{\mathbb{T}}\mathbb{I}_t^{\beta} = {}_a^{\mathbb{T}}\mathbb{I}_t^{\mu+\beta}$$

for $\mu > 0, \beta > 0$.

Proof. By using the generalized definition of nabla derivative of fractional order from the Equation (15), we have

$$\begin{aligned}
\left({}_a^{\mathbb{T}}\mathbb{I}_t^{\mu} \circ {}_a^{\mathbb{T}}\mathbb{I}_t^{\beta}\right)(g(t)) &= {}_a^{\mathbb{T}}\mathbb{I}_t^{\mu}\left({}_a^{\mathbb{T}}\mathbb{I}_t^{\beta}(g(t))\right) \\
&= \frac{1}{\Gamma(\mu)} \int_a^t \left(t-\rho(s)\right)^{\mu-1} \left({}_a^{\mathbb{T}}\mathbb{I}_t^{\beta}(g(s))\right)\nabla s \\
&= \frac{1}{\Gamma(\mu)} \int_a^t (t-\rho(s))^{\mu-1} \frac{1}{\Gamma(\beta)} \int_a^s (s-\rho(v))^{\beta-1} g(v)\nabla v\Big)\nabla s \\
&= \frac{1}{\Gamma(\mu)\Gamma(\beta)} \int_a^t \int_a^s \left((t-\rho(s))^{\mu-1}(s-\rho(v))^{\beta-1} g(v)\nabla v\right)\nabla s.
\end{aligned}$$

By using Definition 7, we obtain

$$\left({}_a^{\mathbb{T}}\mathbb{I}_t^{\mu} \circ {}_a^{\mathbb{T}}\mathbb{I}_t^{\beta}\right)(g(t)) = \frac{1}{\Gamma(\mu)\Gamma(\beta)} \int_a^t \left[\int_v^t (t-\rho(s))^{\mu-1}(s-\rho(v))^{\beta-1}\nabla s\right] g(v)\nabla u.$$

Let $s = \rho(v) + x(t-\rho(v))$ for $x \in \mathbb{R}$. We have

$$\begin{aligned}
&\left({}_a^{\mathbb{T}}\mathbb{I}_t^{\mu} \circ {}_a^{\mathbb{T}}\mathbb{I}_t^{\beta}\right)(g(t)) \\
&= \frac{1}{\Gamma(\mu)\Gamma(\beta)} \int_a^t \left[\int_0^1 (1-x)^{\mu-1}(t-\rho(v))^{\mu-1} x^{\beta-1}(t-\rho(v))^{\beta-1}(t-\rho(v))dx\right] g(v)\nabla v \\
&= \frac{1}{\Gamma(\mu)\Gamma(\beta)} \int_0^1 (1-x)^{\mu-1} x^{\beta-1} dx \int_a^t (t-\rho(v))^{\mu+\beta-1} g(v)\nabla v \\
&= \frac{B(\mu,\beta)}{\Gamma(\mu)\Gamma(\beta)} \int_a^t (t-\rho(v))^{\mu+\beta-1} g(v)\nabla v \\
&= \frac{1}{\Gamma(\mu+\beta)} \int_a^t (t-\rho(v))^{\mu+\beta-1} g(v)\nabla v \\
&= {}_a^{\mathbb{T}}\mathbb{I}_t^{\mu+\beta} g(t).
\end{aligned}$$

This completes the proof. □

Next definition uses integration as an anti-derivative process.

Definition 10. *(Riemann–Liouville fractional derivative on time scales) For $t \in \mathbb{T}$ and $g : \mathbb{T} \to \mathbb{R}$, the (left) Riemann–Liouville fractional derivative of order $\mu \in (0,1]$ is defined by*

$$ {}_a^{RL}D_t^{\mu} g(t) = \frac{1}{\Gamma(1-\mu)} \int_a^t \Big(t - \rho(s))^{-\mu} g(s) \nabla s \Big)^{\nabla}. $$

Remark 2. *If $\mathbb{T} = \mathbb{R}$, then Definition 10 gives the classical (left) Riemann–Liouville derivative of fractional order μ. Here, we are only studying the derivative in terms of left operators, the analogous right operators are easily acquired by changing the limit of integration.*

A different extension to time scales is obtained by using the nabla fractional derivative in terms of Caputo sense, that will be more effective for integer order initial conditions and are more easy to obtain in real world problems [3,11].

Definition 11. *(Nabla derivative on time scales in a Caputo sense) For $t, t_0 \in \mathbb{T}$, let us assume a finite time scale interval $[t_0, t] \cap \mathbb{T}^{\kappa^n} = [t_0, t]_{\mathbb{T}^{\kappa^n}}$. Then, for any $g \in AC_{\nabla}^{(n)}[t_0, t]_{\mathbb{T}^{\kappa^n}}$ of absolutely continuous function as in Definition 8, we define the Caputo nabla fractional derivative of order $0 < \mu \leq 1$ as*

$$ {}_a^{C}D_t^{\mu} g(t) = \frac{1}{\Gamma(n-\mu)} \int_a^t (t - \rho(s))^{n-\mu-1} g_{\nabla}^{(n)}(s) \nabla s, $$

where $n = [\mu] + 1$.

Definition 12. *If $\mu < 0$, then the nabla derivative of order μ in terms of Riemann–Liouville, is the fractional integral of order $-\mu$, that is, ${}_a^{RL}D_t^{\mu} = {}_a^{\mathbb{T}} \mathbb{I}_t^{-\mu}$.*

Definition 13. *If $\mu < 0$, then the nabla fractional integral of order μ is the nabla fractional derivative of order $-\mu$, i.e., ${}_a^{\mathbb{T}}\mathbb{I}_t^{\mu} = {}_a^{RL}D_t^{-\mu}$.*

Proposition 4. *Let $g : \mathbb{T}^{\kappa} \to \mathbb{R}$ be a nabla fractional differentiable function. Then, for any $0 < \mu \leq 1$,*

$$ {}_a^{RL}D_t^{\mu} g(t) = \nabla \circ {}_a^{\mathbb{T}}\mathbb{I}_t^{1-\mu} g(t). $$

Proof. Let $g : \mathbb{T}^{\kappa} \to \mathbb{R}$ be a nabla fractional differentiable function. Then, from Definition 10 and Equation (15), we have

$$ {}_a^{RL}D_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t ((t - \rho(t))^{-\alpha} g(s) \nabla s)^{\nabla} = ({}_a^{\mathbb{T}}\mathbb{I}_t^{1-\alpha} g(t)) = (\nabla \circ {}_a^{\mathbb{T}}\mathbb{I}_t^{1-\alpha}) g(t). $$

The proof is complete. □

Proposition 5. *For any integrable function g defined on a time scales interval $[a, b]$ one has*

$$ {}_a^{RL}D_t^{\mu} \circ {}_a^{RL}\mathbb{I}_t^{\mu} g(t) = g(t). $$

Proof. From Propositions 1 and 4, we obtain

$$ \begin{aligned} {}_a^{RL}D_t^{\mu} \circ {}_a^{\mathbb{T}}\mathbb{I}_t^{\mu} g(t) &= \Big[{}_a^{\mathbb{T}}\mathbb{I}_t^{1-\mu} \Big({}_a^{\mathbb{T}}\mathbb{I}_t^{\mu}(g(t)) \Big) \Big]^{\nabla} \\ &= \Big[{}_a^{\mathbb{T}}\mathbb{I}_t g(t) \Big]^{\nabla} = g(t). \end{aligned} $$

This concludes the proof. □

Corollary 2. *For $0 < \mu \leq 1$, we have ${}_a^{RL}D_t^{\mu} \circ {}_a^{RL}D_t^{-\mu} = I$ and ${}_a^{\mathbb{T}}\mathbb{I}_t^{-\mu} \circ {}_a^{\mathbb{T}}\mathbb{I}_t^{\mu} = I$, where I denotes the identity operator.*

Proof. From Definition 12 and Proposition 5, we find that

$$ {}^{RL}_{a}D^{\mu}_{t} \circ {}^{\mathbb{T}}_{a}D^{-\mu}_{t} = {}^{RL}_{a}D^{\mu}_{t} \circ {}^{\mathbb{T}}_{a}\mathbb{I}^{\mu}_{t} = I. $$

Again, from Definition 13 and Proposition 5, we have

$$ {}^{\mathbb{T}}_{a}\mathbb{I}^{-\mu}_{t} \circ {}^{\mathbb{T}}_{a}\mathbb{I}^{\mu}_{t} = {}^{RL}_{a}D^{\mu}_{t} \circ {}^{\mathbb{T}}_{a}\mathbb{I}^{\mu}_{t} = I, $$

which concludes the proof. □

5. Conclusions

In this paper, we discussed the nabla fractional derivative on time scales in a unified approach by using Grünwald–Letnikov and Riemann–Liouville derivative, respectively. Then, we have initiated the generalized definition of nabla derivative in fractional order in a pure sense of Riemann–Liouville and Caputo. We claim that a lot of further work can be completed by using this new idea. The aim of formulating the derivative is to solve fractional dynamic equations, stochastic dynamic equations, fuzzy dynamic equations, and one can think to extend the concept in a complex dynamic setting. About applications, it has great prospect in mathematical modeling, for example in epidemiology, anomalous diffusion in magnetic resonance imaging [29], fractal derivatives modeling [30], and consensus problems in time scales on fractional calculus.

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