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Quadruple Integral Involving the Logarithm and Product of Bessel Functions Expressed in Terms of the Lerch Function

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Abstract: In this paper, we have derived and evaluated a quadruple integral whose kernel involves the logarithm and product of Bessel functions of the first kind. A new quadruple integral representation of Catalan's G and Apéry's $\zeta(3)$ constants are produced. Some special cases of the result in terms of fundamental constants are evaluated. All the results in this work are new.

Keywords: Bessel function; quadruple integral; Lerch function; Catalan's constant; Aprèy's constant

MSC: Primary 30E20; 33-01; 33-03; 33-04; 33-33B; 33E20

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1. Significance Statement

Bessel functions were first studied by Daniel Bernoulli [1] and then generalized by Friedrich Bessel [2] and are canonical solutions of Bessel's differential equation (see section (10.13) in [3]). Bessel functions are often used as approximants in the construction of uniform asymptotic approximations and expansions for solutions of linear second-order differential equations containing a parameter (see section (10.72) in [3]). Bessel functions are also used in the physical problem involving small oscillations of a uniform heavy flexible chain (see section (10.73) in [3]). Bessel functions arise in the application of cylindrical symmetry in which the physics is described by Laplace's equation (see section (10.73) in [3]).

The definite integral of the product of Bessel functions, which find importance in many branches of mathematical physics, elasticity, potential theory and applied probability, is studied in the works of Glasser [4] and Chaudhry et al. [5]. Multiple integrals of Bessel functions are used in the geometry of fractal sets and studied in the works of Falconer [6] and Ragab [7].

In this work, our goal is to expand upon the current literature of multiple integrals involving the product of Bessel functions by providing a formal derivation in terms of the Lerch function. It is our hope that researchers will find this new integral formula useful for current and future research work where applicable. Consequently, any new result on multiple integrals of the product of Bessel functions is important because of their many applications in applied and pure mathematics.

2. Introduction

In this paper, we derive the quadruple definite integral given by

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty x^{-m} r^{3v-m} y^{m+v} z^{m-2v} J_v(x) J_v(z) e^{-by^2-cr^2} \log^k\left(\frac{ayz}{rx}\right) dx dy dz dr \tag{1}$$

where the parameters k, a, b, p, q, v and m are general complex numbers. The derivations follow the method used by us in [8]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw. \tag{2}$$

where C is in general an open contour in the complex plane, where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of x, y, z and r, and then take a definite quadruple integral of both sides. This yields a definite integral in terms of a contour integral. Then, we multiply both sides of Equation (2) by another function of x, y, z and r and take the infinite sum of both sides such that the contour integrals of both equations are the same.

3. Definite Integral of the Contour Integral

We use the method in [8]. The variable of integration in the contour integral is $\alpha = w + m$. The cut and contour are in the first quadrant of the complex α-plane. The cut approaches the origin from the interior of the first quadrant, and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula, we form the quadruple integral by replacing y by $\log\left(\frac{ayz}{rx}\right)$ and multiplying by $x^{-m}r^{3v-m}y^{m+v}z^{m-2v}J_v(x)J_v(z)e^{-by^2-cr^2}$, then taking the definite integral with respect to $x \in [0,\infty)$, $y \in [0,\infty)$, $z \in [0,\infty)$ and $r \in [0,\infty)$ to obtain

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^{-m}r^{3v-m}y^{m+v}z^{m-2v}J_v(x)J_v(z)e^{-by^2-cr^2} \\
&\log^k\left(\frac{ayz}{rx}\right)dxdydzdr \\
&= \frac{1}{2\pi i}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_C a^w w^{-k-1}x^{-m-w}J_v(x)J_v(z)e^{-by^2-cr^2} \\
&r^{-m+3v-w}y^{m+v+w}z^{m-2v+w}dwdxdydzdr \\
&= \frac{1}{2\pi i}\int_C\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty a^w w^{-k-1}x^{-m-w}J_v(x)J_v(z)e^{-by^2-cr^2} \\
&r^{-m+3v-w}y^{m+v+w}z^{m-2v+w}dxdydzdrdw \\
&= \frac{1}{2\pi i}\int_C \pi 4^{-v-1}a^w w^{-k-1}b^{\frac{1}{2}(-m-v-w-1)}c^{\frac{1}{2}(m-3v+w-1)} \\
&\sec\left(\frac{1}{2}\pi(m-v+w)\right)dw
\end{aligned} \tag{3}$$

from Equation (3.326.2) in [9] and Equation (521.1) in [10], where $-1/2 < Re(m) < Re(v)+1$, $-1/2 < Re(w+m), Re(b,c) > 0$, and using the reflection Formula (8.334.3) in [9] for the Gamma function. We are able to switch the order of integration over α, x, y, z and r using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C}\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)$.

4. The Lerch Function and Infinite Sum of the Contour Integral

In this section, we use Equation (2) to derive the contour integral representations for the Lerch function.

4.1. The Lerch Function

The Lerch function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{4}$$

where $|z| < 1, v \neq 0, -1, \ldots$ and is continued analytically by its integral representation, given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{5}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4.2. Infinite Sum of the Contour Integral

Using Equation (2) and replacing y by $\log(a)-\frac{\log(b)}{2}+\frac{\log(c)}{2}+\frac{1}{2}i\pi(2y+1)$ and then multiplying both sides by $\pi 2^{-2v-1}(-1)^y b^{\frac{1}{2}(-m-v-1)}c^{\frac{1}{2}(m-3v-1)}\exp(\frac{1}{2}i\pi m(2y+1)-\frac{1}{2}i\pi v$ $(2y+1))$ taking the infinite sum over $y\in[0,\infty)$ and simplifying in terms of the Lerch function, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(k+1)}\pi^{k+1}2^{-2v-1}b^{\frac{1}{2}(-m-v-1)}c^{\frac{1}{2}(m-3v-1)}e^{\frac{1}{2}i\pi(k+m-v)} \\
&\Phi\left(-e^{i\pi(m-v)},-k,\frac{-2i\log(a)+i\log(b)-i\log(c)+\pi}{2\pi}\right) \\
&=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C(-1)^y w^{-k-1}\exp\left(\frac{1}{2}(w(2\log(a)-\log(b)+\log(c))+i\pi(2y+1)(m-v+w))\right)dw \\
&=\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}(-1)^y w^{-k-1}\exp\left(\frac{1}{2}(w(2\log(a)-\log(b)+\log(c))+i\pi(2y+1)(m-v+w))\right)dw \\
&=\frac{1}{2\pi i}\int_C\pi 4^{-v-1}a^w w^{-k-1}b^{\frac{1}{2}(-m-v-w-1)}c^{\frac{1}{2}(m-3v+w-1)} \\
&\sec\left(\frac{1}{2}\pi(m-v+w)\right)dw
\end{aligned}
\tag{6}
$$

from Equation (1.232.2) in [9], where $Im(w+m)>0$ in order for the sum to converge.

5. Definite Integral in Terms of the Lerch Function

Theorem 1. *For all* $k,a\in\mathbb{C},Re(b,c)>0,-1/2<Re(m)<Re(v)+1,$

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^{-m}r^{3v-m}y^{m+v}z^{m-2v}J_v(x)J_v(z)e^{-by^2-cr^2}\log^k\left(\frac{ayz}{rx}\right)dxdydzdr \\
&=\pi^{k+1}2^{-2v-1}b^{\frac{1}{2}(-m-v-1)}c^{\frac{1}{2}(m-3v-1)}e^{\frac{1}{2}i\pi(k+m-v)} \\
&\Phi\left(-e^{i\pi(m-v)},-k,\frac{-2i\log(a)+i\log(b)-i\log(c)+\pi}{2\pi}\right)
\end{aligned}
\tag{7}
$$

Proof. Since the right-hand side of Equation (3) is equal to the right-hand side of Equation (6), we can equate the left-hand sides and simplify the gamma function to obtain the stated result. □

6. Special Cases

Example 1. *The degenerate case.*

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^{-m}r^{3v-m}y^{m+v}z^{m-2v}J_v(x)J_v(z)e^{-by^2-cr^2}dxdydzdr \\
&=\pi 4^{-v-1}b^{\frac{1}{2}(-m-v-1)}c^{\frac{1}{2}(m-3v-1)}\sec\left(\frac{1}{2}\pi(m-v)\right)
\end{aligned}
\tag{8}
$$

Proof. Use Equation (7) and set $k=0$ and simplify using entry (2) in the table below (64:12:7) in [11]. □

Example 2.

$$
\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{r^{9/2}y^{7/2}e^{-r^2-y^2}J_2(x)J_2(z)}{x^{3/2}z^{5/2}\left(\log^2\left(\frac{yz}{rx}\right)+\pi^2\right)}dxdydzdr=\frac{\sqrt{2}-\coth^{-1}\left(\sqrt{2}\right)}{32\pi}
\tag{9}
$$

and

$$
\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{r^{9/2}y^{7/2}e^{-r^2-y^2}J_2(x)J_2(z)\log\left(\frac{yz}{rx}\right)}{x^{3/2}z^{5/2}\left(\log^2\left(\frac{yz}{rx}\right)+\pi^2\right)}dxdydzdr=\frac{1}{64}\left(2\sqrt{2}-\pi\right)
\tag{10}
$$

Proof. Use Equation (7) and set $k = -1, a = -1, b = c = 1, v = 2, m = 3/2$; then, rationalize the denominator and compare the real and imaginary parts and simplify in terms of the incomplete Beta function $B(n, m, x)$ using Equation (9.559) in [9] and entry (3) in the table below (58:4:8) and entry (1) in the table below (64:12:7) in [11]. □

Lemma 1.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{-m} e^{-2\left(r^2+y^2\right)} r^{3v-m} y^{m+v} z^{m-2v} J_v(x) J_v(z)}{\log\left(-\frac{yz}{rx}\right)} dxdydzdr$$
$$= 2^{-4v-1} e^{-\frac{1}{2} i\pi(2m-2v+1)} \left(e^{\frac{1}{2} i\pi(m-v)} - \tan^{-1}\left(e^{\frac{1}{2} i\pi(m-v)}\right)\right) \tag{11}$$

Proof. Use Equation (7) and set $a = -1, b = c = 2, k = -1$ and simplify using entry (5) in the table below (64:12:7) in [11]. □

Example 3.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{x^{17/6} z^{19/6} \left(\log^2\left(\frac{yz}{rx}\right) + \pi^2\right)}$$
$$r^{19/6} y^{17/6} e^{-2\left(r^2+y^2\right)} (x\cos(x) - \sin(x))(z\cos(z) - \sin(z)) \log\left(\frac{yz}{rx}\right) dxdydzdr \tag{12}$$
$$= -\frac{\pi\left(\pi - 2\left(-\sqrt{2} + \sqrt{6} + \sqrt{3}\coth^{-1}\left(2\sqrt{2+\sqrt{3}}\right)\right)\right)}{2048}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{x^{17/6} z^{19/6} \left(\log^2\left(\frac{yz}{rx}\right) + \pi^2\right)}$$
$$r^{19/6} y^{17/6} e^{-2\left(r^2+y^2\right)} (x\cos(x) - \sin(x))(z\cos(z) - \sin(z)) dxdydzdr \tag{13}$$
$$= \frac{4\sqrt{2+\sqrt{3}} - \sqrt{3}\pi - 2\coth^{-1}\left(2\sqrt{2+\sqrt{3}}\right)}{2048}$$

Proof. Use Equation (11) and set $v = 3/2, m = 4/3$; then, rationalize the denominator to equate real and imaginary parts and simplify. □

Lemma 2.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty x^{-m} e^{-\pi r^2 - \pi y^2} r^{3v-m} y^{m+v} z^{m-2v} J_v(x) J_v(z)$$
$$\log^k\left(\frac{iyz}{rx}\right) dxdydzdr \tag{14}$$
$$= -2^{-2v-1} e^{\frac{1}{2} i\pi(k+m-v) - i\pi(m-v)} \pi^{k+\frac{1}{2}(m-3v-1)+\frac{1}{2}(-m-v-1)+1} Li_{-k}\left(-e^{i\pi(m-v)}\right)$$

Proof. Use Equation (7) and set $a = i, b = c = \pi$ and simplify in terms of the Polylogarithm function $Li_k(z)$ using Equation (64:12:2) in [11]. □

Example 4.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{x^{5/2} z^{7/2} \left(4\log^2\left(\frac{yz}{rx}\right) + \pi^2\right)^2}$$
$$r^{7/2} y^{5/2} e^{-\pi\left(r^2+y^2\right)} (x\cos(x) - \sin(x))(z\cos(z) - \sin(z)) \left(\pi^2 - 4\log^2\left(\frac{yz}{rx}\right)\right) dxdydzdr \tag{15}$$
$$= \frac{48G + \pi^2}{6144\sqrt{2}\pi^4}$$

and

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{x^{5/2}z^{7/2}\log^2\left(\frac{iyz}{rx}\right)}$$
$$r^{7/2}y^{5/2}e^{-\pi\left(r^2+y^2\right)}(x\cos(x)-\sin(x))(z\cos(z)-\sin(z))dxdydzdr \tag{16}$$
$$=-\frac{\sqrt[4]{-1}\left(\pi^2-48iG\right)}{1536\pi^4}$$

Proof. Use Equation (14) and set $k=-2, m=1, v=3/2$; then, rationalize the denominator and simplify in terms of Catalan's constant G using Equation (2.2.1.2.7) in [12]. □

Lemma 3.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{\log\left(\frac{yz}{rx}\right)} e^{-r^2-y^2} y^v z^{-2v} x^{-m-n} J_v(x) J_v(z) r^{-m-n+3v}$$
$$(r^m x^m y^n z^n - y^m z^m r^n x^n) dxdydzdr \tag{17}$$
$$= 2^{-2v-1}\left(2e^{\frac{1}{2}i\pi(n-v-1)-\frac{1}{2}i\pi(n-v)}\tan^{-1}\left(e^{\frac{1}{2}i\pi(n-v)}\right)\right.$$
$$\left.-2e^{\frac{1}{2}i\pi(m-v-1)-\frac{1}{2}i\pi(m-v)}\tan^{-1}\left(e^{\frac{1}{2}i\pi(m-v)}\right)\right)$$

Proof. Use Equation (7) and form a second equation by replacing $m \to n$ and take their difference. Next, set $k=-1, a=1, b=c=1$ and simplify using entry (2) in the table below (64:12:7) in [11]. □

Example 5.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{x^{27/10}z^{8/3}\log\left(\frac{yz}{rx}\right)} r^{13/10} y^{4/3} e^{-r^2-y^2} J_{\frac{4}{3}}(x) J_{\frac{4}{3}}(z)$$
$$\left(r^{3/2}x^{3/2}y^{6/5}z^{6/5}-r^{6/5}x^{6/5}y^{3/2}z^{3/2}\right)dxdydzdr \tag{18}$$
$$=\frac{1}{16\ 2^{2/3}}\log\left(\frac{\left(-4-\sqrt{2}+\sqrt{6}\right)\left(\sin\left(\frac{\pi}{15}\right)-1\right)}{\left(4-\sqrt{2}+\sqrt{6}\right)\left(1+\sin\left(\frac{\pi}{15}\right)\right)}\right)$$

Proof. Use Equation (17) and set $v=4/3, m=3/2, n=6/5$ and simplify. □

Example 6.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{r^5y^3e^{-\pi\left(r^2+y^2\right)}J_2(x)J_2(z)}{xz^3\log^3\left(\frac{iyz}{rx}\right)}dxdydzdr=\frac{\zeta(3)}{32\pi^7} \tag{19}$$

Proof. Use Equation (14) and set $k=-3, m=1, v=2$ and simplify in terms of Apéry's constant $\zeta(3)$. □

7. Discussion

In this paper, we have presented a novel method for deriving a new quadruple integral involving the product of Bessel functions along with some interesting special cases using contour integration. We will use our method to expand upon this current work and derive other multiple integrals involving other special functions. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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