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Hankel Transform of the Type 2 (p,q)-Analogue of r-Dowling Numbers

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Abstract: In this paper, type 2 (p,q)-analogues of the r-Whitney numbers of the second kind is defined and a combinatorial interpretation in the context of the A-tableaux is given. Moreover, some convolution-type identities, which are useful in deriving the Hankel transform of the type 2 (p,q)-analogue of the r-Whitney numbers of the second kind are obtained. Finally, the Hankel transform of the type 2 (p,q)-analogue of the r-Dowling numbers are established.

Keywords: r-Whitney numbers; r-Dowling numbers; A-tableaux; convolution identities; binomial transform; Hankel transform

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1. Introduction

Several mathematicians were attracted to work on Hankel matrices because of their connections and applications to some areas in mathematics, physics, and computer science. Several theories and applications of these matrices were established including the Hankel determinant and Hankel transform. The notion of Hankel transform was first introduced in Sloane's sequence A055878 [1] and was later on studied by Layman [2].

The Hankel matrix H_n of order n of a sequence $A = \{a_0, a_1, \ldots, a_n\}$ is defined by

$$H_n = (a_{i+j})_{0\le i,j\le n}.$$

On the other hand, the Hankel determinant h_n of order n of A is defined to be the determinant of the corresponding Hankel matrix of order n. That is, $h_n = \det(H_n)$. The Hankel transform of the sequence A, denoted by $H(A)$, is the sequence $\{h_n\}$ of Hankel determinants of A. For instance, the Hankel transform of the sequence of Catalan numbers $C = \{\frac{1}{n+1}\binom{2n}{n}\}_{n=1}^{\infty}$ is given by

$$H(C) = \{1, 1, 1, \ldots\}$$

and the sequence of the sum of two consecutive Catalan numbers, $a_n = c_n + c_{n+1}$ with c_n, the nth Catalan numbers, has the Hankel transform

$$H(a_n) = \{F_{2n+1}\}_{n=0}^{\infty},$$

where F_n is the nth Fibonacci numbers [2].

One remarkable property of the Hankel transform was established by Layman [2], which states that the Hankel transform of an integer sequence is invariant under binomial and inverse transforms. That is, if A is an integer sequence, B is the binomial transform of A and C is the inverse transform of A, then,

$$H(B(A)) = H(A) \text{ and } H(C(A)) = H(A).$$

This property played an important role in proving that the Hankel transform of the sequence of Bell number $\{B_n\}$ [3] and that of r-Bell numbers $\{B_{n,r}\}$ [4] are equal. Corcino and Corcino [5] also used this property in proving that the Hankel transform of the sequence of generalized Bell numbers $\{G_{n,r,\beta}\}$, known as the (r,β)-Bell numbers. It is important to note that the numbers $G_{n,r,\beta}$ are equivalent to the r-Dowling numbers $D_{m,r}(n)$, which were defined in [6] as:

$$D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n,k),$$

where $W_{m,r}(n,k)$ denotes the r-Whitney numbers of the second kind introduced in [7].

In the same paper, the authors tried to establish the Hankel transform for the q-analogue of (r,β)-Bell numbers by using the q-analogue defined in [8]. However, their attempt was not fruitful. In another paper, Corcino et al. [9] introduced a new way of defining the q-analogue of Stirling-type and Bell-type numbers and established the Hankel transforms for the q-analogue of non-central Bell numbers using the method of Mező. On the other hand, the Hankel transforms of the non-central Dowling numbers and the translated Dowling numbers were investigated using the property established by Layman [2] in [10,11].

Recently, a definition of q-analogue of r-Whitney numbers of the second kind $W_{m,r}[n,k]_q$ was introduced in [12,13] by means of the following triangular recurrence relation:

$$W_{m,r}[n,k]_q = q^{m(k-1)+r} W_{m,r}[n-1,k-1]_q + [mk+r]_q W_{m,r}[n-1,k]_q, \tag{1}$$

where n and k are nonnegative integers, and the parameters m and r may be real or complex numbers and $W_{m,r}[n,k]_q = 1$ if $n = k$ and $n \geq 0$ and $W_{m,r}[n,k]_q = 0$ if $n < k$ or $n,k < 0$. From this definition, two more forms of q-analogue were defined in [12,13] as:

$$W^*_{m,r}[n,k]_q := q^{-krm\binom{k}{2}} W_{m,r}[n,k]_q, \tag{2}$$

and

$$\widetilde{W}_{m,r}[n,k]_q := q^{kr} W^*_{m,r}[n,k]_q = q^{-m\binom{k}{2}} W_{m,r}[n,k]_q, \tag{3}$$

where $W^*_{m,r}[n,k]_q$ and $\widetilde{W}_{m,r}[n,k]_q$ denote the second and third form of the q-analogue of the r-Whitney numbers of the second kind, respectively. In line with this, three forms of q-analogue of r-Dowling numbers were defined in [12,13] as follows:

$$D_{m,r}[n]_q := \sum_{k=0}^{n} W_{m,r}[n,k]_q, \tag{4}$$

$$D^*_{m,r}[n]_q := \sum_{k=0}^{n} W^*_{m,r}[n,k]_q, \tag{5}$$

and

$$\widetilde{D}_{m,r}[n]_q := \sum_{k=0}^{n} \widetilde{W}_{m,r}[n,k]_q. \tag{6}$$

The r-Dowling numbers $D_{m,r}(n)$ were defined in [6] as:

$$D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n,k),$$

where $W_{m,r}(n,k)$ denotes the r-Whitney numbers of the second kind introduced in [7]. One can easily verify that the r-Whitney numbers of the second kind and r-Dowling numbers are equivalent to (r,β)-Stirling numbers and (r,β)-Bell numbers, respectively. The Hankel transform of the q-analogue of r-Whitney numbers of the second kind was established in [12], while the Hankel transforms of the three forms of the q-analogues of the r-Dowling numbers were derived in [13–15], which are given as follows:

$$H(\widetilde{D}_{m,r}[n]_q) := q^{m\binom{n+1}{3}-rn(n+1)}[0]_{q^m}![1]_{q^m}!\cdots[n]_{q^m}![m]_q^{\binom{n+1}{2}},$$
$$H(D^*_{m,r}[n]_q) := [m]_q^{\binom{n}{2}}q^{\binom{n}{3}+r\binom{n}{2}}\prod_{k=0}^{n-1}[k]_q^m!,$$

and

$$H(D_{m,r}[n]_q) := q^{2r\binom{n}{2}+(m+1)\binom{n}{3}}\prod_{i=0}^{n-1}((1-q)a;q)_i\prod_{j=1}^{i}[mj]_q.$$

The (p,q)-analogues of some mathematical concepts, special functions, polynomials, numbers and their generalizations have been the object of investigations of several mathematicians and physicists since 1991. For instance, the (p,q)-analogues of binomial coefficients, derivative operator, Volkenborn integration, Stirling and Bell numbers and their generalizations, Apostol type Bernoulli, Euler, Genocchi, Frobenious–Euler, Fubini, Appell polynomials have been extensively studied by researchers in the papers [16–29]. Moreover, (p,q)-analogues of the Bernstein–Durrmeyer operators, multifarious formulas and properties of the derivation and the integration have been defined and studied in [30,31], which gave two (p,q)-Taylor formulas for polynomials, the formula of (p,q)-integration by part and the fundamental theorem of (p,q)-calculus.

Ehrenborg [32] has established the Hankel transform of q-Stirling numbers and that of q-exponential polynomials or q-Bell polynomials. In the final remark of his paper, he has posed a question whether these Hankel transforms can be extended to (p,q)-analogues of Stirling numbers and exponential polynomials. This question has not been considered and answered in the papers [18,26,27,29], which deal with (p,q)-analogues of Stirling and Bell numbers and their generalizations. With this, the present authors have been motivated to establish such Hankel transforms using the methods employed in [12–15]. In the previous paper of the present authors (see [33]), they have made an initiative to define a (p,q)-analogue of r-Whitney numbers of the second kind by means of the following horizontal generating function:

$$[mt+r]^n_{p,q} = \sum_{k=0}^{n} W_{m,r}[n,k]_{p,q}[mt|m]^{\underline{k}}_{p,q} \tag{7}$$

where

$$[t|m]^{\underline{n}}_{p,q} = \prod_{j=0}^{n-1}[t-jm]_{p,q}. \tag{8}$$

In the desire to establish the Hankel transform of the (p,q)-analogue of r-Whitney numbers of the second kind, they have derived some fundamental properties of $W_{m,r}[n,k]_{p,q}$. However, they have failed to obtain the properties that are necessary for deriving the Hankel transform of $W_{m,r}[n,k]_{p,q}$. To address this shortcoming, it is necessary to introduce and investigate another type of (p,q)-analogue of r-Whitney numbers of the second kind, which is parallel to the definition of the q-analogue of r-Whitney numbers of the second kind in three different forms as shown in (1)–(3). It is also necessary to investigate another type of (p,q)-analogue of r-Dowling numbers, which is parallel to the definition of the q-analogue of r-Dowling numbers as shown in (4)–(6). With this, the (p,q)-analogues considered in [33] may then be called a type 1 (p,q)-analogue of r-Whitney numbers of the second kind and a type 1 (p,q)-analogue of r-Dowling numbers.

This research aims to attain the following objectives:

1. Introduce type 2 (p,q)-analogue of r-Whitney numbers of the second kind is introduced;

2. Derive some combinatorial properties of the type 2 (p,q)-analogue of r-Whitney numbers of the second kind;
3. Establish an explicit formula of the type 2 (p,q)-analogue of the r-Whitney numbers of the second kind in symmetric function form;
4. Construct a combinatorial interpretation of the type 2 (p,q)-analogue of r-Whitney numbers of the second kind in the terms of A-tableaux;
5. Obtain some convolution identities of the type 2 (p,q)-analogue of r-Whitney numbers of the second kind;
6. Derive the Hankel transform of the type 2 (p,q)-analogue of r-Whitney numbers of the second kind;
7. Define type 2 (p,q)-analogues of the r-Dowling numbers in three forms;
8. Establish the Hankel transform of one of these forms.

2. Material and Method

This research was facilitated with the methods employed in the previous study on (p,q)-analogue of Stirling-type and Bell-type numbers and the Hankel transform of some special numbers and functions [2,4,5,8,12,18,27,32,34]. Particularly, the generating function method was applied to obtain some properties for the (p,q)-analogue of r-Whitney numbers of the second kind as well as the properties for (p,q)-analogue of r-Dowling numbers.

3. Results

In this section, results of the investigation are presented, which are based on the above-mentioned research objectives. Detailed discussion of the proofs are provided, which justify the validity of the results.

3.1. Type 2 (p,q)-Analogue of the r-Whitney numbers of the second kind: First Form

In this section, another type of (p,q)-analogue of the r-Whitney numbers of the second kind, denoted by $W_{m,r}[n,k;t]_{p,q}$, will be introduced, which is called the type 2 (p,q)-analogue of the r-Whitney numbers of the second kind. This is the first form of the type 2 (p,q)-analogue. Some necessary properties will be obtained including the vertical recurrence relation, rational generating function and a certain explicit formula in the symmetric function form.

Now, let us define the type 2 (p,q)-analogue of the r-Whitney numbers of the second kind.

Definition 1. *For nonnegative integers n and k, and real number r, the (p,q)-analogue $W_{m,r}[n,k;t]_{p,q}$ is defined by:*

$$W_{m,r}[n+1,k;t]_{p,q} = q^{m(k-1)+r} W_{m,r}[n,k-1;t]_{p,q} + [mk+r]_{p,q} p^{mt-km} W_{m,r}[n,k;t]_{p,q}. \quad (9)$$

Remark 1. *Using the above recurrence relation, it can be verified that:*

$$W_{m,r}[n,n;t]_{p,q} = q^{m\binom{n}{2}+nr},$$

and

$$W_{m,r}[n,0;t]_{p,q} = p^{nmt}[r]^n_{p,q}.$$

By applying (9), we obtain some properties for $W_{m,r}[n,k;t]_{p,q}$, which are necessary for deriving the desired Hankel transforms. These properties are given in the following theorems:

Theorem 1. *For nonnegative integers n and k, and real number r, the (p,q)-analogue of r-Whitney numbers of the second kind satisfies the following vertical recurrence relation:*

$$W_{m,r}[n+1,k+1;t]_{p,q} = q^{mk+r} \sum_{j=k}^{n} [m(k+1)+r]_{p,q}^{n-j} p^{(n-j)[mt-(k+1)m]} W_{m,r}[j,k;t]_{p,q}. \quad (10)$$

Proof. Replacing k by $k+1$ in (9) gives:

$$W_{m,r}[n+1,k+1;t]_{p,q} = q^{mk+r} W_{m,r}[n,k;t]_{p,q} + [m(k+1)+r]_{p,q} p^{mt-(k+1)m} W_{m,r}[n,k+1;t]_{p,q}.$$

Applying this repeatedly to (9), gives us:

$W_{m,r}[n+1,k+1;t]_{p,q}$

$$\begin{aligned} = \quad & q^{mk+r} W_{m,r}[n,k;t]_{p,q} + q^{mk+r} p^{mt-(k+1)m} [m(k+1)+r]_{p,q} W_{m,r}[n-1,k;t]_{p,q} \\ & + q^{mk+r} p^{2mt-2(k+1)m} [m(k+1)+r]_{p,q}^2 W_{m,r}[n-2,k;t]_{p,q} + \cdots \\ & + [m(k+1)+r]_{p,q}^{n-k} q^{mk+r} p^{(n-k)[mt-(k+1)m]} W_{m,r}[k+1,k+1;t]_{p,q}. \end{aligned}$$

Using the fact that $W_{m,r}[k+1,k+1;t]_{p,q} = W_{m,r}[k,k;t]_{p,q}$ gives (10). □

Theorem 2. *For nonnegative integers n and k, and real number r, the (p,q)-analogue $W_{m,r}[n,k;t]_{p,q}$ satisfies the following rational generating function:*

$$\Psi_k(x) = \sum_{n=k}^{\infty} W_{m,r}[n,k;t]_{p,q} x^{n-k} = \frac{q^{m\binom{k}{2}+kr}}{\prod_{j=0}^{k}(1 - xp^{m(t-j)}[mj+r]_{p,q})}. \quad (11)$$

Proof. When $k=0$,

$$\Psi_0(x) = \sum_{n=0}^{\infty} W_{m,r}[n,0;t]_{p,q} x^n = \frac{1}{(1 - xp^{mt}[r]_{p,q})}.$$

When $k>0$ and applying the triangular recurrence relation in (9), we have

$$\Psi_k(x) = q^{m(k-1)+r} \Psi_{k-1}(x) + xp^{m(t-k)}[mk+r]_{p,q} \Psi_k(x).$$

Solving for $\Psi_k(t)$

$$\Psi_k(x) = \frac{q^{m(k-1)+r}}{1 - xp^{m(t-k)}[mk+r]_{p,q}} \Psi_{k-1}(x).$$

By backward substitution, we obtain:

$$\Psi_k(t) = \frac{q^{m\binom{k}{2}+kr}}{\prod_{j=0}^{k}(1 - xp^{m(t-j)}[mj+r]_{p,q})}.$$

□

Now, to establish a homogeneous symmetric function form for $W_{m,r}[n,k;t]_{p,q}$, note that (11) can be expressed as follows:

$$\begin{aligned}\sum_{n\geq k} W_{m,r}[n,k;t]_{p,q}x^{n-k} =& q^{m\binom{k}{2}+kr}\prod_{j=0}^{k}\sum_{n\geq 0}(p^{nm(t-j)}[mj+r]^{n}_{p,q}x^{n})\\ =& q^{m\binom{k}{2}+kr}\sum_{n\geq k}\sum_{S_1+S_2+\cdots+S_k=n-k}\prod_{j=0}^{k}(p^{m(t-j)}[mj+r]_{p,q})^{S_j}x^{n-k}.\end{aligned}$$

Comparing the coefficient of x^{n-k}, we have:

$$W_{m,r}[n,k;t]_{p,q} = q^{m\binom{k}{2}+kr}\sum_{S_0+S_1+\cdots+S_k=n-k}\prod_{j=0}^{k}(p^{m(t-j)}[mj+r]_{p,q})^{S_j}. \tag{12}$$

This is equivalent with the next theorem.

Theorem 3. *For nonnegative integers n and k, the explicit formula for $W_{m,r}[n,k;t]_{p,q}$ in the homogeneous symmetric function form is given by:*

$$W_{m,r}[n,k;t]_{p,q} = \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{n-k}\leq k} q^{m\binom{k}{2}+kr}\prod_{i=1}^{n-k}p^{m(t-j_i)}[mj_i+r]_{p,q}. \tag{13}$$

Proof. Note that each term of the sum in (12) is of the form:

$$(p^{m(t-j)}[m(0)+r]_{p,q})^{S_0}(p^{m(t-j)}[m(1)+r]_{p,q})^{S_1}\ldots(p^{m(t-j)}[m(k)+r]_{p,q})^{S_k},$$

where $S_0, S_1, \cdots, S_k$ are nonnegative integers satisfying

$$S_0+S_1+\cdots+S_k = n-k.$$

This means that the each term is just a product $n-k$ factors which are not necessarily distinct factors. That is, we can write each term as a product:

$$(p^{m(t-j)}[mj_1+r]_{p,q})(p^{m(t-j)}[mj_2+r]_{p,q})\ldots(p^{m(t-j)}[mj_{n-k}+r]_{p,q}).$$

where $j_1, j_2, \ldots, j_k \in \{0,1,\ldots,k\}$ and j_i's are not necessarily distinct, i.e.

$$0\leq j_1\leq j_2\leq\cdots\leq j_{n-k}\leq k.$$

Thus,

$$\sum_{S_0+S_1+\cdots+S_k=n-k}\prod_{j=0}^{k}(p^{m(t-j)}[mj+r]_{p,q})^{S_j} = \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{n-k}\leq k}\prod_{i=1}^{n-k}p^{m(t-j_i)}[mj_i+r]_{p,q}.$$

With this, (12) implies (13). □

3.2. Type 2 (p,q)-Analogue of r-Whitney Numbers of the Second kind: Second Form

In this section, the second form of the type 2 (p,q)-analogue of r-Whitney numbers of the second kind, denoted by $W^*_{m,r}[n,k]_{p,q}$, will be introduced. This is defined in terms of the first form of type 2 (p,q)-analogue of r-Whitney numbers of the second kind as follows:

$$W^*_{m,r}[n,k;t]_{p,q} := q^{-kr-m\binom{k}{2}}W_{m,r}[n,k;t]_{p,q}. \tag{14}$$

From the explicit formula in symmetric form in Theorem 3, we have:

$$W^*_{m,r}[n,k;t]_{p,q} = \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} p^{m(t-j_i)}[mj_i+r]_{p,q}. \tag{15}$$

Definition 2 ([18])**.** *An A-tableau is a list ϕ of column c of a Ferrer's diagram of a partition λ (by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A=(r_i)_{i\ge 0}$, a strictly increasing sequence of nonnegative integers.*

Now, let ω be a function from the set of nonnegative integers N to a ring K and suppose Φ is an A-tableau with l columns of lengths $|c|\le h$. We use $T^A_r(h,l)$ to denote the set of such A-tableaux. Then, we set:

$$\omega_A(\Phi) = \prod_{c\in\phi} \omega|c|.$$

Note that Φ might contain a finite number of columns whose lengths are zero since $0\in A=\{0,1,2,\ldots,k\}$. From this point onward, whenever an A-tableau is mentioned, it is always associated with the sequence $A=\{0,1,2,\ldots,k\}$.

The next theorem expresses $W^*_{m,r}[n,k;t]_{p,q}$ in terms of a sum of weights of A-tableaux.

Theorem 4. *Let $\omega : N\to K$ denote a function from N to a ring K over $\mathbb{C}$ (column weights according to length), which is defined by $\omega(|c|) = p^{m(t-|c|)}[m|c|+r]_{p,q}$, where r is a complex number, and $|c|$ is the length of column c of an A-tableau in $T^A_r(k,n-k)$. Then,*

$$W^*_{m,r}[n,k;t]_{p,q} = \sum_{\Phi\in T^A_r(k,n-k)} \prod_{c\in\Phi} \omega(|c|). \tag{16}$$

Proof. Let $\Phi\in T^A_r(k,n-k)$. This implies that Φ has exactly $n-k$ columns, say $c_1,c_2,\ldots,c_{n-k}$, whose lengths are $j_1,j_2,\ldots,j_{n-k}$, respectively. Moreover, for each column $c_i\in\Phi, i=1,2,\ldots,n-k$, we have $|c_i|=j_i$ and $\omega(|c_i|)=p^{m(t-j_i)}[mj_i+r]_{p,q}$. Hence, we get:

$$\prod_{c\in\Phi}\omega(|c|) = \prod_{i=1}^{n-k}\omega(|c_i|) = \prod_{i=1}^{n-k} p^{m(t-j_i)}[mj_i+r]_{p,q}.$$

Since $\Phi\in T^A_r(k,n-k)$, then:

$$\begin{aligned}\sum_{\Phi\in T^A_r(k,n-k)}\prod_{c\in\Phi}\omega(|c|) &= \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k}\prod_{i=1}^{n-k}\omega(|c_i|)\\ &= \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k}\prod_{i=1}^{n-k} p^{m(t-j_i)}[mj_i+r]_{p,q}\\ &= W^*_{m,r}[n,k;t]_{p,q}.\end{aligned}$$

□

Suppose that $r=r_1+r_2$, for some numbers r_1 and r_2. Then the explicit formula in (15) can be rewritten as:

$$W^*_{m,r}[n,k;t]_{p,q} = \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} p^{m(t-j_i)}[(mj_i+r_1)+r_2]_{p,q}.$$

For any $\phi \in T_r^A(n, n-k)$,

$$\omega_A(\phi) = \prod_{c \in \phi} p^{m(t-j_i)}[(mj_i + r_1) + r_2]_{p,q},$$

where $|c| \in \{0, 1, 2, \ldots, k\}$. Note that:

$$\begin{aligned}
p^{m(t-j_i)}[(mj_i + r_1) + r_2]_{p,q} &= p^{mt-mj_i} p^{mj_i+r_1+r_2-1}[mj_i + r_1 + r_2]_{\frac{q}{p}} \\
&= p^{mt+r_1+r_2-1}[(mj_1 + r_1) - (-r_2)]_{\frac{q}{p}} \\
&= p^{mt+r_1+r_2-1}\left(\frac{q}{p}\right)^{r_2}\left([mj_i + r_1]_{\frac{q}{p}} - [-r_2]_{\frac{q}{p}}\right) \\
&= p^{mt+r_1+r_2-1}\left(\frac{q}{p}\right)^{r_2}\left([mj_i + r_1]_{\frac{q}{p}} - \left(-\left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}\right)\right) \\
&= p^{mt+r_1-1}q^{r_2}\left([mj_i + r_1]_{\frac{q}{p}} + \left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\omega_A(\phi) &= \prod_{i=1}^{n-k} p^{mt+r_1-1}q^{r_2}\left([mj_i + r_1]_{\frac{q}{p}} + \left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}\right) \\
&= \left(p^{mt+r_1-1}\right)^{n-k} q^{(n-k)r_2} \prod_{i=1}^{n-k}\left([mj_i + r_1]_{\frac{q}{p}} + \left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}\right) \\
&= \left(p^{mt+r_1-1}q^{r_2}\right)^{n-k} \sum_{l=0}^{n-k}\left\{\left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}\right\}^{n-k-l} \sum_{j_1 \le q_1 \le q_2 \le \cdots \le q_l \le j_{n-k}} \prod_{i=1}^{l} \omega^*(q_i),
\end{aligned}$$

where $\omega^*(q_i) = [mj_i + r_1]_{\frac{q}{p}}$.

Suppose that B_ϕ is the set of all A-tableaux corresponding to ϕ such that for each $\psi \in B_\phi$, one of the following is true:

ψ has no columns whose weight is $[r_2]_{\frac{q}{p}}$;

ψ has one columns whose weight is $[r_2]_{\frac{q}{p}}$;

ψ has two columns whose weight is $[r_2]_{\frac{q}{p}}$;

$\vdots$

ψ has $n-k$ columns whose weight is $[r_2]_{\frac{q}{p}}$.

Thus, we may write:

$$\omega_A(\phi) = \sum_{\psi \in B_\phi} \omega_A(\psi).$$

Now, if l columns in ψ with weights other than $\left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}$, then:

$$\omega_A(\psi) = \prod_{c \in \psi} \omega^*(|c|) = \left(p^{mt+r_1-1}q^{r_2}\right)^{n-k}\left(\left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}\right)^{n-k-l} \prod_{i=1}^{r} \omega^*(q_i),$$

where $q_1, q_2, \ldots, q_l \in \{j_1, j_2, \ldots, j_{n-k}\}$. Note that, for each l, there corresponds an $\binom{n-k}{l}$ tableaux with l columns having weights $\omega^*(q_i) = [mq_i + r_1]_{\frac{q}{p}}$. It can easily be verified that

$$|T_r^A(k, n-k)| = \binom{n-k+k}{n-k} = \binom{n}{n-k} = \binom{n}{k}.$$

Thus, for all $\phi \in T_r^A(k, n-k)$, B_ϕ contains a total of $\binom{n}{k}\binom{n-k}{l}$ A-tableaux with l columns of weights $\omega^*(q_i)$. However, only $\binom{l+k}{l}$ tableaux with l columns in B_ϕ are distinct. Hence, every distinct tableaux ψ with l columns of weight other than $\left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}$ appears

$$\frac{\binom{n}{k}\binom{n-k}{l}}{\binom{l+k}{l}} = \binom{n}{l+k}$$

times in the collection. Thus, we have:

$$\sum_{\phi \in T_r^A(k,n-k)} \omega_A(\phi) = \sum_{l=0}^{n-k} \binom{n}{l+k} \left(p^{mt+r_1-1}q^{r_2}\right)^{n-k} \left(\left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}\right)^{n-k-l} \sum_{\varphi \in \bar{B}_l} \prod_{c \in \varphi} \omega^*(|c|),$$

where $\bar{B}_l$ denotes the set of all tableaux φ having l columns of weights $\omega^*(q_i) = [mq_i + r_2]_{\frac{q}{p}}$. Reindexing the sum, we get:

$$W^*_{m,r}[n,k;t]_{p,q} = \sum_{j=k}^{n} \binom{n}{j} \left(p^{mt+r_1-1}q^{r_2}\right)^{n} \left(\left(\frac{p}{q}\right)^{r_2}[r_2]_{\frac{q}{p}}\right)^{n-j} \sum_{\varphi \in \bar{B}_{j-k}} \prod_{c \in \varphi} \omega^*(|c|),$$

where $\bar{B}_{j-k}$ is the set of all tableaux with $j-k$ columns of weights $\omega^*(q_i) = [mq_i + r_1]_{\frac{q}{p}}$, for each $i = 1, 2, \ldots, j-k$. Clearly, $\bar{B}_{j-k} = T^A_{r_1}(k, j-k)$. Hence,

$$\begin{aligned} W^*_{m,r}[n,k;t]_{p,q} &= \sum_{j=k}^{n} \binom{n}{j} \left(p^{mt+r_1-1}q^{r_2}\right)^{n} \left(\frac{p}{q}\right)^{(n-j)r_2} \left([r_2]_{\frac{q}{p}}\right)^{n-j} \sum_{\varphi \in \bar{B}_{j-k}} \prod_{c \in \varphi} \omega^*(|c|) \\ &= \sum_{j=k}^{n} \binom{n}{j} \left(p^{mt+r_1-1}q^{r_2}\right)^{n} \left(\frac{p}{q}\right)^{(n-j)r_2} \left([r_2]_{\frac{q}{p}}\right)^{n-j} W^*_{m,r_1}[j,k;t]_{\frac{q}{p}}. \end{aligned}$$

This result is formally stated in the next theorem.

Theorem 5. *The (p,q)-analogue $W^*_{m,r}[n,k;t]_{p,q}$ satisfies the following identity:*

$$W^*_{m,r}[n,k;t]_{p,q} = \sum_{j=k}^{n} \binom{n}{j} \left(p^{mt+r_1-1}q^{r_2}\right)^{n} \left(\frac{p}{q}\right)^{(n-j)r_2} \left([r_2]_{\frac{q}{p}}\right)^{n-j} W^*_{m,r_1}[j,k;t]_{\frac{q}{p}}, \tag{17}$$

where $r - r_1 + r_2$ for some numbers r_1 and r_2.

Now, suppose:

ϕ_1 is a tableau with $k-l$ columns whose lengths are in the set $\{0, 1, \ldots, l\}$; and
ϕ_2 is a tableau with $n-k-j$ columns whose lengths are in the set $\{l+1, l+2, \ldots, l+j+1\}$.

Then,

$$\phi_1 \in T^{A_1}(l, k-l) \text{ and } \phi_2 \in T^{A_2}(j, n-k-j),$$

where $A_1 = \{0, 1, \ldots, l\}$ and $A_2 = \{l+1, l+2, \ldots, l+j+1\}$. We can generate an A-tableau ϕ with $n-l-j$ columns whose lengths are in the set $A = A_1 \cup A_2 = \{0, 1, \ldots, l+j+1\}$ by joining the columns of ϕ_1 and ϕ_2. Hence, for $\phi \in T^A(l+j+1, n-l-j)$, we have:

$$\sum_{\phi \in T^A(l+j+1,n-l-j)} \omega_A(\phi) = \sum_{k=l}^{n-j} \left\{ \sum_{\phi_1 \in T^{A_1}(l,k-l)} \omega_{A_1}(\phi_1) \right\} \left\{ \sum_{\phi_2 \in T^{A_2}(j,n-k-j)} \omega_{A_2}(\phi_2) \right\}.$$

Note that:

$$\sum_{\phi_2 \in T^{A_2}(j,n-k-j)} \omega_{A_2}(\phi_2) = \sum_{0 \leq g_1 \leq \leq \cdots \leq g_{n-k-j} \leq j} \prod_{i=1}^{n-k-j} p^{m(t-g_i)}[mg_i + m(l+1) + r]_{p,q}.$$

Thus, using Equation (15), we obtain the following theorem.

Theorem 6. *The $W^*_{m,r}[n,k;t]_{p,q}$ satisfy the following convolution-type identity:*

$$W^*_{m,r}[n+1, l+j+1; t]_{p,q} = \sum_{k=0}^{n} W^*_{m,r}[k,l;t]_{p,q} W^*_{m,r+m(l+1)}[n-k,j;t]_{p,q}. \tag{18}$$

The next theorem provides another convolution-type identity.

Theorem 7. *The $W^*_{m,r}[n,k;t]_{p,q}$ satisfy the following second form convolution-type identity:*

$$W^*_{m,r}[j+l, n; t]_{p,q} = \sum_{k=l}^{n-j} W^*_{m,r}[l,k;t]_{p,q} W^*_{m,r+km}[j,n-k;t]_{p,q}. \tag{19}$$

Proof. Let
ϕ_1 be a tableau with $l-k$ columns whose lengths are in the set $A_1 = \{0,1,\ldots,k\}$, and
ϕ_2 be a tableau with $j-n+k$ columns whose lengths are in the set $A_2 = \{k, k+1, \ldots, n\}$;
then, $\phi_1 \in T^{A_1}(k, l-k)$ and $\phi_2 \in T^{A_2}(n-k, j-n+k)$. Using the same argument with the proof in the previous theorem, we can obtain the desired convolution formula. □

Notice that (19) can be written as:

$$W^*_{m,r}[s+p, u; t]_{p,q} = \sum_{k=max\{0,u-p\}}^{min\{u,s\}} W^*_{m,r}[s,k;t]_{p,q} W^*_{m,r+mk}[p,u-k;t]_{p,q}.$$

Replacing s with $s+i$, p with j, and u with $s+j$, we get:

$$W^*_{m,r}[s+i+j, s+j; t]_{p,q} = \sum_{k=s}^{min\{s+j,s+i\}} W^*_{m,r}[s+i,k;t]_{p,q} W^*_{m,r+mk}[j,s+j-k;t]_{p,q}. \tag{20}$$

This gives the LU factorization of the matrix:

$$\begin{pmatrix} W^*_{m,r}[s,s;t]_{p,q} & W^*_{m,r}[s+1,s+1;t]_{p,q} & \cdots & W^*_{m,r}[s+n,s+n;t]_{p,q} \\ W^*_{m,r}[s+1,s;t]_{p,q} & W^*_{m,r}[s+2,s+1;t]_{p,q} & \cdots & W^*_{m,r}[s+n+1,s+n;t]_{p,q} \\ \vdots & \vdots & \cdots & \vdots \\ W^*_{m,r}[s+n,s;t]_{p,q} & W^*_{m,r}[s+n,s+1;t]_{p,q} & \cdots & W^*_{m,r}[s+2n,s+n;t]_{p,q} \end{pmatrix}$$
$$= \begin{pmatrix} W^*_{m,r}[s,s;t]_{p,q} & 0 & \cdots & 0 \\ W^*_{m,r}[s+1,s;t]_{p,q} & W^*_{m,r}[s+1,s+1;t]_{p,q} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ W^*_{m,r}[s+n,s;t]_{p,q} & W^*_{m,r}[s+n,s+1;t]_{p,q} & \cdots & W^*_{m,r}[s+n,s+n;t]_{p,q} \end{pmatrix}$$
$$\times \begin{pmatrix} W^*_{m,r+ms}[0,0;t]_{p,q} & W^*_{m,r+ms}[1,1;t]_{p,q} & \cdots & W^*_{m,r+ms}[n,n;t]_{p,q} \\ 0 & W^*_{m,r+m(s+1)}[1,0;t]_{p,q} & \cdots & W^*_{m,r+m(s+1)}[n,n-1;t]_{p,q} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & W^*_{m,r+m(s+n)}[n,0;t]_{p,q} \end{pmatrix}.$$

This implies that:

$$\det(W^*_{m,r}[s+i+j,s+j;t]_{p,q})_{0\le i,j\le n}$$
$$= \left(\prod_{k=0}^{n} W^*_{m,r}[s+k,s+k;t]_{p,q}\right)\left(\prod_{k=0}^{n} W^*_{m,m(s+k)+r}[k,0;t]_{p,q}\right).$$

Since

$$W^*_{m,r}[n,n;t]_{p,q} = q^{-nr-m\binom{n}{2}} W_{m,r}[n,n;t]_{p,q} = q^{-nr-m\binom{n}{2}} q^{nr+m\binom{n}{2}} = 1$$

and

$$W^*_{m,r}[n,0;t]_{p,q} = q^{-0r-m\binom{0}{2}} W_{m,r}[0,0;t]_{p,q} = p^{nmt}[r]^n_{p,q},$$

we have the following theorem.

Theorem 8. *For nonnegative integers n and k, the Hankel transform for $W^*_{m,r}[n,k;t]_{p,q}$ is given by:*

$$\det(W^*_{m,r}[s+i+j,s+j;t]_{p,q})_{0\le i,j\le n} = \prod_{k=0}^{n} p^{nmt}[m(s+k)+r]^k_{p,q}. \tag{21}$$

Recall that:

$$W_{m,r}[n,k;t]_{p,q} = q^{m\binom{k}{2}+kr} W^*_{m,r}[n,k;t]_{p,q}.$$

Multiplying both sides of Equation (20) by $q^{m\binom{s+j}{2}+(s+j)r}$, we have

$$W_{m,r}[s+i+j,s+j;t]_{p,q} =$$
$$\sum_{k=s}^{min\{s+j,s+i\}} q^{m\binom{s+j}{2}+(s+j)r} W^*_{m,r}[s+i,k;t]_{p,q} W^*_{m,r+mk}[j,s+j-k;t]_{p,q}.$$

Note that,

$$q^{m\binom{k}{2}+kr} q^{m\binom{s+j-k}{2}+(s+j-k)+(s+j-k)(r+mk)} = q^{m\left[\binom{k}{2}+\binom{s+j-k}{2}\right]+kr+(s+j-k)(r+mk)}$$

$$m\left[\binom{k}{2}+\binom{s+j-k}{2}\right] = m\binom{s+j}{2} + mk^2 - mk(s+j).$$

Hence,

$$\begin{aligned} q^{m\binom{k}{2}+kr} q^{m\binom{s+j-k}{2}+(s+j-k)(r+mk)} &= q^{m\binom{s+j}{2}+mk^2-mk(s+j)+kr+(s+j-k)(r+mk)} \\ &= q^{m\binom{s+j}{2}+kr+mk^2-mk(s+j)(s+j)r-kr+mk(s+j)-mk^2} \\ &= q^{m\binom{s+j}{2}+(s+j)r}. \end{aligned}$$

Thus, we have:

$$W_{m,r}[s+i+j,s+j;t]_{p,q} = \sum_{k=s}^{min\{s+j,s+i\}} W_{m,r}[s+i,k;t]_{p,q} W_{m,r+mk}[j,s+j-k;t]_{p,q}$$

This implies that:

$$\det\left(W_{m,r}[s+i+j,s+j;t]_{p,q}\right)_{0\le i,j\le n} =$$
$$\left(\prod_{k=0}^{n} W_{m,r}[s+k,s+k;t]_{p,q}\right)\left(\prod_{k=0}^{n} W_{m,m(s+k)+r}[k,0;t]_{p,q}\right).$$

Recall that
$$W_{m,r}[s+k,s+k;t]_{p,q} = q^{m\binom{s+k}{2}+(s+k)r}$$
and
$$W_{m,r+m(s+k)}[k,0;t]_{p,q} = p^{nmt}[m(s+k)+r]^k_{p,q}.$$

Thus, we have:
$$\det\left(W_{m,r}[s+i+j,s+j;t]_{p,q}\right)_{0\le i,j\le n} = \left(\prod_{k=0}^{n} q^{m\binom{s+k}{2}+(s+k)r}\right)\left(\prod_{k=0}^{n} p^{nmt}[m(s+k)+r]^k_{p,q}\right).$$

Hence, we established the following corollary.

Corollary 1. *For nonnegative integers n and k, the Hankel transform for $W_{m,r}[n,k;t]_{p,q}$ is given by:*
$$\det\left(W_{m,r}[s+i+j,s+j;t]_{p,q}\right)_{0\le i,j\le n} = \prod_{k=0}^{n} q^{m\binom{s+k}{2}+(s+k)r} p^{nmt}[m(s+k)+r]^k_{p,q}. \tag{22}$$

3.3. Hankel Transform of Type 2 (p,q)-Analogue of r-Dowling Numbers

A (p,q)-analogue of r-Dowling Numbers that has been investigated in [33] is defined as the sum of type 1 (p,q)-Analogue of r-Whitney numbers of the second kind. We may also call this a type 1 (p,q)-analogue of r-Dowling numbers. In this section, the type 2 (p,q)-analogue of the r-Dowling numbers will be defined in three different forms. Moreover, the Hankel transform of the second form of the type 2 (p,q)-analogue of r-Dowling numbers will be derived.

Now, let us define the three forms of type 2 (p,q)-analogue of the r-Dowling numbers.

Definition 3. *The first, second and third forms of type 2 (p,q)-analogue of the r-Dowling numbers, denoted by $D_{m,r}[n]_{p,q}$, $D^*_{m,r}[n]_{p,q}$ and $\widetilde{D}_{m,r}[n]_{p,q}$, respectively, are defined as follows:*
$$D_{m,r}[n]_{p,q} := \sum_{k=0}^{n} W_{m,r}[n,k;t]_{p,q}, \tag{23}$$
$$D^*_{m,r}[n]_{p,q} := \sum_{k=0}^{n} W^*_{m,r}[n,k;t]_{p,q}, \tag{24}$$
$$\widetilde{D}_{m,r}[n]_{p,q} := \sum_{k=0}^{n} \widetilde{W}_{m,r}[n,k;t]_{p,q}, \tag{25}$$
where $W_{m,r}[n,k;t]_{p,q}$,
$$W^*_{m,r}[n,k;t]_{p,q} = q^{-kr-m\binom{k}{2}} W_{m,r}[n,k;t]_{p,q},$$
and
$$\widetilde{W}_{m,r}[n,k;t]_{p,q} = q^{kr} W^*_{m,r}[n,k;t]_{p,q},$$
denote the first, second and third forms of the (p,q)-analogue of the r-Whitney numbers of the second kind, respectively.

Our focus in this section is on the second form of type 2 (p,q)-analogue of the r-Dowling numbers in (24), particularly, its Hankel transform. The other forms will be considered in separate papers.

In deriving the Hankel transform of $D^*_{m,r}[n]_{p,q}$, the following theorem is necessary.

Theorem 9. *The (p,q)-analogue of r-Dowling numbers satisfy the following relation:*

$$D^*_{m,r+1}[n]_{p,q} = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{q}{p}\right)^j D^*_{m,r}[j]_{\frac{q}{p}}.$$

Proof. Letting $r_2 = 1, r = r_1 + 1$ in Theorem 5 yields:

$$\begin{aligned} W^*_{m,r}[n,k;t]_{p,q} &= \sum_{j=k}^{n} \binom{n}{j} \left(p^{mt+r_1-1}\right)^n q^n \left(\frac{p}{q}\right)^{(n-j)} W^*_{m,r_1}[j,k;t]_{\frac{q}{p}} \\ &= \sum_{j=k}^{n} \binom{n}{j} \left(p^{mt+r-2} \cdot p\right)^n \frac{q^n}{q^n} \left(\frac{q}{p}\right)^j W^*_{m,r_1}[j,k;t]_{\frac{q}{p}} \\ &= \sum_{j=k}^{n} \binom{n}{j} \left(p^{mt+r-1}\right)^n \left(\frac{q}{p}\right)^j W^*_{m,r-1}[j,k;t]_{\frac{q}{p}}. \end{aligned}$$

Then, summing up both sides of the preceding sum over k yields:

$$\begin{aligned} \sum_{k=0}^{n} W^*_{m,r}[n,k;t]_{p,q} &= \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} \left(p^{mt+r-1}\right)^n \left(\frac{q}{p}\right)^j W^*_{m,r-1}[j,k;t]_{\frac{q}{p}} \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(p^{mt+r-1}\right)^n \left(\frac{q}{p}\right)^j \sum_{k=0}^{j} W^*_{m,r-1}[j,k;t]_{\frac{q}{p}}. \end{aligned}$$

Hence,

$$D^*_{m,r}[n]_{p,q} = \sum_{j=0}^{n} \binom{n}{j} \left(p^{mt+r-1}\right)^n \left(\frac{q}{p}\right)^j D^*_{m,r-1}[j]_{\frac{q}{p}}.$$

Taking $t = \frac{1-r}{m}$ gives $\left(p^{mt+r-1}\right)^n = 1$. Thus,

$$D^*_{m,r}[n]_{p,q} = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{q}{p}\right)^j D^*_{m,r-1}[j]_{\frac{q}{p}},$$

which is exactly the desired relation. □

As a direct consequence of Theorem 9, we have the following corollary.

Corollary 2. *The (p,q)-analogue of r-Dowling numbers satisfy the following relation:*

$$\left(\frac{q}{p}\right)^n D^*_{m,r}[n]_{\frac{q}{p}} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} D^*_{m,r+1}[j]_{p,q}.$$

To establish the Hankel tranform of $D^*_{m,r}[n]_{p,q}$, the concept of rising k-binomial transform by Spivey and Steil [34] as well as its property in relation to the Hankel transform are necessary. In Spivey and Steil [34], the rising k-binomial transform R of a sequence $A = \{a_n\}$ is the sequence $R(A;k) = \{r_n\}$, where r_n is given by:

$$r_n = \sum_{j=0}^{n} \binom{n}{j} k^j a_j, \quad k \neq 0.$$

Hence, we use $R(A,k)$ to denote the set of rising k-binomial transform of A. Then, given a sequence $A = \{a_0, a_1, \ldots\}$ and $H(A) = \{h_n\}$,

$$H(R(A,0)) = \{a_0, 0, 0, \ldots\}.$$

If $k \neq 0$,

$$H(R(A,k)) = \{k^{n(n+1)} h_n\},$$

as seen in [34]. We are now ready to state the final theorem of this paper.

Theorem 10. *The Hankel transform of the sequence of (p,q)-analogue of r-Dowling numbers $\{D^*_{m,r}[n]_{p,q}\}$ is given by:*

$$H(D^*_{m,r}[n]_{p,q}) = \left(\frac{q}{p}\right)^{\frac{n(n^2+3n+8)}{6}+r-1\binom{n}{2}} ([m]_{\frac{q}{p}})^{\binom{n}{2}} \prod_{k=0}^{n-1} [k]_{\left(\frac{q}{p}\right)^m}!. \tag{26}$$

Proof. From Corollary 2, we say that $D^*_{m,r}[n]_{p,q}$ is the binomial transform of $\left(\frac{q}{p}\right)^n D^*_{m,r}[n]_{\frac{q}{p}}$, that means that:

$$B\left(\left(\frac{q}{p}\right)^n D^*_{m,r}[n]_{\frac{q}{p}}\right) = D^*_{m,r+1}[n]_{p,q}.$$

By Layman's theorem [2],

$$H\left(B\left(\left(\frac{q}{p}\right)^n D^*_{m,r}[n]_{\frac{q}{p}}\right)\right) = H\left(\left(\frac{q}{p}\right)^n D^*_{m,r}[n]_{\frac{q}{p}}\right),$$

that is,

$$H(D^*_{m,r+1}[n]_{p,q}) = H\left(\left(\frac{q}{p}\right)^n D^*_{m,r}[j]_{\frac{q}{p}}\right).$$

Now, Theorem 9 can be stated, as $D^*_{m,r+1}[n]_{p,q}$ is the rising $\frac{q}{p}$-binomial transform of $D^*_{m,r}[j]_{\frac{q}{p}}$. Using the Spivey–Steil Theorem, with $A = \{D^*_{m,r}[j]_{\frac{q}{p}}\}, h_n = H(D^*_{m,r}[j]_{\frac{q}{p}})$ and $r_n = D^*_{m,r+1}[n]_{p,q}$, we have:

$$H(D^*_{m,r+1}[n]_{p,q}) = \left(\frac{q}{p}\right)^{n(n+1)} H(D^*_{m,r}[j]_{\frac{q}{p}}).$$

Recall that the Hankel transform of $D^*_{m,r}[n]_q$ in [15] is given by:

$$H(D^*_{m,r}[n]_q) = ([m]_q)^{\binom{n}{2}} q^{\binom{n}{3}+r\binom{n}{2}} \prod_{k=0}^{n-1} [k]_{q^m}!.$$

Hence,

$$H(D^*_{m,r}[n]_{\frac{q}{p}}) = ([m]_{\frac{q}{p}})^{\binom{n}{2}} \left(\frac{q}{p}\right)^{\binom{n}{3}+r\binom{n}{2}} \prod_{k=0}^{n-1} [k]_{\left(\frac{q}{p}\right)^m}!.$$

Thus,

$$\begin{aligned} H(D^*_{m,r+1}[n]_{p,q}) &= \left(\frac{q}{p}\right)^{n(n+1)} ([m]_{\frac{q}{p}})^{\binom{n}{2}} \left(\frac{q}{p}\right)^{\binom{n}{3}+r\binom{n}{2}} \prod_{k=0}^{n-1} [k]_{\left(\frac{q}{p}\right)^m}! \\ &= \left(\frac{q}{p}\right)^{\frac{n(n^2+3n+8)}{6}+r\binom{n}{2}} ([m]_{\frac{q}{p}})^{\binom{n}{2}} \prod_{k=0}^{n-1} [k]_{\left(\frac{q}{p}\right)^m}!. \end{aligned}$$

□

Remark 2. *When $m = 1, r = 0, p = 1$ and $q \to 1$, the Hankel transform in (26) reduces to:*

$$H(D_{1,0}) = \prod_{k=0}^{n-1} k!,$$

which is the Hankel transform for the noncentral Bell numbers in [35].

Remark 3. *When $r = 0, p = 1$ and $q \to 1$, the Hankel transform in (26) yields:*

$$H(D_{m,0}) = m^{\binom{n}{2}} \prod_{k=0}^{n-1} k!,$$

which is the Hankel transform of the (r, β)-Bell numbers $G_{n,\beta,r}$ with $\beta = m$ in [5].

Illustration 1. *When $n = 3, m = 1$ and $r = 2$ in Theorem 10, we have:*

$$\begin{aligned} H(D^*_{1,2}[3]_{p,q}) &= \left(\frac{q}{p}\right)^{\frac{3(3^2+3(3)+8)}{6}+2\binom{3}{2}} ([1]_{\frac{q}{p}})^{\binom{3}{2}} \prod_{k=0}^{3-1} [k]_{\left(\frac{q}{p}\right)^1}! \\ &= \left(\frac{q}{p}\right)^{13+6} \prod_{k=0}^{2} [k]_{\left(\frac{q}{p}\right)}! \\ &= \left(\frac{q}{p}\right)^{19} [0]_{\left(\frac{q}{p}\right)}![1]_{\left(\frac{q}{p}\right)}![2]_{\left(\frac{q}{p}\right)}!. \end{aligned}$$

When $p = \frac{1}{3}$ and $q = \frac{2}{3}$, this further gives:

$$\begin{aligned} H(D^*_{1,2}[3]_{\frac{1}{3},\frac{2}{3}}) &= 2^{19}[0]_2![1]_2![2]_2! \\ &= 2^{19}(1)\left(\frac{2^1-1}{2-1}\right)\left(\frac{2^2-1}{2-1}\frac{2^1-1}{2-1}\right) \\ &= 2^{19}(1)(1)(3) = 1572864. \end{aligned}$$

4. Discussion

An additional parameter p to q-analogue of some special numbers and polynomials can widen the scope or coverage of representation in terms of their combinatorial and physical applications. For instance, the (p, q)-analogue defined by Remmel and Wachs [27] for generalized Stirling numbers are given application to rook theory. Results obtained in this paper, particularly in Theorem 8, Corollary 1 and Theorem 10, have extended the Hankel transforms of the q-analogues of Stirling-type and Bell-type numbers to their (p, q)-analogues. These give a positive answer to the question posed by Ehrenborg in the final remark of his paper [32] as stated in the introduction. However, these results are simply the translations of the existing results obtained in [12,15], which are good examples that confirm Srivastava's observations in (p. 340, [36]) and (pp. 1511–1512, [37]).

5. Conclusions

In this paper, the type 2 (p, q)-analogue of the r-Whitney numbers of the second kind were defined by means of triangular recurrence relation in three different forms and some combinatorial properties were obtained. A combinatorial interpretation in the context of A-tableaux was also given and convolution-type identities were consequently obtained. One of these convolution-type identities was used to derive the Hankel transform of the second form of the type 2 (p, q)-analogue of the r-Whitney numbers of the second kind. Furthermore, the Hankel transform of the second form of the type 2 (p, q)-analogue of the r-Dowling numbers was established.

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