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Torsion for Homological Complexes of Nonassociative Algebras with Metagroup Relations

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Abstract: The article is devoted to homological complexes and modules over nonassociative algebras with metagroup relations. Smashed tensor products of them are studied. Their torsions and homomorphisms are investigated.

Keywords: nonassociative algebra; homological complex; cohomology; metagroup; torsion; smashed; tensor product

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1. Introduction

Nonassociative algebras play a very important role in different branches of mathematics and its applications including physics, quantum mechanics, informatics, and biology (see, for example, [1–4] and references therein). In particular, octonions and generalized Cayley–Dickson algebras are widely used in noncommutative analysis, partial differential equations (PDEs), operator theory, particle physics, mathematical physics, and quantum field theory [5–21].

On the other hand, generalized Cayley–Dickson algebras are particular cases of nonassociative metagroup algebras [12,22–24].

It is worth mentioning that studies of PDEs are tightly related with cohomologies and deformed cohomologies [25]. This means that it is important to develop this area over metagroup algebras.

This article is devoted to investigations of smashed torsion products and torsions of homological complexes and modules over nonassociative algebras with metagroup relations. Certainly, a class of metagroups principally differs from a class of groups since a metagroup may be nonassociative, power nonassociative, or nonalternative, and left or right inverse elements in the metagroup may not exist or may contain elements for which left and right inverse elements do not coincide.

The (co)homology theory is one of the main tools for studying the structure of algebras, their modules, and their complexes. The previously developed traditional cohomology theory operates with associative algebras [1,26–28], so it is not worthwhile for nonassociative algebras. For Lie algebras, pre-Lie algebras, flexible algebras, and alternative algebras, (co)homology theory was advanced for the needs of their structure studies [2,29–31]. However, the latter algebras differ significantly from generalized Cayley–Dickson algebras and nonassociative algebras with metagroup relations.

Earlier cohomologies of loop spaces on quaternion and octonion manifolds were studied in [11], which have specific features in comparison with complex manifolds. Then, the basics of (co)homology theory for nonassociative algebras with metagroup relations were described in [12,32].

We recall the definition of the metagroup.

Definition 1. *Assume that G is a set with a single-valued binary operation (multiplication) $G^2 \ni (a,b) \mapsto ab \in G$ defined on G. Then $\mathcal{C}(G) := Com(G) \cap N(G)$ is called the center $\mathcal{C}(G)$ of G, where the set of all elements $h \in G$ commuting (or associating) with G is denoted by $Com(G)$ (or $N(G)$, respectively). That is,*

$Com(G) := \{a \in G : \forall b \in G,\ ab = ba\}$,
$N_l(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ab)c = a(bc)\}$,
$N_m(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ba)c = b(ac)\}$,
$N_r(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (bc)a = b(ca)\}$,
$N(G) := N_l(G) \cap N_m(G) \cap N_r(G)$;
$\mathcal{C}(G) := Com(G) \cap N(G)$.

We consider the following conditions:

$$\text{for each } a \text{ and } b \text{ in } G, \text{ there is a unique } x \in G \text{ with } ax = b, \text{ and} \quad (1)$$

$$\text{a unique } y \in G \text{ exists satisfying } ya = b, \quad (2)$$

which are denoted by $x = a \setminus b = Div_l(a,b)$ and $y = b/a = Div_r(a,b)$ respectively;

$$\text{there exists a neutral (that is, unit) element } e_G = e \in G :\ eg = ge = g \text{ for each } g \in G. \quad (3)$$

If a set G possesses a single-valued binary operation satisfying conditions (1)–(3) and

$$(ab)c = \mathrm{t}_3(a,b,c)a(bc) \text{ for each } a, b, \text{ and } c \text{ in } G, \quad (4)$$

then G is called a metagroup, where $\mathrm{t}_3(a,b,c) \in \mathbf{\Psi}$, $\mathbf{\Psi} \subset \mathcal{C}(G)$, where t_3 shortens a notation $\mathrm{t}_{3,G}$, and where $\mathbf{\Psi}$ denotes a (proper or improper) subgroup of $\mathcal{C}(G)$.

In this article, torsions for homological complexes of nonassociative algebras with metagroup relations are studied. Torsion products of modules over metagroup algebras are investigated in Propositions 2–4. Homomorphisms of torsion products are scrutinized in Theorems 1 and 2, Proposition 5. Connecting homomorphisms for torsion products are studied in Theorem 3, Proposition 6. Relations of flat modules over metagroup algebras with the torsion products are investigated in Theorem 4. Homomorphisms of homological complex torsion products are scrutinized in Theorem 5 and Lemmas 1 and 2. Retractions of canonical homomorphisms for smashed tensor products of homomological complexes over metagroup algebras are studied in Theorem 6. In the Appendix A, necessary properties of homomorphisms of metagroup algebras and modules over them are provided.

All the main results of this paper are obtained for the first time. They can be useful for further studies of nonassociative algebra (co)homologies, nonassociative algebra structures, operator theory, and spectral theory over Cayley–Dickson algebras, PDEs, noncommutative analysis, noncommutative geometry, mathematical physics, their applications in the sciences, etc.

2. Torsion Functor of Complexes for Nonassociative Algebras with Metagroup Relations

Remark 1. *Let $\mathcal{T}$ be a commutative associative unital ring, G be a metagroup, and $\mathcal{T}[G]$ be a metagroup algebra of G over $\mathcal{T}$; let also B be a unital smashly G-graded A-algebra (see the notation and definitions in [12,32]). Let X be a smashly G-graded B-bimodule, where "smashly" may be omitted for brevity. We consider a free $\mathcal{T}[\mathsf{C}(G)]$-module $(\mathcal{T}[\mathsf{C}(G)])^{(X)}$ with a canonical basis f_x, where $x \in X$. Certainly, $\mathcal{T}[\mathsf{C}(G)]1_B \hookrightarrow \mathsf{C}(B)$. We put $L_0(X) := L_{0,B}(X) := B^{(X)} = B^X \otimes_{\mathcal{T}} (\mathcal{T}[\mathsf{C}(G)])^{(X)}$, where $B^X = \prod_{x \in X} B_x$ with $B_x = B$ for each $x \in X$. That is, $L_0(X)$ is a free G-graded B-bimodule with base X. Then we define $p_X : L_0(X) \to X$ to be a $\mathcal{T}[\mathsf{C}(G)]$-linear map such that $p_X(f_x) = x$ and $p_X(bf_x) = bx$ and $p_X(f_x b) = xb$ for each $x \in X$ and $b \in B$. Then we put $\mathcal{Z}_0(X) = Ker(p_X)$, $i_X : \mathcal{Z}_0(X) \hookrightarrow L_0(X)$ to be a canonical injection. By induction, let $\mathcal{Z}_n = \mathcal{Z}_0(\mathcal{Z}_{n-1}(X))$, $L_n(X) = L_0(\mathcal{Z}_{n-1}(X))$ for each $n \in \mathbf{N}$, $L_k(X) = (0)$ for each $k < 0$*

in $\mathbf{Z}$*, where* $\mathbf{Z} = \{0, -1, 1, -2, 2, \dots\}$ *denotes the ring of integers,* $\mathbf{N} = \{1, 2, 3, \dots\}$ *is a set of positive integers. This induces the following exact sequence:*

$$0 \to \mathcal{Z}_0(X) \xrightarrow[i_X]{} L_0(X) \xrightarrow[p_X]{} X \to 0. \tag{5}$$

This construction provides a $\mathbf{Z}$*-graded G-graded B-bimodule* $L(X)$ *with* $(L(X))_n = L_n(X)$ *for each* $n \in \mathbf{Z}$*. By virtue of Lemmas 2 and 3 in [32],* $L_n(X)$ *can be supplied with the* $G^{|n|+2}$*-graded B-bimodule structure for each* $n \in \mathbf{Z}$*.*

We define the following left and right B-homomorphisms of B-bimodules $L(X)$ *and X:*

$$d_n^X : L_n(X) \to L_{n-1}(X) \tag{6}$$

$\forall n \leq 0,\ d_n^X = 0;$
$d_1^X = i_X \circ p_{\mathcal{Z}_0(X)};$
$\forall n > 1,\ d_n^X = i_{\mathcal{Z}_{n-2}(X)} \circ p_{\mathcal{Z}_{n-1}(X)},$

where $i \circ p$ *as usually denotes the composition of maps i and p, such that* $i \circ p(x) = i(p(x))$ *for an argument x.*

This construction induces the following exact sequence

$$\xrightarrow[d_{n+1}^X]{} L_n \xrightarrow[d_n^X]{} L_{n-1} \xrightarrow[d_{n-1}^X]{} \dots \xrightarrow[d_1^X]{} L_0 \xrightarrow[p_X]{} X \to 0, \tag{7}$$

where $L_n = L_n(X)$ *for each* $n \in \mathbf{Z}$*. There exists an extension*

$$p_X : (L(X), d^X) \to X \tag{8}$$

which is a B-exact homomorphism of the G-graded B-complexes. The exact sequence (7) is called a canonical free resolution of X.

Assume that Y is a smashly ${}^1G$*-graded* ${}^1B$*-bimodule, where* ${}^1A = \mathcal{T}[{}^1G]$ *is a metagroup algebra,* ${}^1G$ *is a metagroup, and* ${}^1B$ *is a unital smashly* ${}^1G$*-graded* ${}^1A$*-algebra. Suppose that* $h : X \to Y$ *is an* $((B, {}^1B), (B, {}^1B))$*-epigeneric (or -exact) homomorphism (see also Definitions 2 and 5 in [32]). It is convenient to define a unique* $((B, {}^1B), (B, {}^1B))$*-epigeneric (or -exact, respectively) homomorphism*

$$L_0(h) : L_0(X) \to L_0(Y) \tag{9}$$

such that

$$L_0(h)(f_x) = f_{h(x)} \tag{10}$$

for each $x \in X$ *and* $L_0(h'_B)(f'_b) = f'_{h'_B(b)}$ *for each* $b \in B$ *and* $f'_b \in L_0(B)$*, where* $h'_B : B \to B$*, where* f'_b *is the canonical basic element in* $L_0(B)$*. Therefore,*

$$p_Y \circ L_0(h) = h \circ p_X \text{ and } (p_Y)'_{{}^1B} \circ L_0(h'_B) = h'_B \circ (p_X)'_B, \tag{11}$$

where $(p_X)'_B(f'_b) = b$ *and* $(p_Y)'_{{}^1B}(f'_{{}^1b}) = {}^1b$ *for each* $b \in B$ *and* ${}^1b \in {}^1B$*. Henceforth, a* $(B, {}^1B)$*-epigeneric homomorphism is supposed to also be* $(A, {}^1A)$*-epigeneric, and an* $(A, {}^1A)$*-epigeneric homomorphism is supposed to also be* $(G, {}^1G)$*-epigeneric.*

Then $L_0(h)$ *induces a* $((B, {}^1B), (B, {}^1B))$*-epigeneric (or -exact, respectively) homomorphism* $\mathcal{Z}_0(h) : \mathcal{Z}_0(X) \to \mathcal{Z}_0(Y)$ *such that*

$$i_Y \circ \mathcal{Z}_0(h) = L_0(h) \circ i_X, \quad (i_Y)'_{{}^1B} \circ \mathcal{Z}_0(h'_B) = L_0(h'_B) \circ (i_X)'_B. \tag{12}$$

Putting $L_n(h) = 0$ *for each* $n < 0$*, it is useful to define, by induction,* $((B, {}^1B), (B, {}^1B))$*-epigeneric (or -exact respectively) homomorphisms*

$L_n(h) : L_n(X) \to L_n(Y)$ *and* $\mathcal{Z}_n(h) : \mathcal{Z}_n(X) \to \mathcal{Z}_n(Y)$ *such that*

$$L_n(h) = L_0(\mathcal{Z}_{n-1}(h)) \text{ and } \mathcal{Z}_n(h) = \mathcal{Z}_0(\mathcal{Z}_{n-1}(h)) \tag{13}$$

such that $L_n(h'_B) = L_0(\mathcal{Z}_{n-1}(h'_B))$, $\mathcal{Z}_n(h'_B) = \mathcal{Z}_0(\mathcal{Z}_{n-1}(h'_B))$.

Similarly smashly G-graded left or right B-modules can be considered. Examples of abundant families of metagroups, their modules, and their complexes are described in [11,12,22–24,32].

Proposition 1. *If conditions of Remark 1 are satisfied, then* $L(h) : L(X) \to L(Y)$ *is a* $((B, {}^1B), (B, {}^1B))$*-epigeneric homomorphism of complexes such that*

$$p_Y \circ L(h) = h \circ p_X. \tag{14}$$

Moreover, if h *is* $((B, {}^1B), (B, {}^1B))$*-exact, then* $L(h)$ *is* $((B, {}^1B), (B, {}^1B))$*-exact.*

Proof. From Formulas (6) and (13) above, it follows that

$d_1^Y \circ L_1(h) = i_Y \circ p_{\mathcal{Z}_0(Y)} \circ L_0(\mathcal{Z}_0(h))$ and consequently,
$(d_1^Y \circ L_1(h))'(B) = {}^1B$,

since the homomorphisms i_Y, p_Y and hence $p_{\mathcal{Z}_0(Y)}$ are ${}^1B$-exact, $\mathcal{Z}_0(h)$ and hence $L_0(\mathcal{Z}_0(h))$ are $((B, {}^1B), (B, {}^1B))$-epigeneric (see also Definitions 2 and 5 in [32]).

Using (11), we infer that
$i_Y \circ p_{\mathcal{Z}_0(Y)} \circ L_0(\mathcal{Z}_0(X)) = i_Y \circ \mathcal{Z}_0(h) \circ p_{\mathcal{Z}_0(X)}$.

Then (12) implies that
$i_Y \circ \mathcal{Z}_0(h) \circ p_{\mathcal{Z}_0(X)} = L_0(h) \circ i_X \circ p_{\mathcal{Z}_0(X)}$.

From (6), it follows that
$L_0(h) \circ i_X \circ p_{\mathcal{Z}_0(X)} = L_0(h) \circ d_1^X$ and
$(L_0(h) \circ d_1^X)'(B) = {}^1B$.

Then, by induction on $n > 1$, we deduce from Formulas (6) and (13) that
$d_n^Y \circ L_n(h) = i_{\mathcal{Z}_{n-2}(Y)} \circ p_{\mathcal{Z}_{n-1}(Y)} \circ L_0(\mathcal{Z}_{n-1}(h))$ and hence
$(d_n^Y \circ L_n(h))'(B) = {}^1B$,

since by induction the homomorphisms $i_{\mathcal{Z}_{n-2}(Y)}$ and $p_{\mathcal{Z}_{n-1}(Y)}$ are ${}^1B$-exact, and $\mathcal{Z}_{n-1}(h)$ and hence $L_0(\mathcal{Z}_{n-1}(h))$ are $((B, {}^1B), (B, {}^1B))$-epigeneric.

Utilizing (11), we deduce that
$i_{\mathcal{Z}_{n-2}(Y)} \circ p_{\mathcal{Z}_{n-1}(Y)} \circ L_0(\mathcal{Z}_{n-1}(h)) = i_{\mathcal{Z}_{n-2}(Y)} \circ \mathcal{Z}_{n-1}(h) \circ p_{\mathcal{Z}_{n-1}(X)}$.

From (13) it follows that
$i_{\mathcal{Z}_{n-2}(Y)} \circ \mathcal{Z}_{n-1}(h) \circ p_{\mathcal{Z}_{n-1}(X)} = i_{\mathcal{Z}_{n-2}(Y)} \circ \mathcal{Z}_0(\mathcal{Z}_{n-2}(h)) \circ p_{\mathcal{Z}_{n-1}(Y)}$.

Taking into account (12), we get
$i_{\mathcal{Z}_{n-2}(Y)} \circ \mathcal{Z}_0(\mathcal{Z}_{n-2}(h)) \circ p_{\mathcal{Z}_{n-1}(Y)} = L_0(\mathcal{Z}_{n-2}(h)) \circ i_{\mathcal{Z}_{n-2}(X)} \circ p_{\mathcal{Z}_{n-1}(X)}$.

Formulas (6) and (13) imply that
$L_0(\mathcal{Z}_{n-2}(h)) \circ i_{\mathcal{Z}_{n-2}(X)} \circ p_{\mathcal{Z}_{n-1}(X)} = L_{n-1}(h) \circ d_n^X$ and
$(L_{n-1}(h) \circ d_n^X)'(B) = {}^1B$. Thus,

$$d_n^Y \circ L_n(h) = L_{n-1}(h) \circ d_n^X \tag{15}$$

for each natural number n,

$$L(1_X) = 1_{L(X)}. \tag{16}$$

If $u : Y \to P$ is a $(({}^1B, {}^2B), ({}^1B, {}^2B))$-epigeneric homomorphism, where P is a ${}^2G$-graded ${}^2A$-bimodule, then $L_0(u \circ h)(f_x) = f_{u \circ h(x)}$ for each $x \in X$. Hence, $L_0(u \circ h)(f_x) = L_0(u)(f_{h(x)}) = L_0(u) \circ L_0(h)(f_x)$. Therefore, $L_0(u \circ h) = L_0(u) \circ L_0(h)$ and $(L_0(u \circ h))'(B) = {}^2B$. Consequently, $\mathcal{Z}_0(u \circ h) = \mathcal{Z}_0(u) \circ \mathcal{Z}_0(h)$ and by induction $L_n(u \circ h) = L_n(u) \circ L_n(h)$ for each $n \geq 0$. Therefore, $L(u \circ h) = L(u) \circ L(h)$ and $(L(u \circ h))'(B) = {}^2B$. If h is

$((B, {}^1B), (B, {}^1B))$-exact, then from Remark 1 and the proof above, it follows that $L(h)$ also is $((B, {}^1B), (B, {}^1B))$-exact. □

Definition 2. *Assume that $\mathcal{T}$ is a commutative associative unital ring, G is a metagroup, $\mathcal{T}[G]$ is a metagroup algebra of G over $\mathcal{T}$, and B is a unital G-graded A-algebra. Suppose that X and Y are G-graded B-bimodules. Assume also that the acyclic G-graded B-complexes $L(X)$ and $L(Y)$ are, as in Remark 1, such that $d_n = 0$ for each $\mathbf{Z} \ni n \leq 0$ with $p_X : L(X) \to X$ and $p_Y : L(Y) \to Y$ the free canonical left resolution of X and Y, respectively. By a torsion product, X and Y are called a G-graded and $\mathbf{Z}$-graded B-bimodule $Tor^B(X, Y) := H(L(X) \bigotimes_B L(Y))$ with n-homogeneous components $Tor^B_n(X, Y) := H_n(L(X) \bigotimes_B L(Y))$ for each $n \in \mathbf{Z}$, where $L(X) \bigotimes_B L(Y)$ is the G-smashed tensor product of $L(X)$ with $L(Y)$ over B (see Definition 7 in [32]).*

Similarly, the case is considered in which X is a G-graded B-bimodule (or a right B-module) and Y is a G-graded left B-module (or a B-bimodule, respectively) providing a G-graded and $\mathbf{Z}$-graded left B-module (or a right B-module, respectively) $Tor^B(X, Y)$.

Remark 2. *Definition 2 and Remark 1 imply that $Tor^B_n(X, Y) = 0$ for $n < 0$, since the G-graded B-complexes $L(X)$ and $L(Y)$ are zero on the right.*

For a G-graded left B-module X and a ${}^1G$-graded left ${}^1B$-module Y by $Hom_{l,s;B,{}^1B}(X, Y)$ will be denoted a family of all left $\mathcal{T}$-linear homomorphisms which are $(B, {}^1B)$-epigeneric if $s = eg$, $(B, {}^1B)$-exact if $s = e$ (see also Remark 1). Then by $Hom_{l,s;G,{}^1G}(X, Y)$ will be denoted a family of all left $\mathcal{T}$-linear homomorphisms, which are $(G, {}^1G)$-epigeneric if $s = eg$, $(G, {}^1G)$-exact if $s = e$. For right modules, $Hom_{r,s;B,{}^1B}(X, Y)$ and $Hom_{r,s;G,{}^1G}(X, Y)$ will be used instead of them, respectively. Then for a $(G, {}^2G)$-graded $(B, {}^2B)$-bimodule X, and a $({}^1G, {}^3G)$-graded $({}^1B, {}^3B)$-bimodule Y,

$Hom_{s;(B,{}^2B),({}^1B,{}^3B)}(X, Y) = \{f : X \to Y | f \in Hom_{l,s;B,{}^1B}(X, Y) \& f \in Hom_{r,s;{}^2B,{}^3B}(X, Y)\}$,

$Hom_{s;(G,{}^2G),({}^1G,{}^3G)}(X, Y) = \{f : X \to Y | f \in Hom_{l,s;G,{}^1G}(X, Y) \& f \in Hom_{r,s;{}^2G,{}^3G}(X, Y)\}$.

Proposition 2. *Let X and Y be G-graded B-bimodules. Then there are G-epigeneric bijective homomorphisms*

$\eta^X_Y : Tor^B(X, Y) \to H(L(X) \otimes_B Y)$ *and*
$\tilde{\eta}^Y_X : Tor^B(X, Y) \to H(X \otimes_B L(Y))$.

Proof. Let $f : X \to {}^1X$ and $g : Y \to {}^1Y$ be (B, B)-epigeneric homomorphisms of G-graded B-bimodules X, ${}^1X$, and Y, ${}^1Y$, respectively. This induces a G-epigeneric homomorphism of G-graded $\mathbf{Z}$-graded B-bimodules $Tor^B(f, g) : Tor^B(X, Y) \to Tor^B({}^1X, {}^1Y)$, where

$$Tor^B(f, g) := H(L(f) \otimes_B L(g)) \tag{17}$$

with homogeneous components $Tor^B_n(f, g) : Tor^B_n(X, Y) \to Tor^B_n({}^1X, {}^1Y)$ being $G^{|n|+2}$-epigeneric, such that $(Tor^B_n(f, g))' : Tor^B_n(B, B) \to Tor^B_n(B, B)$, where A is embedded into the unital algebra B as $A1_B$. From Proposition 9 and Corollary 2 in [32], it follows that the canonical $\mathcal{T}$-linear homomorphism $\hat{h}_{0,0} : H_0(L(X)) \otimes_B H_0(L(Y)) \to H_0(L(X) \otimes_B L(Y))$ is bijective, where $\hat{h} = \hat{h}(L(X), L(Y))$. In view of Lemma 2 in [32] and the conditions imposed on $L_0(X)$, $L_0(Y)$ and the maps p_X, p_Y in Remark 1, there are G-epigeneric isomorphisms of the G-graded B-bimodules $\theta_X : X \to H_0(L(X))$ and $\theta_Y : Y \to H_0(L(Y))$. This induces a canonical G-epigeneric isomorphism

$$\hat{h}_{X,Y} : X \otimes_B Y \to Tor^B_0(X, Y). \tag{18}$$

This implies that

$$\hat{h}_{{}^1X,{}^1Y} \circ (f \otimes g) = Tor^B_0(f, g) \circ \hat{h}_{X,Y}. \tag{19}$$

Notice that the G-epigeneric homomorphism of complexes

$p_X \otimes p_Y : L(X) \bigotimes_B L(Y) \to X \bigotimes_B Y$ induces a G-epigeneric 0 isomorphism $\hat{h}^{-1}_{X,Y} : Tor^B_0(X,Y) \to X \otimes_B Y$

inverse to $\hat{h}_{X,Y}$. Then one gets $L(1_X) = 1_{L(x)}$ and $L(1_Y) = 1_{L(Y)}$; consequently, $Tor^B(1_X, 1_Y) = 1_{Tor^B(X,Y)}$. Assume that ${}^1f : {}^1X \to {}^2X$ and ${}^1g : {}^1Y \to {}^2Y$ are (B,B)-epigeneric homomorphisms of G-graded B-bimodules ${}^1X$, ${}^2X$, and ${}^1Y$, ${}^2Y$, respectively, then $L({}^1f \circ f) = L({}^1f) \circ L(f)$, $L({}^1g \circ g) = L({}^1g) \circ L(g)$ and $(L({}^1f \circ f))' = L({}^1f' \circ f') = L({}^1f') \circ L(f') = (L({}^1f) \circ L(f))'$ and $(L({}^1g \circ g))' = L({}^1g' \circ g') = L({}^1g') \circ L(g') = (L({}^1g) \circ L(g))'$. Hence $Tor^B({}^1f \circ f, {}^1g \circ g) = Tor^B({}^1f, {}^1g) \circ Tor^B(f,g)$, $(Tor^B({}^1f \circ f, {}^1g \circ g))' = (Tor^B({}^1f, {}^1g))' \circ (Tor^B(f,g))'$ and $({}^1f \otimes {}^1g) \circ (f \otimes g) = ({}^1f \circ f) \otimes ({}^1g \circ g)$. There are natural G-epigeneric homomorphisms of G-graded B-bimodules

$1 \otimes p_Y : L(X) \bigotimes_B L(Y) \to L(X) \bigotimes_B Y$ and
$p_X \otimes 1 : L(X) \bigotimes_B L(Y) \to X \bigotimes_B L(Y)$.

In view of Proposition 4 in [32] $1 \otimes p_Y$ and $p_X \otimes 1$ are (B,B)-epigeneric homologisms. They induce G-epigeneric bijective homomorphisms
$\eta^X_Y : Tor^B(X,Y) \to H(L(X) \bigotimes_B Y)$ and $\tilde{\eta}^Y_X : Tor^B(X,Y) \to H(X \bigotimes_B L(Y))$.

This implies the assertion of this proposition. □

Corollary 1. *If the conditions of Proposition 2 are satisfied and either X or Y is flat, then $Tor^B_n(X,Y) = 0$ for each $n > 0$.*

Proof. Assume that a module either X or Y is flat, then either $p_X \otimes 1 : L(X) \bigotimes_B Y \to X \bigotimes_B Y$ or $1 \bigotimes_B p_Y : X \bigotimes_B L(Y) \to X \bigotimes_B Y$, respectively, is a (B,B)-epigeneric homologism by Proposition 4 in [32]. Therefore a module either $H_n(L(X) \bigotimes_B Y)$ or $H_n(X \bigotimes_B L(Y))$, respectively, is null for each $n > 0$. Then, from Proposition 2, it follows that $Tor^B_n(X,Y) = 0$ for each $n > 0$. □

Proposition 3. *Let X, Y be G-graded B-bimodules and let ${}^1X$, ${}^1Y$ be G-graded left B-modules. Then a map $\Psi : (f,g) \mapsto Tor^B(f,g)$ is $\mathcal{T}$-bilinear, where*
$\Psi : Hom_{s;(B,B),(B,B)}(X, {}^1X) \times Hom_{l,s;B,B}(Y, {}^1Y) \to$
$\to Hom_{l,s;B,B}(Tor^B(X,Y), Tor^B({}^1X, {}^1Y))$.

Proof. For any f and p in $Hom_{s;(B,B),(B,B)}(X, {}^1X)$, g and v in $Hom_{l,s;B,B}(Y, {}^1Y)$, and a and b in $\mathcal{T}$, we infer that homomorphisms $a(L(f) \otimes g) + b(L(f) \otimes v)$ and $L(f) \otimes (ag + bv)$ from $L(X) \bigotimes_B Y$ into $L({}^1X) \bigotimes_B {}^1Y$ coincide, and $a(f \bigotimes L(g)) + b(p \bigotimes L(g))$ and $(af + bp) \bigotimes L(g)$ from $X \bigotimes_B L(Y)$ into ${}^1X \bigotimes_B L({}^1Y)$ coincide. From Proposition 2 and Remarks 1 and 2, it follows that $Tor^B(f, ag + bv) = aTor^B(f,g) + bTor^B(f,v)$ and $Tor^B(af + bp, g) = aTor^B(f,g) + bTor^B(p,g)$. □

Corollary 2. *Assume that X, Y are G-graded B-bimodules. If $b \in \mathcal{T}$ annihilates X or Y, then b annihilates $Tor^B(X,Y)$.*

Proof. The assertion of this corollary follows from Proposition 3 and $b1_{Tor^B(X,Y)} = Tor^B(b1_X, 1_Y)$ and $b1_{Tor^B(X,Y)} = Tor^B(1_X, b1_Y)$. □

Proposition 4. *Assume that $\{{}^jX : j \in J\}$ is a family of G-graded B-bimodules and $\{{}^kY : k \in K\}$ is a family of left G-graded B-modules, ${}^jf : {}^jX \to X$ with $X = \bigoplus_{i \in J} {}^iX$, ${}^kg : {}^kY \to Y$ with $Y = \bigoplus_{l \in K} {}^lY$ are canonical homomorphisms for each $j \in J$ and $k \in K$, where J and K are sets. Then there exists a G-epigeneric bijective homomorphism $\bigoplus_{i \in J, l \in K} Tor^B({}^iX, {}^lY) \to Tor^B(X,Y)$.*

Proof. Certainly X is a G-graded B-bimodule, and Y is a G-graded left B-module. Evidently, the canonical homomorphism ${}^jf$ is injective and (B,B)-exact for each $j \in J$, ${}^kg$ is injective and B-exact for each $k \in K$. This induces G-epigeneric isomorphisms $\bigoplus_{k\in K}(L(X) \bigotimes_B {}^kY) \to (L(X) \bigotimes_B Y)$ and $\bigoplus_{j\in J}({}^jX \bigotimes_B L(Y)) \to (X \bigotimes_B L(Y))$. Therefore, in view of Propositions 2 and A2, there exist G-epigeneric bijective homomorphisms

$$\bigoplus_{k\in K} Tor^B(X, {}^kY) \to Tor^B(X,Y) \text{ and } \bigoplus_{j\in J} Tor^B({}^jX, Y) \to Tor^B(X,Y).$$

This implies the assertion of this proposition. □

Theorem 1. *Let J and K be directed sets and let $D_X = \{{}^jX, {}^j_lf, J\}$ be a direct system of G-graded B-bimodules with B-epigeneric homomorphisms ${}^j_lf : {}^jX \to {}^lX$ for each $j < l$ in J, and let $D_Y = \{{}^kY, {}^k_lg, K\}$ be a direct system of left G-graded B-modules with B-epigeneric homomorphisms ${}^k_ig : {}^kY \to {}^iY$ for each $k < i$ in K. Then there exists a G-epigeneric bijective homomorphism $\underrightarrow{\lim}_{(i,k)\in J\times K} Tor^B({}^iX, {}^kY) \to Tor^B(X,Y)$ with $X = \underrightarrow{\lim} D_X$, $Y = \underrightarrow{\lim} D_Y$.*

Proof. The limits of directed systems D_X and D_Y are a G-graded B-bimodule X and a G-graded left B-module Y, respectively. For the limits of direct systems there are natural injective (B,B)-epigeneric homomorphisms ${}^jf : {}^jX \to X$ for each $j \in J$, injective B-epigeneric ${}^kg : {}^kY \to Y$ for each $k \in K$. On the other hand, A is embedded into the unital A-algebra B as $A1_B$, while G is embedded into A as $G1_A$, where 1_B is the unit element in B. Hence there are G-epigeneric isomorphisms $\underrightarrow{\lim}_{k\in K}(L(X) \bigotimes_B {}^kY) \to (L(X) \bigotimes_B Y)$ and $\underrightarrow{\lim}_{j\in J}({}^jX \bigotimes_B L(Y)) \to (X \bigotimes_B L(Y))$. In view of Propositions 2 and A2, there exist G-epigeneric bijective homomorphisms

$\underrightarrow{\lim}_{k\in K} Tor^B(X, {}^kY) \to Tor^B(X,Y)$ and $\underrightarrow{\lim}_{j\in J} Tor^B({}^jX, Y) \to Tor^B(X,Y)$. This induces the G-epigeneric bijective homomorphism $\underrightarrow{\lim}_{(i,k)\in J\times K} Tor^B({}^iX, {}^kY) \to Tor^B(X,Y)$. □

Remark 3. *Let X be a G-graded B-bimodule, where $A = \mathcal{T}[G]$ is a metagroup algebra, G is a metagroup, $\mathcal{T}$ is a commutative associative unital ring, and B is a unital G-graded A-algebra. Let also*

$$0 \to {}^1Y \xrightarrow[p]{} Y \xrightarrow[s]{} {}^2Y \to 0 \tag{20}$$

be an exact sequence of G-graded left B-modules with B-epigeneric (or B-exact) homomorphisms p and s. The sequence in (20) will be denoted by $\mathcal{P} = \mathcal{P}({}^1Y, Y, {}^2Y; p, s)$. In view of Proposition 10 and Lemma 5 in [32] and Proposition 1 above, a sequence of G-graded left B-complexes

$$0 \to L(X) \bigotimes_B {}^1Y \xrightarrow[1\otimes p]{} L(X) \bigotimes_B Y \xrightarrow[1\otimes s]{} L(X) \bigotimes_B {}^2Y \to 0 \tag{21}$$

is exact with B-epigeneric (or B -exact respectively) homomorphisms $1 \otimes p$ and $1 \otimes s$. We denote the sequence in (21) by ${}^X\mathcal{P}$. In view of Lemma 4 in [32] and (20), there exists a B-epigeneric homomorphism

$$\partial({}^X\mathcal{P}) : H(L(X) \bigotimes_B {}^2Y) \to H(L(X) \bigotimes_B {}^1Y). \tag{22}$$

Definition 3. *A composition of the G-epigeneric bijective homomorphisms $\eta^X_{{}^1Y}$ and $\eta^X_{{}^2Y}$ and the B-epigeneric homomorphism $\partial({}^X\mathcal{P})$*

$$\partial(X, \mathcal{P}) = (\eta^X_{{}^1Y})^{-1} \circ \partial({}^X\mathcal{P}) \circ \eta^X_{{}^2Y} : Tor^B(X, {}^2Y) \to Tor^B(X, {}^1Y) \tag{23}$$

is called a connecting homomorphism of torsion products relative to the module X and the exact sequence $\mathcal{P}$.

Theorem 2. *Assume that modules are as in Remark 3. Then there exists no boundary on the left sequence of G-epigeneric homomorphisms* $Tor_n^B(1,p)$, $Tor_n^B(1,s)$, $\partial_n(X,\mathcal{P})$, $1\otimes p$, $1\otimes s$ *of G-graded left B-modules*

$$... \to Tor_n^B(X,{}^1Y)\xrightarrow[Tor_n^B(1,p)]{} Tor_n^B(X,Y)\xrightarrow[Tor_n^B(1,s)]{} Tor_n^B(X,{}^2Y)$$
$$\xrightarrow[\partial_n(X,\mathcal{P})]{} Tor_{n-1}^B(X,{}^1Y)\xrightarrow[Tor_{n-1}^B(1,p)]{} ...\xrightarrow[Tor_1^B(1,s)]{} Tor_1^B(X,{}^2Y)$$
$$\xrightarrow[\partial_1(X,\mathcal{P})]{} X\bigotimes_B {}^1Y \xrightarrow[1\otimes p]{} X\bigotimes_B Y \xrightarrow[1\otimes s]{} X\bigotimes_B {}^2Y \to 0. \tag{24}$$

Proof. Notice that $\partial(X,\mathcal{P})$ is the graded homomorphism of degree -1 with components relative to the **Z**-gradation

$$\partial_n(X,\mathcal{P}) : Tor_n^B(X,{}^2Y) \to Tor_{n-1}^B(X,{}^1Y).$$

The latter homomorphism is B-epigeneric. The homomorphisms $(\eta^X_{{}^1Y})'$ and $(\eta^X_{{}^2Y})'$ from G into G are bijective by Proposition 2. There exists the commutative diagram

$$\begin{array}{ccccc} Tor^B(X,{}^1Y) & \xrightarrow[Tor^B(1,p)]{} & Tor^B(X,Y) & \xrightarrow[Tor^B(1,s)]{} & Tor^B(X,{}^2Y) \to \\ \eta^X_{{}^1Y}\downarrow & & \eta^X_Y\downarrow & & \eta^X_{{}^2Y}\downarrow \\ H(L(X)\otimes_B {}^1Y) & \xrightarrow[H(1\otimes p)]{} & H(L(X)\otimes_B Y) & \xrightarrow[H(1\otimes s)]{} & H(L(X)\otimes_B {}^2Y) \to \end{array}$$

$$\begin{array}{ccc} \xrightarrow[\partial(X,\mathcal{P})]{} Tor^B(X,{}^1Y) & \xrightarrow[Tor^B(1,p)]{} & Tor^B(X,Y) \\ \eta^X_{{}^1Y}\downarrow & & \eta^X_Y\downarrow \\ \xrightarrow[\partial({}^X\mathcal{P})]{} H(L(X)\otimes_B {}^1Y) & \xrightarrow[H(1\otimes p)]{} & H(L(X)\otimes_B Y). \end{array}$$

In view of Proposition 2, the homomorphisms η^X_Y, $\eta^X_{{}^1Y}$, and $\eta^X_{{}^2Y}$ are G-epigeneric bijective. This diagram is commutative based on Remarks 1 and 3. The lower line of this commutative diagram is exact by Theorem 1 in [32]. □

Corollary 3. *If the conditions of Remark 3 are satisfied and* $Tor_1^B(X,{}^2Y)=0$, *then the following sequence*

$$0 \to X\bigotimes_B {}^1Y \xrightarrow[1\otimes p]{} X\bigotimes_B Y \xrightarrow[1\otimes s]{} X\bigotimes_B {}^2Y \to 0$$

is exact with G-epigeneric homomorphisms $1\otimes p$, $1\otimes s$.

Corollary 4. *Let the following sequence of complexes of G-graded left B-modules be exact with B-epigeneric homomorphisms p and s*

$$0 \to {}^1\mathcal{C} \xrightarrow[p]{} \mathcal{C} \xrightarrow[s]{} {}^2\mathcal{C} \to 0$$

and let ${}^3\mathcal{C}$ *be a complex of G-graded B-bimodules. If* ${}^2\mathcal{C}$ *or* ${}^3\mathcal{C}$ *is flat, then the following sequence*

$$0 \to {}^3\mathcal{C}\bigotimes_B {}^1\mathcal{C} \xrightarrow[1\otimes p]{} {}^3\mathcal{C}\bigotimes_B \mathcal{C} \xrightarrow[1\otimes s]{} {}^3\mathcal{C}\bigotimes_B {}^2\mathcal{C} \to 0$$

is exact with G-epigeneric homomorphisms $1\otimes p$ *and* $1\otimes s$.

Proof. This follows from Theorem 2 and Corollary 1. □

Proposition 5. *Assume that there is the following commutative diagram of G-graded left B-modules with B-epigeneric homomorphisms p, s,* ${}^1p$, ${}^1s$, ${}^1h$, h, ${}^2h$

$$\begin{array}{ccccccccc} 0 \to & {}^1Y & \xrightarrow[p]{} & Y & \xrightarrow[s]{} & {}^2Y & \to 0 \\ & {}^1h\downarrow & & h\downarrow & & {}^2h\downarrow & \\ 0 \to & {}^1W & \xrightarrow[{}^1p]{} & W & \xrightarrow[{}^1s]{} & {}^2W & \to 0 \end{array} \tag{25}$$

with exact (horizontal) lines.
Assume also that there is a B-epigeneric homomorphism of G-graded B-bimodules $f : X \to {}^1X$. *Then the following diagram of G-graded left B-modules is commutative with G-epigeneric* **Z**-*graded homomorphisms:*

$$\begin{array}{ccc} Tor^B(X, {}^2Y) & \xrightarrow[\partial(X,\mathcal{P})]{} & Tor^B(X, {}^1Y) \\ Tor^B(f, {}^2h) \downarrow & & Tor^B(f, {}^1h) \downarrow \\ Tor^B({}^1X, {}^2W) & \xrightarrow[\partial({}^1X,{}^1\mathcal{P})]{} & Tor^B({}^1X, {}^1W). \end{array} \tag{26}$$

Proof. There is the following commutative diagram with G-epigeneric homomorphisms $1 \otimes p$, $1 \otimes s$, $1 \otimes {}^1p$, $1 \otimes {}^1s$, $L(f) \otimes {}^1h$, $L(f) \otimes h$, $L(f) \otimes {}^2h$:

$$\begin{array}{ccccc} 0 \to L(X) \bigotimes_B {}^1Y & \xrightarrow[1\otimes p]{} & L(X) \bigotimes_B Y & \xrightarrow[1\otimes s]{} & L(X) \bigotimes_B {}^2Y \to 0 \\ L(f) \otimes {}^1h \downarrow & & L(f) \otimes h \downarrow & & L(f) \otimes {}^2h \downarrow \\ 0 \to L({}^1X) \bigotimes_B {}^1W & \xrightarrow[1\otimes {}^1p]{} & L({}^1X) \bigotimes_B W & \xrightarrow[1\otimes {}^1s]{} & L({}^1X) \bigotimes_B {}^2W \to 0. \end{array}$$

From the latter commutative diagram and Proposition A3, the assertion of this proposition follows. □

Remark 4. *Symmetrically to the case considered above, let X be a G-graded B-bimodule, where* $A = \mathcal{T}[G]$ *is a metagroup algebra, G is a metagroup,* $\mathcal{T}$ *is a commutative associative unital ring, and B is a unital G-graded A-algebra. Let also*

$$0 \to {}^1Y \xrightarrow[p]{} Y \xrightarrow[s]{} {}^2Y \to 0 \tag{27}$$

be an exact sequence of G-graded right B-modules with B-epigeneric (or B-exact) homomorphisms p and s. We denote the sequence in (27) *by* $\mathcal{S} = \mathcal{S}({}^1Y, Y, {}^2Y; p, s)$. *From Proposition 10 and Lemma 5 in [32] and Proposition 1 above, it follows that a sequence of G-graded right B-complexes*

$$0 \to {}^1Y \bigotimes_B L(X) \xrightarrow[p\otimes 1]{} Y \bigotimes_B L(X) \xrightarrow[s\otimes 1]{} {}^2Y \bigotimes_B L(X) \to 0 \tag{28}$$

is exact with B-epigeneric (or B -exact, respectively) homomorphisms $p \otimes 1$ *and* $s \otimes 1$. *We denote the sequence in* (28) *by* ${}^X\mathcal{S}$. *By virtue of Lemma 4 in [32], there exists a B-epigeneric homomorphism*

$$\partial({}^X\mathcal{S}) : H({}^2Y \bigotimes_B L(X)) \to H({}^1Y \bigotimes_B L(X)) \tag{29}$$

Definition 4. *A composition of G-epigeneric bijective homomorphisms* $\tilde{\eta}^X_{{}^1Y}$ *and* $\tilde{\eta}^X_{{}^2Y}$ *and the B-epigeneric homomorphism* $\partial({}^X\mathcal{S})$

$$\partial(X, \mathcal{S}) = (\tilde{\eta}^X_{{}^1Y})^{-1} \circ \partial({}^X\mathcal{S}) \circ \tilde{\eta}^X_{{}^2Y} : Tor^B({}^2Y, X) \to Tor^B({}^1Y, X) \tag{30}$$

is called a connecting homomorphism of torsion products relative to the module X and the exact sequence $\mathcal{S}$.

Theorem 3. *Let modules be as in Remark 4. Then there exists a sequence unbounded on the left of the G-epigeneric homomorphisms* $Tor^B_n(p, 1)$, $Tor^B_n(s, 1)$, $\partial_n(X, \mathcal{S})$, $p \otimes 1$, $s \otimes 1$ *of G-graded right B-modules*

$$\begin{array}{l} \ldots \to Tor^B_n({}^1Y, X) \xrightarrow[Tor^B_n(p,1)]{} Tor^B_n(Y, X) \xrightarrow[Tor^B_n(s,1)]{} Tor^B_n({}^2Y, X) \\ \xrightarrow[\partial_n(X,\mathcal{S})]{} Tor^B_{n-1}({}^1Y, X) \xrightarrow[Tor^B_{n-1}(p,1)]{} \ldots \xrightarrow[Tor^B_1(s,1)]{} Tor^B_1({}^2Y, X) \\ \xrightarrow[\partial_1(X,\mathcal{S})]{} {}^1Y \otimes_B X \xrightarrow[p\otimes 1]{} Y \otimes_B X \xrightarrow[s\otimes 1]{} {}^2Y \otimes_B X \to 0. \end{array} \tag{31}$$

Proof. From Remark 4 and Definition 4, one gets that the homomorphism $\partial(X,\mathcal{S})$ is $\mathbf{Z}$-graded of degree -1 with components

$$\partial_n(X,\mathcal{S}) : Tor_n^B({}^2Y, X) \to Tor_{n-1}^B({}^1Y, X),$$

which are B-epigeneric. Then the homomorphisms $(\tilde{\eta}^X_{{}^1Y})'$ and $(\tilde{\eta}^X_{{}^2Y})'$ from G into G are bijective. We consider the commutative diagram

$$\begin{array}{ccccc}
Tor^B({}^1Y,X) & \xrightarrow[Tor^B(p,1)]{} & Tor^B(Y,X) & \xrightarrow[Tor^B(s,1)]{} & Tor^B({}^2Y,X) \to \\
\tilde{\eta}^X_{{}^1Y}\downarrow & & \tilde{\eta}^X_{Y}\downarrow & & \tilde{\eta}^X_{{}^2Y}\downarrow \\
H({}^1Y\bigotimes_B L(X)) & \xrightarrow[H(p\otimes 1)]{} & H(Y\bigotimes_B L(X)) & \xrightarrow[H(s\otimes 1)]{} & H({}^2Y\bigotimes_B L(X)) \to
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow[\partial(X,\mathcal{S})]{} Tor^B({}^1Y,X) & \xrightarrow[Tor^B(p,1)]{} & Tor^B(Y,X) \\
\tilde{\eta}^X_{{}^1Y}\downarrow & & \tilde{\eta}^X_{Y}\downarrow \\
\xrightarrow[\partial({}^X\mathcal{S})]{} H({}^1Y\bigotimes_B L(X)) & \xrightarrow[H(p\otimes 1)]{} & H(Y\bigotimes_B L(X)).
\end{array}$$

According to Proposition 2, the homomorphisms $\tilde{\eta}^X_{Y}$, $\tilde{\eta}^X_{{}^1Y}$, and $\tilde{\eta}^X_{{}^2Y}$ are G-epigeneric bijective. In view of Theorem 1 in [32] and Remarks 1 and 4 above the latter diagram is commutative possessing the exact lower (horizontal) line. □

Corollary 5. *Assume that the conditions of Remark 4 are satisfied and* $Tor_1^B({}^2Y,X) = 0$. *Then the following sequence*

$$0 \to {}^1Y\bigotimes_B X \xrightarrow[p\otimes 1]{} Y\bigotimes_B X \xrightarrow[s\otimes 1]{} {}^2Y\bigotimes_B X \to 0$$

is exact with G-epigeneric homomorphisms $p\otimes 1$, $s\otimes 1$.

Corollary 6. *Suppose that the following sequence of complexes of G-graded right B-modules is exact with B-epigeneric homomorphisms p and s*

$$0 \to {}^1\mathcal{C} \xrightarrow[p]{} \mathcal{C} \xrightarrow[s]{} {}^2\mathcal{C} \to 0$$

and let ${}^3\mathcal{C}$ *be a complex of G-graded B-bimodules. If* ${}^2\mathcal{C}$ *or* ${}^3\mathcal{C}$ *is flat, then the following sequence*

$$0 \to {}^1\mathcal{C}\bigotimes_B {}^3\mathcal{C} \xrightarrow[p\otimes 1]{} \mathcal{C}\bigotimes_B {}^3\mathcal{C} \xrightarrow[s\otimes 1]{} {}^2\mathcal{C}\bigotimes_B {}^3\mathcal{C} \to 0$$

is exact with G-epigeneric homomorphisms $p\otimes 1$ *and* $s\otimes 1$.

Proof. This follows from Theorem 3 and Corollary 1. □

Proposition 6. *Let the following commutative diagram of G-graded right B-modules*

$$\begin{array}{ccccccccc}
0 \to & {}^1Y & \xrightarrow[p]{} & Y & \xrightarrow[s]{} & {}^2Y & \to 0 \\
& {}^1h\downarrow & & h\downarrow & & {}^2h\downarrow & \\
0 \to & {}^1W & \xrightarrow[{}^1p]{} & W & \xrightarrow[{}^1s]{} & {}^2W & \to 0
\end{array} \tag{32}$$

be with B-epigeneric homomorphisms p, s, ${}^1p$, ${}^1s$, ${}^1h$, h, ${}^2h$ *and with exact (horizontal) lines. Let also a homomorphism* $f : X \to {}^1X$ *of G-graded B-bimodules be B-epigeneric. Then the following diagram of G-graded right B-modules*

$$\begin{array}{ccc}
Tor^B({}^2Y,X) & \xrightarrow[\partial(X,\mathcal{S})]{} & Tor^B({}^1Y,X) \\
Tor^B({}^2h,f)\downarrow & & Tor^B({}^1h,f)\downarrow \\
Tor^B({}^2W,{}^1X) & \xrightarrow[\partial({}^1X,{}^1\mathcal{S})]{} & Tor^B({}^1W,{}^1X)
\end{array} \tag{33}$$

is commutative with G-epigeneric $\mathbf{Z}$*-graded homomorphisms.*

Proof. We take the following commutative diagram

$$\begin{array}{ccccccc} 0 \to {}^1Y \bigotimes_B L(X) & \xrightarrow[p\otimes 1]{} & Y \bigotimes_B L(X) & \xrightarrow[s\otimes 1]{} & {}^2Y \bigotimes_B L(X) \to 0 \\ {}^1h \otimes L(f) \downarrow & & h \otimes L(f) \downarrow & & {}^2h \otimes L(f) \downarrow \\ 0 \to {}^1W \bigotimes_B L({}^1X) & \xrightarrow[{}^1p\otimes 1]{} & W \bigotimes_B L({}^1X) & \xrightarrow[{}^1s\otimes 1]{} & {}^2W \bigotimes_B L({}^1X) \to 0 \end{array}$$

possessing G-epigeneric homomorphisms $p \otimes 1, s \otimes 1, {}^1p \otimes 1, {}^1s \otimes 1, {}^1h \otimes L(f), h \otimes L(f), {}^2h \otimes L(f)$. Then Proposition A3 implies the assertion of this proposition. □

Theorem 4. *Let X be a G-graded B-bimodule. Then the following conditions are equivalent:*

(i) *X is flat;*

(ii) *$Tor_n^B(X, Y) = 0$ for each G-graded left B-module Y and for each positive integer n;*

(iii) *the following sequence*

$$0 \to {}^2X \bigotimes_B Y \xrightarrow[p\otimes 1]{} {}^1X \bigotimes_B Y \xrightarrow[s\otimes 1]{} X \bigotimes_B Y \to 0$$

is exact with B-epigeneric homomorphisms $p \otimes 1$ and $s \otimes 1$ for each exact sequence

$$0 \to {}^2X \xrightarrow[p]{} {}^1X \xrightarrow[s]{} X \to 0$$

of G-graded B-bimodules with B-epigeneric homomorphisms p and s and for each G-graded left B-module Y.

Proof. From (i), it follows that (ii) by Corollary 1. In view of Corollary 4, we get that (i) implies (iii).

Assume that the conditions in (iii) are satisfied. Then from Remark 1 it follows that

$$0 \to \mathcal{Z}_n(X) \xrightarrow[i_{\mathcal{Z}_{n-1}(X)}]{} L_n(X) \xrightarrow[p_{\mathcal{Z}_{n-1}(X)}]{} \mathcal{Z}_{n-1}(X) \to 0$$

is the exact sequence with B-epigeneric homomorphisms $i_{\mathcal{Z}_{n-1}(X)}$ and $p_{\mathcal{Z}_{n-1}(X)}$ for each integer $n \geq 1$, since $L_n(X) = L_0(\mathcal{Z}_{n-1}(X))$ and $\mathcal{Z}_n(X) = \mathcal{Z}_0(\mathcal{Z}_{n-1}(X))$. By virtue of Proposition 2, this induces the following exact sequence

$$0 \to Tor_n^B(X, Y) \to \mathcal{Z}_n(X) \bigotimes_B Y \xrightarrow[i_{\mathcal{Z}_{n-1}(X)}\otimes 1]{} L_n(X) \bigotimes_B Y$$
$$\xrightarrow[p_{\mathcal{Z}_{n-1}(X)}\otimes 1]{} \mathcal{Z}_{n-1}(X) \bigotimes_B Y \to 0$$

with ${}^1B$-epigeneric homomorphisms $i_{\mathcal{Z}_{n-1}(X)} \otimes 1$ and $p_{\mathcal{Z}_{n-1}(X)} \otimes 1$ for each integer $n \geq 1$. Together with the conditions in (iii), this implies that $Tor_n^B(X, Y) = 0$ for each positive integer $n \geq 1$.

From Proposition 1 in [32] and Theorem 2 above, it follows that X is flat if the conditions in (ii) are satisfied. □

Corollary 7. *Assume that there is an exact sequence*

$$0 \to {}^2X \xrightarrow[p]{} {}^1X \xrightarrow[s]{} X \to 0$$

of G-graded B-bimodules with B-epigeneric homomorphisms p and s and X is flat. Then ${}^1X$ is flat if and only if ${}^2X$ is flat.

Proof. There is an exact sequence

$$0 \to Tor_n^B({}^2X, Y) \to Tor_n^B({}^1X, Y) \to 0$$

for each G-graded left B-module Y and each positive integer $n \geq 1$ by $(i) \Leftrightarrow (ii)$ in Theorem 4. This implies the assertion of this corollary. □

Remark 5. *We consider a G-graded B-complex $(\mathcal{C}, d)$ of G-graded B-bimodules and a G-graded B-complex $({}^1\mathcal{C}, {}^1d)$ of G-graded left B-modules. There are exact sequences*

$$0 \to \mathcal{Z}(\mathcal{C}) \xrightarrow[j]{} \mathcal{C} \xrightarrow[\delta]{} \mathcal{B}(\mathcal{C})(-1) \to 0 \tag{34}$$

$$0 \to \mathcal{B}(\mathcal{C}) \xrightarrow[i]{} \mathcal{Z}(\mathcal{C}) \xrightarrow[vs.]{} H(\mathcal{C}) \to 0 \tag{35}$$

with B-epigeneric homomorphisms j, δ, i, v. From (34) *and* (35)*, B-epigeneric homomorphisms are induced*

$$H(\delta \otimes 1) : H(\mathcal{C} \bigotimes_B {}^1\mathcal{C}) \to H(\mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C})(-1) \tag{36}$$

$$\partial(H({}^1\mathcal{C})) : Tor_1^B(H(\mathcal{C}), H({}^1\mathcal{C})) \to \mathcal{B}(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \tag{37}$$

(see Remark 1 and Definition 7 in [32]). Then $Tor_1^B(H(\mathcal{C}), H({}^1\mathcal{C}))$ can be supplied with the **Z***-gradation such that*

$$(Tor_1^B(H(\mathcal{C}), H({}^1\mathcal{C})))_n = \sum_{l+m=n} Tor_1^B(H_l(\mathcal{C}), H_m({}^1\mathcal{C})). \tag{38}$$

This implies that $\partial(H({}^1\mathcal{C}))$ is the **Z***-graded homomorphism of zero degree.*

By virtue of Proposition 9 in [32], there exists the homomorphism

$$\hat{h}(\mathcal{B}(\mathcal{C}), {}^1\mathcal{C}) : \mathcal{B}(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \to H(\mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C}). \tag{39}$$

Theorem 5. *Let the conditions of Remark 5 be satisfied and let G-graded B-bimodules $\mathcal{Z}(\mathcal{C})$ and $\mathcal{B}(\mathcal{C})$ be flat and let the homomorphisms j, δ, i, v be B-epigeneric. Then there exists a unique G-epigeneric* **Z***-graded homomorphism of degree −1*

$$w : H(\mathcal{C} \bigotimes_B {}^1\mathcal{C}) \to Tor_1^B(H(\mathcal{C}), H({}^1\mathcal{C})). \tag{40}$$

such that the following diagram is commutative

$$\begin{array}{ccc} H(\mathcal{C} \otimes_B {}^1\mathcal{C}) & \xrightarrow[w]{} & Tor_1^B(H(\mathcal{C}), H({}^1\mathcal{C}))(-1) \\ H(\delta \otimes 1) \downarrow & & \partial(H({}^1\mathcal{C})) \downarrow \\ H(\mathcal{B}(\mathcal{C}) \otimes_B {}^1\mathcal{C})(-1) & \xleftarrow[\hat{h}(\mathcal{B}(\mathcal{C}), {}^1\mathcal{C})]{} & (\mathcal{B}(\mathcal{C}) \otimes_B H({}^1\mathcal{C}))(-1) \end{array} \tag{41}$$

and the following sequence of G-graded left B-modules

$$\begin{array}{c} 0 \to H(\mathcal{C}) \otimes_B H({}^1\mathcal{C}) \xrightarrow[\hat{h}(\mathcal{C}, {}^1\mathcal{C})]{} H(\mathcal{C} \otimes_B {}^1\mathcal{C}) \xrightarrow[w]{} \\ \to Tor_1^B(H(\mathcal{C}), H({}^1\mathcal{C}))(-1) \to 0 \end{array} \tag{42}$$

is exact with G-epigeneric homomorphisms $\hat{h}(\mathcal{C}, {}^1\mathcal{C})$ and w.

Proof. In view of Corollary 7, $\mathcal{C}$ and $H(\mathcal{C})$ are flat, since the sequences (34) and (35) are exact and since the homomorphisms j, δ, i, v are B-epigeneric. Then the following sequence

$$0 \to \mathcal{Z}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C} \xrightarrow[j \otimes 1]{} \mathcal{C} \bigotimes_B {}^1\mathcal{C} \xrightarrow[\delta \otimes 1]{} \to (\mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C})(-1) \to 0 \tag{43}$$

is exact with G-epigeneric homomorphisms $j \otimes 1$ and $\delta \otimes 1$ by Corollary 4. The sequence in (42) for each n takes the form:

$$\begin{array}{c} 0 \to \bigoplus_{l+m=n} H_l(\mathcal{C}) \otimes_B H_m({}^1\mathcal{C}) \xrightarrow[\hat{h}_n(\mathcal{C}, {}^1\mathcal{C})]{} H_n(\mathcal{C} \otimes_B {}^1\mathcal{C}) \xrightarrow[w]{} \\ \to \bigoplus_{l+m=n-1} Tor_1^B(H_l(\mathcal{C}), H_m({}^1\mathcal{C})) \to 0. \end{array} \tag{44}$$

The lemmas below are used for the proof continuation of this theorem. □

Lemma 1. *The connecting homomorphism $\rho : H(\mathcal{B}(\mathcal{C}) \otimes_B {}^1\mathcal{C}) \to H(\mathcal{Z}(\mathcal{C}) \otimes_B {}^1\mathcal{C})$ associated with the exact sequence* (43) *is $H(i \otimes 1)$.*

Proof. Since $\mathcal{B}(\mathcal{C})$ is flat, then b belongs to the image of $\mathcal{B}(\mathcal{C}) \bigotimes_B \mathcal{Z}(^1\mathcal{C})$ for each $b \in \mathcal{Z}(\mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C})$. Therefore, there exists $c_m \in \mathcal{C}$ and $u_m \in {}^1\mathcal{C}$ such that $b = \sum_m dc_m \otimes u_m$ with $du_m = 0$ for each m. A class of $\rho(b)$ is a class of $D(\sum_m c_m \otimes u_m) = \sum_m dc_m \otimes u_m = (i \otimes 1)(b)$. $\square$

Lemma 2. *Let ${}^1\mathcal{C}$ be a split G-graded B-complex of G-graded left B-modules, and let $H({}^1\mathcal{C})$ be flat. Let also $\mathcal{C}$ be a G-graded B-complex of G-graded B-bimodules. Then the map*

$$\hat{h}(\mathcal{C}, {}^1\mathcal{C}) : H(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \to H(\mathcal{C} \bigotimes_B {}^1\mathcal{C})$$

is G-epigeneric and bijective.

Proof. In view of Proposition 5 and Definition 11 in [32] and Remarks 1 and 5, a B-epigeneric homotopism ${}^1f$ of $({}^1\mathcal{C}, {}^1d)$ onto $(H({}^1\mathcal{C}), 0)$ exists. By virtue of Proposition 10 in [32], there exists a homotopism $1_{\mathcal{C}} \otimes {}^1f : \mathcal{C} \bigotimes_B {}^1\mathcal{C} \to \mathcal{C} \bigotimes_B H({}^1\mathcal{C})$. Notice that

$$H(1_{\mathcal{C}} \otimes {}^1f) \circ \hat{h}(\mathcal{C}, {}^1\mathcal{C}) = \hat{h}(\mathcal{C}, H({}^1\mathcal{C})) \circ H(1_{\mathcal{C}} \otimes H({}^1f)).$$

Therefore, it remains to be proved that $\hat{h}(\mathcal{C}, H({}^1\mathcal{C}))$ is G-epigeneric and bijective, since $H(1_{\mathcal{C}} \otimes {}^1f)$ and $H({}^1f)$ are B-epigeneric and bijective, $G1_A \subset A$, $A1_B \hookrightarrow B$. Hence it is sufficient to consider the case such that ${}^1C$ is flat and with zero ${}^1d$. On the other hand, there are exact sequences

$$0 \to \mathcal{Z}(\mathcal{C}) \underset{i}{\to} \mathcal{C} \underset{\delta}{\to} \mathcal{B}(\mathcal{C}) \to 0; \tag{45}$$

$$0 \to \mathcal{B}(\mathcal{C}) \underset{j}{\to} \mathcal{Z}(\mathcal{C}) \underset{\pi}{\to} H(\mathcal{C}) \to 0; \tag{46}$$

with B-epigeneric homomorphisms i, δ, j, π. For the flat G-graded B-complex ${}^1C$ with zero ${}^1d$ from (45) and (46), it follows that the following sequences are exact with G-epigeneric homomorphisms $i \otimes 1$, $\delta \otimes 1$, $j \otimes 1$, $\pi \otimes 1$:

$$0 \to \mathcal{Z}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C} \xrightarrow[i \otimes 1]{} \mathcal{C} \bigotimes_B {}^1\mathcal{C} \xrightarrow[\delta \otimes 1]{} \mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C} \to 0; \tag{47}$$

$$0 \to \mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C} \xrightarrow[j \otimes 1]{} \mathcal{Z}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C} \xrightarrow[\pi \otimes 1]{} H(\mathcal{C}) \bigotimes_B {}^1\mathcal{C} \to 0. \tag{48}$$

Therefore we infer that $D = d \otimes 1_{{}^1\mathcal{C}}$ and hence $D = (i \otimes 1) \circ (j \otimes 1) \circ (\delta \otimes 1)$, since $d = i \circ j \circ \delta$. This implies that the canonical maps $\mathcal{Z}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C} \to \mathcal{Z}(\mathcal{C} \bigotimes_B {}^1\mathcal{C})$ and $\mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C} \to \mathcal{B}(\mathcal{C} \bigotimes_B {}^1\mathcal{C})$ are bijective and G-epigeneric. Therefore, $\hat{h}(\mathcal{C}, {}^1\mathcal{C})$ is G-epigeneric and bijective. $\square$

Continuation of the Proof of Theorem 5. Notice that the exact homological sequence related with (43) is:

$$H(\mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C}) \xrightarrow[H(i \otimes 1)]{} H(\mathcal{Z}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C}) \xrightarrow[H(j \otimes 1)]{} H(\mathcal{C} \bigotimes_B {}^1\mathcal{C}) \to$$
$$\xrightarrow[H(\delta \otimes 1)]{} H(\mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C})(-1) \xrightarrow[H(i \otimes 1)]{} H(\mathcal{Z}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C})(-1).$$

Since $\mathcal{Z}(\mathcal{C})$ is flat, taking into account (35), we infer that the sequence

$$0 \to Tor_1^B(H(\mathcal{C}), H({}^1\mathcal{C})) \xrightarrow[\partial(H({}^1\mathcal{C}))]{} \mathcal{B}(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \xrightarrow[i \otimes 1]{} \mathcal{Z}(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \to$$
$$\xrightarrow[v \otimes 1]{} H(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \to 0$$

is exact with G-epigeneric homomorphisms $\partial(H({}^1\mathcal{C}))$, $i \otimes 1$, $v \otimes 1$ of $\mathbf{Z}$-graded G-graded left B-modules.

There are G-epigeneric homomorphisms

$$\hat{h}(\mathcal{B}(\mathcal{C}), {}^1\mathcal{C}) : \mathcal{B}(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \to H(\mathcal{B}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C}),$$
$$\hat{h}(\mathcal{Z}(\mathcal{C}), {}^1\mathcal{C}) : \mathcal{Z}(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \to H(\mathcal{Z}(\mathcal{C}) \bigotimes_B {}^1\mathcal{C}),$$
$$\hat{h}(H(\mathcal{C}), H({}^1\mathcal{C})) : H(\mathcal{C}) \bigotimes_B H({}^1\mathcal{C}) \to H(\mathcal{C} \bigotimes_B {}^1\mathcal{C}).$$

This implies that the following diagram is commutative of G-graded left B-modules with exact (horizontal) lines and G-epigeneric homomorphisms

$$\begin{array}{ccccccc} \mathcal{B}(\mathcal{C})\otimes_B H(^1\mathcal{C}) & \xrightarrow[i\otimes 1]{} & \mathcal{Z}(\mathcal{C})\otimes_B H(^1\mathcal{C}) & \xrightarrow[v\otimes 1]{} & H(\mathcal{C})\otimes_B H(^1\mathcal{C}) \to 0 & & \\ \hat{h}(\mathcal{B}(\mathcal{C}),{}^1\mathcal{C})\downarrow & & \hat{h}(\mathcal{Z}(\mathcal{C}),{}^1\mathcal{C})\downarrow & & \hat{h}(H(\mathcal{C}),H(^1\mathcal{C}))\downarrow & & \\ H(\mathcal{B}(\mathcal{C})\otimes_B {}^1\mathcal{C}) & \xrightarrow[H(i\otimes 1)]{} & H(\mathcal{Z}(\mathcal{C})\otimes_B {}^1\mathcal{C}) & \xrightarrow[H(j\otimes 1)]{} & H(\mathcal{C}\otimes_B {}^1\mathcal{C}) & \xrightarrow[H(\delta\otimes 1)]{} & \end{array}$$

$$\begin{array}{ccccc} & \xrightarrow[H(\delta\otimes 1)]{} & H(\mathcal{B}(\mathcal{C})\otimes_B {}^1\mathcal{C})(-1) & \xrightarrow[H(i\otimes 1)]{} & \\ & & \hat{h}(\mathcal{B}(\mathcal{C}),{}^1\mathcal{C})\uparrow & & \\ 0\to Tor_1^B(H(\mathcal{C}),H(^1\mathcal{C}))(-1) & \xrightarrow[\partial(H(^1\mathcal{C}))]{} & \mathcal{B}(\mathcal{C})\otimes_B H(^1\mathcal{C})(-1) & \xrightarrow[i\otimes 1]{} & \end{array}$$

$$\begin{array}{cc} \xrightarrow[H(i\otimes 1)]{} & H(\mathcal{Z}(\mathcal{C})\otimes_B {}^1\mathcal{C})(-1)) \\ & \hat{h}(\mathcal{Z}(\mathcal{C}),{}^1\mathcal{C})\uparrow \\ \xrightarrow[i\otimes 1]{} & (\mathcal{Z}(\mathcal{C})\otimes_B H(^1\mathcal{C}))(-1). \end{array}$$

From Lemma 2, it follows that the maps $\hat{h}(\mathcal{B}(\mathcal{C}),{}^1\mathcal{C})$ and $\hat{h}(\mathcal{Z}(\mathcal{C}),{}^1\mathcal{C})$ are bijective, since the G-graded B-complexes $\mathcal{B}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{C})$ are flat and split. Hence the map $\hat{h}(\mathcal{C},{}^1\mathcal{C})$ is injective with $\hat{h}(\mathcal{C},{}^1\mathcal{C})(H(\mathcal{C})\otimes_B H(^1\mathcal{C}) = KerH(\delta\otimes 1)$ by Corollary A2. Using Corollary A1, we deduce that the map $\hat{h}(\mathcal{B}(\mathcal{C}),{}^1\mathcal{C})\circ\partial(H(^1\mathcal{C}))$ is injective with the image $Im(\hat{h}(\mathcal{B}(\mathcal{C}),{}^1\mathcal{C})\circ\partial(H(^1\mathcal{C}))) = ImH(\delta\otimes 1)$. From this, the assertion of Theorem 5 follows. □

Theorem 6. *Assume that $\mathcal{C}$ and $^1\mathcal{C}$ are G-graded B-complexes of G-graded B-bimodules and G-graded left B-modules, respectively, and that $\mathcal{B}(\mathcal{C})$ and $\mathcal{B}(^1\mathcal{C})$ are projective. Then the canonical homomorphism $\hat{h}(\mathcal{C},{}^1\mathcal{C}): H(\mathcal{C})\otimes_B H(^1\mathcal{C})\to H(\mathcal{C}\otimes_B {}^1\mathcal{C})$ has a G-epigeneric retraction.*

Proof. In view of Propositions 4 and 10 in [32], Remarks 1 and 5, there exist B-epigeneric homologisms $f:\mathcal{C}\to H(\mathcal{C})$ and $^1f: {}^1\mathcal{C}\to H(^1\mathcal{C})$ such that $H(f) = 1_{H(\mathcal{C})}$ and $H(^1f) = 1_{H(^1\mathcal{C})}$. There is the following commutative diagram with G-epigeneric homomorphisms

$$\begin{array}{ccc} H(\mathcal{C})\otimes_B H(^1\mathcal{C}) & \xrightarrow[\hat{h}(\mathcal{C},{}^1\mathcal{C})]{} & H(\mathcal{C}\otimes_B {}^1\mathcal{C}) \\ H(f)\otimes H(^1f)\downarrow & & H(f\otimes {}^1f)\downarrow \\ H(\mathcal{C})\otimes_B H(^1\mathcal{C}) & \xrightarrow[\hat{h}(H(\mathcal{C}),H(^1\mathcal{C}))]{} & H(\mathcal{C})\otimes_B H(^1\mathcal{C}) \end{array}$$

in which $H(f)\otimes H(^1f)$ and $\hat{h}(H(\mathcal{C}),H(^1\mathcal{C}))$ are the identity maps; consequently, $H(f\otimes {}^1f)\circ\hat{h}(\mathcal{C},{}^1\mathcal{C})$ is the identity map. Hence $\hat{h}(\mathcal{C},{}^1\mathcal{C})$ has the G-epigeneric retraction $H(f\otimes {}^1f)$. □

Corollary 8. *Let $\mathcal{C}$ be a G-graded B-complex of G-graded B-bimodules, Y be a G-graded left B-module, and let $\mathcal{B}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{C})$ be flat. Then the following sequence is exact*

$$0\to H_n(\mathcal{C})\bigotimes_B Y\xrightarrow[\hat{h}_n(\mathcal{C},Y)]{} H_n(\mathcal{C}\bigotimes_B Y)\xrightarrow[w_n]{} Tor_1^B(H_{n-1}(\mathcal{C},Y))\to 0 \quad (49)$$

for each integer n with G-epigeneric homomorphisms $\hat{h}_n(\mathcal{C},Y)$ and w_n.

Corollary 9. *Let $\mathcal{C}$ and $^1\mathcal{C}$ be G-graded B-complexes of G-graded B-bimodules and G-graded left B-modules, respectively; let also $\mathcal{B}(\mathcal{C})$ and $\mathcal{B}(^1\mathcal{C})$ be projective, and let $\mathcal{Z}(\mathcal{C})$ be flat. Then the sequences (44) and (49) are exact and split.*

Proof. This follows from Theorems 5 and 6. □

Corollary 10. *Let the conditions of Remark 5 be satisfied, let the homomorphisms j, δ, i, v be B-epigeneric, let $\mathcal{C}$ be bounded on the right, and let $\mathcal{C}$ and $H(\mathcal{C})$ be flat. Then the canonical homomorphism*

$$\hat{h}(\mathcal{C}, {}^1\mathcal{C}) : H(\mathcal{C}) \otimes_B H({}^1\mathcal{C}) \to H(\mathcal{C} \otimes_B {}^1\mathcal{C})$$

is bijective and G-epigeneric.

Proof. By virtue of Theorem 5, it is sufficient to prove that $\mathcal{B}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{C})$ are flat. There are exact sequences (34) and (35). From Corollary 7, it follows that if $\mathcal{B}_{n-1}(\mathcal{C})$ is flat, then $\mathcal{Z}_n(\mathcal{C})$ is flat; if $\mathcal{Z}_n(\mathcal{C})$ is flat, then $\mathcal{B}_n(\mathcal{C})$ is flat. Notice that by the conditions of this corollary there exists $m \in \mathbf{Z}$ such that $\mathcal{B}_k(\mathcal{C}) = 0$ for each $k < m$. This implies the assertion of the corollary. □

3. Conclusions

The results of this article can be used for subsequent studies of (co)homology theory of nonassociative algebras with metagroup relations, modules, and complexes over them and noncommutative manifolds. Other actual applications exist in mathematical coding theory, autonomous mobile robotics, analysis of information flows, and their technical realizations [33–36], because for these purposes, binary systems and algebras are indispensable. This has the natural reason that metagroup relations are weaker than relations in groups. Therefore, coding complexity and routing of autonomous mobile robots can be more effective using algebras with metagroup relations in comparison with group algebras or Lie algebras.

Other applications were outlined in the introduction. This can also be applied to studies of PDEs or their system solutions with boundary conditions using (co)homologies, which can have a practical importance in dynamical systems [25,34]. It is known that the Cayley–Dickson algebras are related with the quasi Hopf algebras. The latter are used for the quantum Yang–Baxter equation. Therefore on the other side, metagroup algebras also provide new possibilities for investigations of the quantum Yang–Baxter equation.

4. Future Work

Besides areas of further studies in this area outlined in the introduction and in the conclusions, it would be interesting to scrutinize different types of products of metagroup algebras and their modules in relation to specific products of complexes over algebras with metagroup relations and their (co)homologies. On the other hand, studies of the structure of metagroups are related with those of metagroup algebras and their modules. This is natural because many structures in physics, quantum field theory, systems of partial differential equations, or systems of integral equations frequently lead to nonassociative structures.

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Appendix A

Definitions of metagroups G, metagroup algebras A, smashly G-graded A-algebras B, smashly G-graded B-modules (left, right or bimodules), and G-graded B-complexes are given in [12,32], where "smashly" may be omitted for shortening. In these works and [11,22–24], examples of families of metagroups, their modules, and their complexes are given.

Proposition A1. *Let ${}^jB$ be a unital ${}^jG$-graded ${}^jA$-algebra for each $j \in \{1, 2, 3\}$, X and ${}^1X$ be left ${}^2B$-modules, Y and ${}^1Y$ be left ${}^1B$-modules, and Z and ${}^1Z$ be left ${}^3B$-modules, where ${}^jA = \mathcal{T}[{}^jG]$ is* *a metagroup algebra, where ${}^jG$ is a metagroup, and where $\mathcal{T}$ is an associative commutative unital ring. Let*

$$\begin{array}{ccccc} X & \xrightarrow[u]{} & Y & \xrightarrow[v]{} & Z \\ f\downarrow & & g\downarrow & & h\downarrow \\ {}^1X & \xrightarrow[{}^1u]{} & {}^1Y & \xrightarrow[{}^1v]{} & {}^1Z \end{array} \tag{A1}$$

be a commutative diagram with exact (horizontal) lines, where homomorphisms u and ${}^1u$ are $({}^2B, {}^1B)$-epigeneric, v and ${}^1v$ are $({}^1B, {}^3B)$-epigeneric, f is ${}^2B$-epigeneric, g is ${}^1B$-epigeneric, and h is ${}^3B$-epigeneric.

$$\text{If } h \text{ is injective, then} \tag{A2}$$

$$Im(g) \cap Im({}^1u) = Im({}^1u \circ f) = Im(g \circ u). \tag{A3}$$

$$\text{If } f \text{ is surjective, then} \tag{A4}$$

$$Ker(g) + Im(u) = Ker({}^1v \circ g) = Ker(h \circ v). \tag{A5}$$

Proof. Note that the homomorphisms $v' \circ u' : {}^2B \to {}^3B$ and ${}^1v' \circ {}^1u' : {}^2B \to {}^3B$ are epimorphic and $\mathcal{T}$-linear and from the conditions of this proposition, we get the commutative diagram

$$\begin{array}{ccccc} {}^2B & \xrightarrow[u']{} & {}^1B & \xrightarrow[v']{} & {}^3B \\ f'\downarrow & & g'\downarrow & & h'\downarrow \\ {}^2B & \xrightarrow[{}^1u']{} & {}^1B & \xrightarrow[{}^1v']{} & {}^3B. \end{array} \tag{A6}$$

Let h be injective. From the diagram (A1), we get that $Im({}^1u \circ f) = Im(g \circ u) \subseteq Im(g) \cap Im({}^1u)$. Assume that ${}^1y \in Im(g) \cap Im({}^1u)$. Then there exists $y \in Y$ such that ${}^1y = g(y)$. From ${}^1v \circ {}^1u = 0$, it follows that $0 = {}^1v({}^1y) = {}^1v(g(y)) = h(v(y))$; consequently, $v(y) = 0$, because h is injective. The exactness of the sequence (u, v) implies that there exists $x \in M$ such that $y = u(x)$; hence, ${}^1y = g(u(x))$ and consequently, $Im(g) \cap Im({}^1u) \subseteq Im(g \circ u) = Im({}^1u \circ f)$.

Let f be surjective. From $v \circ u = 0$ and ${}^1v \circ {}^1u = 0$, it follows that $Ker(g) + Im(u) \subseteq Ker({}^1v \circ g) = Ker(h \circ v)$. On the contrary, $y \in Ker({}^1v \circ g)$ implies that $g(y) \in Ker({}^1v)$, and hence there exists ${}^1x \in {}^1X$ for which ${}^1u({}^1x) = g(y)$, because the sequence $({}^1u, {}^1v)$ is exact. Therefore, there exists $x \in X$ such that $f(x) = {}^1x$, because f is surjective. Hence $g(y) = {}^1u(f(x)) = g(u(x))$; consequently, $y - u(x) \in Ker(g)$. □

Lemma A1. *Assume that X and ${}^1X$ are left ${}^2B$-modules, Y and ${}^1Y$ are left ${}^1B$-modules, where ${}^jA = \mathcal{T}[{}^jG]$ is the metagroup algebra, where ${}^jG$ is the metagroup, $\mathcal{T}$ is an associative commutative unital ring, and ${}^jB$ is the unital ${}^jG$-graded ${}^jA$-algebra for each $j \in \{1, 2\}$. Assume also that*

$$\begin{array}{ccc} X & \xrightarrow[u]{} & Y \\ f\downarrow & & g\downarrow \\ {}^1X & \xrightarrow[{}^1u]{} & {}^1Y \end{array} \tag{A7}$$

is a commutative diagram, where homomorphisms u and ${}^1u$ are $({}^2B, {}^1B)$-epigeneric, f is ${}^2B$-epigeneric, g is ${}^1B$-epigeneric. Then there exist unique $({}^2B, {}^1B)$-epigeneric homomorphisms $w : Ker(f) \to Ker(g)$ and ${}^1w : Coker(f) \to Coker(g)$, for which the following diagrams are commutative

$$\begin{array}{ccc} Ker(f) & \xrightarrow[w]{} & Ker(g) \\ i\downarrow & & j\downarrow \\ X & \xrightarrow[u]{} & Y \end{array} \tag{A8}$$

$$\begin{array}{ccc} {}^1X & \xrightarrow[{}^1u]{\qquad\qquad} & {}^1Y \\ p\downarrow & & q\downarrow \\ Coker(f) & \xrightarrow[{}^1w]{\qquad} & Coker(g), \end{array} \tag{A9}$$

where i and j denote the canonical embeddings, and p and q are the canonical surjections. Moreover, if u and ^1u *are* $(^2B, ^1B)$*-exact, f is* ^2B*-exact, and g is* ^1B*-exact, then w and* ^1w *are* $(^2B, ^1B)$*-exact.*

Proof. By the conditions of this lemma, the homomorphisms $u' : {}^2B \to {}^1B$ and ${}^1u' : {}^2B \to {}^1B$ are epimorphic and $\mathcal{T}$-linear, and the following diagram is commutative:

$$\begin{array}{ccc} {}^2B & \xrightarrow[u']{\qquad} & {}^1B \\ f'\downarrow & & g'\downarrow \\ {}^2B & \xrightarrow[{}^1u']{\qquad} & {}^1B. \end{array} \tag{A10}$$

Notice that $Ker(f)$ is the left ^2B-module, $Im(g)$ is the left ^1B-module, since the homomorphism f is ^2B-epigeneric and the homomorphism g is ^1B-epigeneric. If $x \in Ker(f)$, then $f(x) = 0$ and $g(u(x)) = {}^1u(f(x)) = 0$; consequently, $u(x) \in Ker(g)$ and hence a unique $(^2B, ^1B)$-epigeneric homomorphism w exists. Then ${}^1u(f(X)) = g(u(X)) \subseteq g(Y)$. For the corresponding quotient modules, this induces the homomorphism ${}^1w : Coker(f) \to Coker(g)$, which is $(^2B, ^1B)$-epigeneric by (A10) and unique for which the diagram (A9) is commutative. If u and ^1u are $(^2B, ^1B)$-exact, f is ^2B-exact, g is ^1B-exact, and then, from (A10) and the proof above, it follows that w and ^1w are $(^2B, ^1B)$-exact. □

Remark A1. *By Lemma A1 to the commutative diagram of Proposition A1 the following commutative diagram corresponds*

$$\begin{array}{ccccc} Ker(f) & \xrightarrow[w]{\qquad} & Ker(g) & \xrightarrow[t]{\qquad} & Ker(h) \\ i\downarrow & & j\downarrow & & k\downarrow \\ X & \xrightarrow[u]{\qquad} & Y & \xrightarrow[v]{\qquad} & Z \\ f\downarrow & & g\downarrow & & h\downarrow \\ {}^1X & \xrightarrow[{}^1u]{\qquad} & {}^1Y & \xrightarrow[{}^1v]{\qquad} & {}^1Z \\ p\downarrow & & q\downarrow & & r\downarrow \\ Coker(f) & \xrightarrow[{}^1w]{\qquad} & Coker(g) & \xrightarrow[{}^1t]{\qquad} & {}^1Z, \end{array} \tag{A11}$$

where the homomorphisms w, u, ^1u*,* ^1w *are* $(^2B, ^1B)$*-epigeneric, t, v,* ^1v*,* ^1t *are* $(^1B, ^3B)$*-epigeneric, i, f, p are* ^2B*-epigeneric, j, g, q are* ^1B*-epigeneric, k, h, r are* ^3B*-epigeneric; i, j, k are the canonical injections; p, q, r are the canonical surjections.*

Proposition A2. *Suppose that in the commutative diagram (A1) the (horizontal) lines* (u, v)*,* $(^1u, ^1v)$ *are exact and the conditions of Remark A1 are satisfied. Then*

$$t \circ w = 0, \text{ and, if } {}^1u \text{ is injective,} \tag{A12}$$

then the row (w, t) *is exact;*

$${}^1t \circ {}^1w = 0, \text{ and, if } v \text{ is surjective,} \tag{A13}$$

then the sequence $(^1w, ^1t)$ *is exact;*

$$\text{if } {}^1u \text{ is injective and } v \text{ is surjective, } {}^1B = {}^2B = {}^3B,\ u,\ v,\ {}^1u,\ {}^1v \text{ are } {}^1B\text{-epigeneric,} \tag{A14}$$

then there exists a unique ^1B*-epigeneric homomorphism* $d : Ker(h) \to Coker(f)$ *such that the following sequence is exact*

$$\begin{array}{l} Ker(f) \xrightarrow{w} Ker(g) \xrightarrow{t} Ker(h) \xrightarrow{d} \\ Coker(f) \xrightarrow{^1w} Coker(g) \xrightarrow{^1t} Coker(h) \end{array} \tag{A15}$$

and $d(z) = p(^1x)$ *for each* $z \in Ker(h)$, $y \in Y$, $^1x \in {}^1X$ *satisfying the equalities* $v(y) = k(z)$ *and* $^1u(^1x) = g(y)$. *Moreover, if* u, ^1u, v, ^1v, f, g, h *are* ^1B*-exact, then* w, t, ^1w, ^1t, d *are* ^1B*-exact.*

Proof. By the conditions of Remark A1, the homomorphisms w, u, ^1u, and ^1w are $(^2B, {}^1B)$-epigeneric, t, v, ^1v, ^1t are $(^1B, {}^3B)$-epigeneric, i, f, p are ^2B-epigeneric, j, g, and q are ^1B-epigeneric, k, h, r are ^3B-epigeneric. Therefore, the homomorphisms $t \circ w$ and $^1t \circ {}^1w$ are $(^2B, {}^3B)$-epigeneric.

Since $w = u|_{Ker(f)}$ and $t = v|_{Ker(g)}$, then $t \circ w = 0$. On the other hand, $Ker(t) = Ker(g) \cap Ker(v) = Ker(g) \cap Im(u) = Im(j) \cap Im(u)$. From the injectivity of ^1u and Proposition A1, it follows that $Ker(t) = Im(j \circ w) = Im(w)$.

The homomorphisms ^1w and ^1t are obtained from u and v by using quotient modules; hence $^1t \circ {}^1w = 0$. If v is surjective, then from Proposition A1 and the surjectivity of p and q, and it follows that $Ker(^1t) = q(Ker(^1t \circ q)) = q(Ker(^1v) + Im(g)) = q(Im(^1u)) = Im(^1w \circ p) = Im(^1w)$.

If $z \in Ker(h)$, then there exists $y \in Y$ for which $v(y) = k(z)$, since the homomorphism v is surjective. On the other hand, $^1v(g(y)) = h(k(z)) = 0$; consequently, a unique $^1x \in {}^1X$ exists such that $^1u(^1x) = g(y)$, because the homomorphism ^1u is injective.

If $^1y \in Y$ is such that $v(^1y) = k(z)$, then $^1y = y + u(x)$, where $x \in X$. If $^1_1x \in {}^1X$ and $^1u(^1_1x) = g(^1y)$, then $^1u(^1x + f(x)) = {}^1u(^1x) + {}^1u(f(x)) = g(y) + g(u(x)) = g(y + u(x)) = g(^1y)$; consequently, $p(^1_1x) = p(^1x) + p(f(x)) = p(^1x)$. Thus $^1_1x = {}^1x + f(x)$ and hence $p(^1x) \in Coker(f)$ is independent of $y \in Y$ for which $v(y) = k(z)$. Putting $d(z) = p(^1x)$, we get a homomorphism $d : Ker(h) \to Coker(f)$.

It remains to be verified that the corresponding homomorphism d' also exists. By the conditions of this propositions, the homomorphisms u, v, and ^1u, ^1v are ^1B-epigeneric and $^1B = {}^2B = {}^3B$; hence u', v', $^1u'$, $^1v'$ are the $\mathcal{T}$-linear surjective homomorphisms.

The commmutative diagram (A11) implies the following commutative diagram:

$$\begin{array}{ccccc} ^1B & \xrightarrow{w'} & ^1B & \xrightarrow{t'} & ^1B \\ i' \downarrow & & j' \downarrow & & k' \downarrow \\ ^1B & \xrightarrow{u'} & ^1B & \xrightarrow{v'} & ^1B \\ f' \downarrow & & g' \downarrow & & h' \downarrow \\ ^1B & \xrightarrow{^1u'} & ^1B & \xrightarrow{^1v'} & ^1B \\ p' \downarrow & & q' \downarrow & & r' \downarrow \\ ^1B & \xrightarrow{^1w'} & ^1B & \xrightarrow{^1t'} & ^1B, \end{array} \tag{A16}$$

since the homomorphisms w, ^1w, t, ^1t, i, f, p, j, g, q, k, h, r are ^1B-epigeneric.

If z_1 and z_2 are in $Ker(h)$, a_1 and a_2 belong to ^1B; if $z = a_1z_1 + a_2z_2$, then y_1 and y_2 in Y exist such that $v(y_1) = k(z_1)$ and $v(y_2) = k(z_2)$, $y = b_1y_1 + b_2y_2 \in Y$ with b_1 and b_2 belonging to ^1B such that $v(y) = k(z)$ with $v'(b_1) = k'(a_1)$, $v'(b_2) = k'(a_2)$. Therefore, there exists a ^1B-exact homomorphism d such that $d(z) = d'(a_1)d(z_1) + d'(a_2)d(z_2)$.

It remains to be proven that the sequence (A15) is exact. Suppose that $z = t(\xi)$ for some $\xi \in Ker(g)$. We choose $y \in Y$ such that $y = j(\xi)$. From $g(j(\xi) = 0$, it follows that $d(z) = 0$ and hence $d \circ t = 0$.

On the contrary, assume that $d(z) = 0$. Then as above, $d(z) = p(^1x)$ with $^1x \in {}^1X$ and $^1x = f(x)$ with $x \in X$. Therefore, $g(y) = {}^1u(f(x)) = g(u(x))$; consequently, $g(y - u(x)) = 0$. Hence there exists $\eta \in Ker(g)$ such that $y - u(x) = j(\eta)$; consequently, $k(z) = v(y) = v(u(x) + j(\eta)) = v(j(\eta)) = k(t(\eta))$. The homomorphism k is injective and hence $z = t(\eta)$; consequently, the sequence (A15) is exact in the term $Ker(h)$. Then we deduce that ${}^1w(d(z)) = {}^1w(p({}^1x)) = q({}^1u({}^1x)) = q(g(y)) = 0$; consequently, ${}^1w \circ d = 0$.

Assume that $\xi = p({}^1x)$ belongs to $Coker(f)$ such that ${}^1w(\xi) = {}^1w(p({}^1x)) = 0$, where ${}^1x \in {}^1X$ and hence $q({}^1u({}^1x)) = 0$; consequently, there exists $y \in Y$ such that ${}^1u({}^1x) = g(y)$. From ${}^1v({}^1u({}^1x)) = 0$, it follows that ${}^1v(g(y)) = 0$ and hence $h(v(y)) = 0$. Therefore, there exists $z \in Ker(h)$ such that $v(y) = k(z)$ and hence $\xi = d(z)$. Thus, the sequence (A15) is exact in the term $Coker(f)$. The homomorphisms in (A15) are ${}^1B$-epigeneric as proved above. On the other hand, (A12) means that the sequence (A15) is exact in $Ker(g)$ and (A13) implies that it is exact in $Coker(g)$. Moreover, if the homomorphisms u, ${}^1u$, v, ${}^1v$, f, g, and h in the diagram (A11) are ${}^1B$-exact, then from Lemma A1 (A16) and the proof above, it follows that w, t, ${}^1w$, ${}^1t$, d are ${}^1B$-exact. □

Corollary A1. *Assume that in the commutative diagram (A1), the (horizontal) lines are exact, where homomorphisms u and ${}^1u$ are $({}^2B, {}^1B)$-epigeneric, v and ${}^1v$ are $({}^1B, {}^3B)$-epigeneric, f is ${}^2B$-epigeneric, g is ${}^1B$-epigeneric, and h is ${}^3B$-epigeneric. Then*

(i) *if ${}^1u$, f and h are injective homomorphisms, then g is injective;*
(ii) *if v, f and h are surjective, then g is surjective.*

Proof. This follows from assertions (A12) and (A13) of Proposition A2. □

Corollary A2. *Suppose that in the commutative diagram (A1) (horizontal) lines are exact; ${}^1B = {}^2B = {}^3B$; the homomorphisms u, v, ${}^1u$, ${}^1v$ are ${}^1B$-exact; and f, g, h are ${}^1B$-epigeneric. Then*

(i) *if g is injective, f and v are surjective, then h is injective;*
(ii) *if g is surjective, h and ${}^1u$ are injective, then f is surjective.*

Proof. We consider the following diagram

$$\begin{array}{ccccc} u(X) & \xrightarrow[\xi]{} & Y & \xrightarrow[v]{} & Z \\ {}^1f \downarrow & & g \downarrow & & h \downarrow \\ {}^1u({}^1X) & \xrightarrow[{}^1\xi]{} & {}^1Y & \xrightarrow[{}^1v]{} & {}^1Z, \end{array} \tag{A17}$$

where ${}^1f = g|_{u(X)}$, ξ, and ${}^1\xi$ are the canonical injections. From the condition that u is ${}^1B$-exact, it follows that u is ${}^1B$-epigeneric and hence $u(X)$ is the ${}^1B$-(sub)module. Then ${}^1f$ is ${}^1B$-epigeneric, since g is ${}^1B$-epigeneric. The homomorphisms ξ and ${}^1\xi$ are injective and ${}^1B$-exact, since u and ${}^1u$ are ${}^1B$-exact. By (A14) of Proposition A2, the following sequence is exact:

$$Ker(g) \to Ker(h) \xrightarrow[d]{} Coker({}^1f). \tag{A18}$$

The injectivity of g and the surjectivity of ${}^1f$ imply that $Ker(h) = 0$.

The following diagram proves (ii):

$$\begin{array}{ccccc} X & \xrightarrow[u]{} & Y & \xrightarrow[\eta]{} & v(Y) \\ f \downarrow & & g \downarrow & & {}^1h \downarrow \\ {}^1X & \xrightarrow[{}^1u]{} & {}^1Y & \xrightarrow[{}^1\eta]{} & {}^1v({}^1Y), \end{array} \tag{A19}$$

where ${}^1h = h|_{v(Y)}$, η and ${}^1\eta$ are induced by v and ${}^1v$, respectively. This diagram is commutative, and its (horizontal) lines are exact, where the homomorphisms u, ${}^1u$, η, ${}^1\eta$ are ${}^1B$-exact. From (A14), (A15) of Proposition A2 it follows that the following sequence is exact:

$$Ker({}^1h) \xrightarrow[d]{} Coker(f) \to Coker(g), \tag{A20}$$

since g is surjective and ${}^1h$ is injective; consequently, $Coker(f) = 0$. □

Corollary A3. *In the following commutative diagram, let horizontal lines be exact, the homomorphisms ^ju and ^jv be $(^jB,^{j+1}B)$-epigeneric for each $j \in \{1,\dots,4\}$, and ^jf be ^jB-epigeneric for each $j \in \{1,\dots,5\}$.*

$$\begin{array}{ccccccccc} ^1X & \xrightarrow[^1u]{} & ^2X & \xrightarrow[u^2]{} & ^3X & \xrightarrow[^3u]{} & ^4X & \xrightarrow[^4u]{} & ^5X \\ {}^1f\downarrow & & {}^2f\downarrow & & {}^3f\downarrow & & {}^4f\downarrow & & {}^5f\downarrow \\ ^1Y & \xrightarrow[^1v]{} & ^2Y & \xrightarrow[^2v]{} & ^3Y & \xrightarrow[^3v]{} & ^4Y & \xrightarrow[^4v]{} & ^5Y \end{array} \quad \text{(A21)}$$

Then

$$\text{if homomorphisms } {}^2f \text{ and } {}^4f \text{ are injective and } {}^1f \text{ is surjective,} \quad \text{(A22)}$$

then ^3f is injective;

$$\text{if } B = {}^jB \text{ for each } j \in \{1,\dots,5\}, \text{ the homomorphisms } {}^ju,\ {}^jv \text{ are } B\text{-exact for each } j \in \{1,\dots,4\}; \quad \text{(A23)}$$

if also ^2f and ^4f are surjective, ^5f is injective, then ^3f is surjective. In particular, if ^jf is the isomorphism for each $j \in \{1,2,4,5\}$, then ^3f is an isomorphism.

Proof. We consider $^2Z = Coker(^1u)$ and $^2Q = Coker(^1v)$ and a homomorphism $^2g : {}^2Z \to {}^2Q$ induced by ^2f. Then, we consider the commutative diagram

$$\begin{array}{ccccc} ^2Z & \xrightarrow[^2w]{} & ^3X & \xrightarrow[^3u]{} & ^4X \\ {}^2g\downarrow & & {}^3f\downarrow & & {}^4f\downarrow \\ ^2Q & \xrightarrow[^2\xi]{} & ^3Y & \xrightarrow[^3v]{} & ^4Y \end{array} \quad \text{(A24)}$$

with ^2w and $^2\xi$ induced by ^2u and ^2v. In view of (i) in Corollary A1, the homomorphism ^3f is injective. Now we put $^4Z = Ker(^4u)$ and $^4Q = Ker(^4v)$ and $^4g : {}^4Z \to {}^4Q$ to be induced by the homomorphism ^4f. By virtue of (ii) in Corollary 12, the homomorphism ^4g is surjective. Next we consider the following commutative diagram:

$$\begin{array}{ccccc} ^2X & \xrightarrow[^2w]{} & ^3X & \xrightarrow[^3\eta]{} & ^4Z \\ {}^2f\downarrow & & {}^3f\downarrow & & {}^4g\downarrow \\ ^2Y & \xrightarrow[^2v]{} & ^3Y & \xrightarrow[^3\zeta]{} & ^4Q \end{array} \quad \text{(A25)}$$

with $^3\eta$ and $^3\zeta$ possessing the same graph as ^3u and ^3v, respectively. From (ii) in Corollary A1, it follows that the homomorphism ^3f is surjective. □

Proposition A3. *Assume that there is a commutative diagram of G-graded B-complexes*

$$\begin{array}{ccccccccc} 0 & \to & ^1\mathcal{C} & \xrightarrow[p]{} & \mathcal{C} & \xrightarrow[s]{} & ^2\mathcal{C} & \to & 0 \\ & & {}^1h\downarrow & & h\downarrow & & {}^2h\downarrow & & \\ 0 & \to & ^1_1\mathcal{C} & \xrightarrow[^1p]{} & {}_1\mathcal{C} & \xrightarrow[^1s]{} & ^2_1\mathcal{C} & \to & 0 \end{array} \quad \text{(A26)}$$

with B-epigeneric **Z***-graded homomorphisms p, s, ^1h, h, ^2h, ^1p, ^1s, and exact rows. Then $H(^1h) \circ \partial(p,s) = \partial(^1p,{}^1s) \circ H(^2h)$.*

Proof. Let $^2u \in H(^2\mathcal{C})$ and let ^2v be a cocycle of class ^2u. There exists $x \in \mathcal{C}$ such that $s(x) = {}^2v$. In view of Theorem 1 and Definition 6 in [32],

$(\partial(^1p,{}^1s) \circ H(^2h))(^2u) = \partial(^1p,{}^1s)\overline{(^2h(^2v))} = \overline{(^1p)^{-1}(dh(x))} = \overline{^1h(p^{-1}(dx))} = H(^1h)\overline{(p^{-1}(dx))} = (H(^1h) \circ \partial(p,s))(^2u)$. □

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