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A Note on Hermite–Hadamard–Fejer Type Inequalities for Functions Whose n-th Derivatives Are m-Convex or (α, m)-Convex Functions

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Abstract: In this paper, we develop some Hermite–Hadamard–Fejér type inequalities for n-times differentiable functions whose absolute values of n-th derivatives are (α, m)-convex function. The results obtained in this paper are extensions and generalizations of the existing ones. As a special case, the generalization of the remainder term of the midpoint and trapezoidal quadrature formulas are obtained.

Keywords: weight; function; Hermite–Hadamard; Fejér; onepoint integral formula; trapezoid formula; midpoint formula

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1. Introduction

The main objects of our interest are integral inequalities involving weight functions of Hermite–Hadamard–Fejér type for various classes of convex functions.

Let us start by bringing out the well-known definition of convex functions:

Definition 1. *A function $f : [0, b] \to \mathbb{R}$ is said to be convex on $[0, b]$ if*

$$f(tx + (1-t)x) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$.

In [1], Toader established the class of m-convex functions as the following:

Definition 2. *A function $f : [0, b] \to \mathbb{R}$ is said to be an m-convex function (for $m \in (0, 1]$) if*

$$f(\lambda x + m(1-\lambda)y) \leq \lambda f(x) + m(1-\lambda)f(y) \tag{2}$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$.

V.G. Mihesan ([2]) has introduced the generalization of the m-convexity in the following definition:

Definition 3. *A function $f : [0, b] \to \mathbb{R}$ is said to be an (α, m)-convex function (for α, $m \in (0, 1]$) if*

$$f(\lambda x + m(1-\lambda)y) \leq \lambda^{\alpha} f(x) + m(1-\lambda^{\alpha})f(y) \tag{3}$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$.

Remark 1. *It is easy to show that $(1, m)$-convex function is m-convex function for every $m \in (0, 1]$, and 1-convex function is convex. Thus $(1, 1)$-convex function is actually a convex function. It is interesting to mention that there are continuous and differentiable functions which are m-convex, but which are not convex in the classical sense (see [3]).*

Fejér [4] has obtained the weighted version of famous Hermite–Hadamard inequality:

Theorem 1. *If $f : [a,b] \to \mathbb{R}$ is convex, and $w : [a,b] \to [0,+\infty)$ integrable and symmetric about $\frac{a+b}{2}$ then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \le \int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx. \quad (4)$$

Many researchers have been interested in improving and refining Hermite–Hadamard inequalities for various types of convex functions (see for instance [5–9]). In [10], authors established a new integral inequalitites of Hermite–Hadamard type for (α, m)-function.

Theorem 2. *Let $f : [0,\infty) \to \mathbb{R}$ be an n-time differentiable function for $n \in \mathbb{N}$ and let $0 \le a < b < \infty$ and $\alpha, m \in (0,1]$. If $f^{(n)} \in L\left[a, \frac{b}{m}\right]$ and $\left|f^{(n)}\right|^q$ for $q \le 1$ is (α,m)-convex on $\left[0, \frac{b}{m}\right]$, then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(b-t)^{k+1}+(-1)^k(t-a)^{k+1}}{(k+1)!}f^{(k)}(t)\right| \\
\le\ & \frac{1}{(b-a)(n+1)!}\left\{(t-a)^{n+1}\left[\alpha B(n+2,\alpha)\left|f^{(n)}(a)\right|^q\right.\right. \\
+\ & \left. m(1-\alpha B(n+2,\alpha))\left|f^{(n)}\left(\frac{t}{m}\right)\right|^q\right]^{1/q} \\
+\ & \left.(b-t)^{n+1}\left[\frac{1}{n+\alpha+1}\left((n+1)\left|f^{(n)}(t)\right|^q + \alpha m\left|f^{(n)}(\frac{b}{m})\right|^q\right)\right]^{1/q}\right\},
\end{aligned}$$

where $t \in [a,b]$ and $B(\alpha,\beta)$ is the Beta function

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \quad \alpha,\beta > 0.$$

Let us recall the weighted one-point integral formula of Matić, Pečarić and Ujević [11] which is introduced from the general m-point integral identity in [12] and states:

$$\int_a^b w(t)f(t)dt = \sum_{j=1}^n A_{w,j}(x)f^{(j-1)}(x) + (-1)^n\int_a^b W_{n,w}(t,x)f^{(n)}(t)dt, \quad (5)$$

where $f : [a,b] \to \mathbf{R}$ is such that $f^{(n-1)}$ is absolutely continuous function, $w : [a,b] \to [0,\infty)$ is weight function, $x \in [a,b]$

$$A_{w,j}(x) = \frac{(-1)^{j-1}}{(j-1)!}\int_a^b (x-s)^{j-1}w(s)ds, \quad \text{for } j = 1,\dots,n \quad (6)$$

and

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}w(s)ds & \text{for } t \in [a,x], \\ w_{2n}(t) = \frac{1}{(n-1)!}\int_b^t (t-s)^{n-1}w(s)ds & \text{for } t \in (x,b]. \end{cases} \quad (7)$$

In [12] is also given the following L_p-inequality:
If $f^{(n)} \in L_p$ for some $1 \le p \le \infty$, then we have

$$\left|\int_a^b w(t)f(t)dt - \sum_{j=1}^n A_{j,w}(x)f^{(j-1)}(x)\right| \le C_1(n,p,x,w)\cdot\|f^{(n)}\|_p,$$

for $\frac{1}{p}+\frac{1}{q}=1$, where

$$C_1(n,p,x,w)=\frac{1}{(n-1)!}\left[\int_a^x\left|\int_a^t(t-s)^{n-1}w(s)ds\right|^q dt+\int_x^b\left|\int_b^t(t-s)^{n-1}w(s)ds\right|^q dt\right]^{\frac{1}{q}}, \tag{8}$$

for $1<p\leq\infty$, and

$$C_1(n,1,x,w)=\frac{1}{(n-1)!}\max\{\sup_{t\in[a,x]}\left|\int_a^t(t-s)^{n-1}w(s)ds\right|,\sup_{t\in[x,b]}\left|\int_b^t(t-s)^{n-1}w(s)ds\right|\}. \tag{9}$$

The inequality is the best possible for $p=1$ and sharp for $1<p\leq\infty$.

In this paper we establish weighted version of Theorem 2, i.e., Hermite–Hadamard–Fejér type inequalities for n-times differentiable functions which are m-convex and (α,m)-convex. In addition, some special cases for different choices of weight functions are given.

2. Main Results

We shall start by technical lemma which will be useful in proving the main theorems.

Lemma 1. *For $0\leq a<x<b<+\infty$ and $\alpha,m\in\langle 0,1]$ we have*

(a)

$$\int_a^x w_{1n}(t)\cdot\left(\frac{x-t}{x-a}\right)^\alpha dt=\frac{B(\alpha+1,n)\int_a^x(x-s)^{n+\alpha}w(s)ds}{(n-1)!(x-a)^\alpha}$$

(b)

$$\int_x^b w_{2n}(t)\cdot\left(\frac{t-x}{b-x}\right)^\alpha dt=\frac{(-1)^n B(\alpha+1,n)\int_b^x(s-x)^{n+\alpha}w(s)ds}{(n-1)!(b-x)^\alpha}.$$

Proof. (a) Apply the definition of the function w_{1n} to obtain

$$\begin{aligned}
&\int_a^x w_{1n}(t)\left(\frac{x-t}{x-a}\right)^\alpha dt=\frac{1}{(n-1)!}\cdot\int_a^x\left(\int_a^t(t-s)^{n-1}w(s)ds\right)\left(\frac{x-t}{x-a}\right)^\alpha dt\\
=&\frac{1}{(x-a)^\alpha(n-1)!}\cdot\int_a^x\left(\int_s^x(x-t)^\alpha(t-s)^{n-1}dt\right)w(s)ds=\\
=&\left\{\text{substitution: } u=\frac{t-s}{x-s},\quad du=\frac{dt}{x-s}\right\}\\
=&\frac{1}{(x-a)^\alpha(n-1)!}\cdot\int_a^x\left(\int_0^1(1-u)^\alpha u^{n-1}du\right)(x-s)^{\alpha+n}w(s)ds\\
=&\frac{B(\alpha+1,n)\int_a^x(x-s)^{n+\alpha}w(s)ds}{(n-1)!(x-a)^\alpha}.
\end{aligned}$$

(b) Apply the definition of the function w_{2n} to obtain

$$\begin{aligned}
&\int_x^b w_{2n}(t)\left(\frac{t-x}{b-x}\right)^\alpha dt=\frac{-1}{(n-1)!}\cdot\int_x^b\left(\int_t^b(t-s)^{n-1}w(s)ds\right)\left(\frac{t-x}{b-x}\right)^\alpha dt\\
=&\frac{(-1)^n}{(b-x^\alpha(n-1)!}\cdot\int_x^b\left(\int_x^s(t-x)^\alpha(s-t)^{n-1}dt\right)w(s)ds=\\
=&\left\{\text{substitution: } u=\frac{s-t}{s-x},\quad du=\frac{-dt}{s-x}\right\}\\
=&\frac{(-1)^n}{(b-x)^\alpha(n-1)!}\cdot\int_x^b\left(\int_0^1(1-u)^\alpha u^{n-1}du\right)(s-x)^{\alpha+n}w(s)ds\\
=&\frac{(-1)^n B(\alpha+1,n)\int_x^b(s-x)^{n+\alpha}w(s)ds}{(n-1)!(b-x)^\alpha}
\end{aligned}$$

which finishes the proof.
□

Now, we can introduce and prove the following new Hermite–Hadamard–Fejér type inequality for the class of (α, m)-convex functions.

Theorem 3. *Let $f : [0, \infty) \to \mathbb{R}$ be an n-times differentiable function for $n \in \mathbb{N}$ and $0 \le a \le x \le b < +\infty$, $\alpha, m \in \langle 0, 1]$. If $f^{(n)} \in L_1\left[a, \frac{b}{m}\right]$ and $\left|f^{(n)}\right|^q$ for $q \ge 1$ is (α, m)-convex on $\left[0, \frac{b}{m}\right]$, then*

$$\begin{aligned} & \left| \int_a^b w(t)f(t)dt - \sum_{j=1}^n A_{j,w}(x)f^{(j-1)}(x) \right| \\ \le\ & w_{1,n+1}(x)^{1-1/q} \cdot \left[K_w(x,a,\alpha,n) \cdot \left|f^{(n)}(a)\right|^q + m \cdot (1 - K_w(x,a,\alpha,n)) \cdot \left|f^{(n)}(\tfrac{x}{m})\right|^q \right]^{1/q} \\ +\ & w_{2,n+1}(x)^{1-1/q} \cdot \left[M_w(b,x,\alpha,n) \cdot \left|f^{(n)}(x)\right|^q + m \cdot (1 - M_w(b,x,\alpha,n)) \cdot \left|f^{(n)}(\tfrac{b}{m})\right|^q \right]^{1/q}, \end{aligned} \tag{10}$$

where

$$K_w(a,x,\alpha,n) := \frac{B(\alpha+1,n)\int_a^x (x-s)^{n+\alpha} w(s)ds}{(n-1)!(x-a)^\alpha w_{1,n+1}(x)}, \quad x > a, \quad K_w(a,a,\alpha,n) := 0$$

and

$$M_w(b,x,\alpha,n) := \frac{(-1)^n B(\alpha+1,n)\int_b^x (s-x)^{n+\alpha} w(s)ds}{(n-1)!(b-x)^\alpha w_{2,n+1}(x)}, \quad x < b, \quad M_w(b,b,\alpha,n) := 0.$$

Proof. For $a < x < b$ we apply triangle inequality and Hölder's inequality to identity

$$\int_a^b w(t)f(t)dt - \sum_{j=1}^n A_{w,j}(x)f^{(j-1)}(x) = (-1)^n \int_a^b W_{n,w}(t,x)f^{(n)}(t)dt.$$

Therefore, for $p \ge 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we obtain

$$\begin{aligned} & \left| \int_a^b w(t)f(t)dt - \sum_{j=1}^n A_{w,j}(x)f^{(j-1)}(x) \right| = \left| (-1)^n \int_a^b W_{n,w}(t,x)f^{(n)}(t)dt \right| \\ \le\ & \int_a^x w_{1n}(t) \cdot \left|f^{(n)}(t)\right| dt + \int_x^b w_{2n}(t) \cdot \left|f^{(n)}(t)\right| dt \\ \le\ & \left(\int_a^x w_{1n}(t)dt \right)^{1/p} \cdot \left[\int_a^x w_{1n}(t) \cdot \left|f^{(n)}(t)\right|^q dt \right]^{1/q} \\ +\ & \left(\int_x^b w_{2n}(t)dt \right)^{1/p} \cdot \left[\int_x^b w_{2n}(t) \cdot \left|f^{(n)}(t)\right|^q dt \right]^{1/q} \\ =\ & w_{1,n+1}(x)^{1/p} \cdot \left[\int_a^x w_{1n}(t) \cdot \left| f^{(n)}\left(\frac{x-t}{x-a} \cdot a + m \cdot \left(1 - \frac{x-t}{x-a}\right) \cdot \frac{x}{m}\right) \right|^q dt \right]^{1/q} \\ +\ & w_{2,n+1}(x)^{1/p} \cdot \left[\int_x^b w_{2n}(t) \cdot \left| f^{(n)}\left(\frac{t-x}{b-x} \cdot x + m \cdot \left(1 - \frac{t-x}{b-x}\right) \cdot \frac{b}{m}\right) \right|^q dt \right]^{1/q} \end{aligned}$$

Then, by using $(\alpha, m)-$convexity and Lemma 1 we have

$$\leq \quad w_{1,n+1}(x)^{1/p}\left[\int_a^x\left(w_{1n}(t)\left(\frac{x-t}{x-a}\right)^{\alpha}\left|f^{(n)}(a)\right|^q+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f^{(n)}\left(\frac{x}{m}\right)\right|^q\right)dt\right]^{1/q}$$
$$+ \quad w_{2,n+1}(x)^{1/p}\left[\int_x^b\left(w_{2n}(t)\left(\frac{t-x}{b-x}\right)^{\alpha}\left|f^{(n)}(x)\right|^q+m\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right)\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)dt\right]^{1/q}$$
$$= \quad w_{1,n+1}(x)^{1/p}\cdot\left[K_w(x,a,\alpha,n)\cdot\left|f^{(n)}(a)\right|^q+m\cdot(1-K_w(x,a,\alpha,n))\cdot\left|f^{(n)}(\frac{x}{m})\right|^q\right]^{1/q}$$
$$+ \quad w_{2,n+1}(x)^{1/p}\cdot\left[M_w(b,x,\alpha,n)\cdot\left|f^{(n)}(x)\right|^q+m\cdot(1-M_w(b,x,\alpha,n))\cdot\left|f^{(n)}(\frac{b}{m})\right|^q\right]^{1/q}.$$

For $x=a$, we have by definition $w_{1,n+1}(a)=0$ and $K_w(a,a,\alpha,n)=0$, so the inequality follows easily. Similarly, for $x=b$, we have by definition $w_{2,n+1}(b)=0$ and $M_w(b,b,\alpha,n)=0$, so the inequality also follows easily. □

In the following corollary, we include the endpoints of the interval $[a,b]$ to the inequality (10) in order to get the related inequality for weight trapezoidal quadrature formula.

Corollary 1. *Under the conditions of Theorem 3*
(i) for $x=a$ we have

$$\left|\int_a^b w(t)f(t)dt-\sum_{j=1}^n A_{j,w}(a)f^{(j-1)}(a)\right| \tag{11}$$
$$\leq \quad w_{2,n+1}(a)^{1-1/q}\cdot\left[M_w(b,a,\alpha,n)\cdot\left|f^{(n)}(a)\right|^q+m\cdot(1-M_w(b,a,\alpha,n))\cdot\left|f^{(n)}(\frac{b}{m})\right|^q\right]^{1/q},$$

(ii) for $x=b$ we have

$$\left|\int_a^b w(t)f(t)dt-\sum_{j=1}^n A^n_{j=1}f^{(j-1)}(b)\right| \tag{12}$$
$$\leq \quad w_{1,n+1}(b)^{1-1/q}\cdot\left[K_w(b,a,\alpha,n)\cdot\left|f^{(n)}(a)\right|^q+m\cdot(1-K_w(b,a,\alpha,n))\cdot\left|f^{(n)}(\frac{b}{m})\right|^q\right]^{1/q}$$

Remark 2. *If we combine inequalities (11) and (12), then we get the weight generalization of the trapezoid formula*

$$\left|\int_a^b w(t)f(t)dt-\frac{1}{2}\left(\sum_{j=1}^n A_{j,w}(a)f^{(j-1)}(a)+\sum_{j=1}^n A_{j,w}(b)f^{(j-1)}(b)\right)\right|$$
$$\leq \quad \frac{1}{2}w_{1,n+1}(b)^{1-1/q}\cdot\left[K_w(b,a,\alpha,n)\cdot\left|f^{(n)}(a)\right|^q+m\cdot(1-K_w(b,a,\alpha,n))\cdot\left|f^{(n)}(\frac{b}{m})\right|^q\right]^{1/q}$$
$$+ \quad \frac{1}{2}w_{2,n+1}(a)^{1-1/q}\cdot\left[M_w(b,a,\alpha,n)\cdot\left|f^{(n)}(a)\right|^q+m\cdot(1-M_w(b,a,\alpha,n))\cdot\left|f^{(n)}(\frac{b}{m})\right|^q\right]^{1/q}.$$

Specially, for $w(t)=\frac{1}{b-a}$ and $\alpha=m=n=1$ we get the result obtained in [13].

The following special case is a generalization of the weight midpoint formula.

Corollary 2. *Under the conditions of the Theorem 3 and for $x=\frac{a+b}{2}$ we have*

$$\left|\int_a^b w(t)f(t)dt-\sum_{oddj=1}^n A_{j,w}(\frac{a+b}{2})f^{(j-1)}(\frac{a+b}{2})\right| \tag{13}$$

$$\begin{aligned} &\leq w_{1,n+1}(\tfrac{a+b}{2})^{1-1/q} \cdot \Big[K_w(\tfrac{a+b}{2},a,\alpha,n) \cdot \Big|f^{(n)}(a)\Big|^q \\ &+ m \cdot \Big(1-K_w(\tfrac{a+b}{2},a,\alpha,n)\Big) \cdot \Big|f^{(n)}(\tfrac{a+b}{2m})\Big|^q\Big]^{1/q} \\ &+ w_{2,n+1}(\tfrac{a+b}{2})^{1-1/q} \cdot \Big[M_w(b,\tfrac{a+b}{2},\alpha,n) \cdot \Big|f^{(n)}(\tfrac{a+b}{2})\Big|^q \\ &+ m \cdot \Big(1-M_w(b,\tfrac{a+b}{2},\alpha,n)\Big) \cdot \Big|f^{(n)}(\tfrac{b}{m})\Big|^q\Big]^{1/q}, \end{aligned} \tag{14}$$

Remark 3. *Specially, for uniform weight function $w(t) = \frac{1}{b-a}$ we get the result obtained in [14], while for $m = n = \alpha = 1$ and uniform weight we get result from [15].*

Now, we shall give special inequalities of the weighted type for $q = 1$, $\alpha = 1$ (m-convex functions), $m = 1$ and finally $q = \alpha = m = 1$.

Corollary 3. *Under the conditions of Theorem 3*

(i) for $q = 1$ we have

$$\begin{aligned} &\Big|\int_a^b w(t)f(t)dt - \sum_{j=1}^n A_{j,w}(x)f^{(j-1)}(x)\Big| \\ \leq\ & w_{1,n+1}(x) \cdot \Big[K_w(x,a,\alpha,n) \cdot \Big|f^{(n)}(a)\Big| + m \cdot (1-K_w(x,a,\alpha,n)) \cdot \Big|f^{(n)}(\tfrac{x}{m})\Big|\Big] \\ +\ & w_{2,n+1}(x) \cdot \Big[M_w(b,x,\alpha,n) \cdot \Big|f^{(n)}(x)\Big| + m \cdot (1-M_w(b,x,\alpha,n)) \cdot \Big|f^{(n)}(\tfrac{b}{m})\Big|\Big], \end{aligned} \tag{15}$$

(ii) for $\alpha = 1$, we have

$$\begin{aligned} &\Big|\int_a^b w(t)f(t)dt - \sum_{j=1}^n A_{j,w}(x)f^{(j-1)}(x)\Big| \\ \leq\ & w_{1,n+1}(x)^{1-1/q} \cdot \Big[K_w(x,a,1,n) \cdot \Big|f^{(n)}(a)\Big|^q + m \cdot (1-K_w(x,a,1,n)) \cdot \Big|f^{(n)}(\tfrac{x}{m})\Big|^q\Big]^{1/q} \\ +\ & w_{2,n+1}(x)^{1-1/q} \cdot \Big[M_w(b,x,1,n) \cdot \Big|f^{(n)}(x)\Big|^q + m \cdot (1-M_w(b,x,1,n)) \cdot \Big|f^{(n)}(\tfrac{b}{m})\Big|^q\Big]^{1/q}, \end{aligned} \tag{16}$$

(iii) for $m = 1$, i.e., $\Big|f^{(n)}\Big|^q$ is α-convex on $[0,b]$, we have

$$\begin{aligned} &\Big|\int_a^b w(t)f(t)dt - \sum_{j=1}^n A_{j,w}(x)f^{(j-1)}(x)\Big| \\ \leq\ & w_{1,n+1}(x)^{1-1/q} \cdot \Big[K_w(x,a,\alpha,n) \cdot \Big|f^{(n)}(a)\Big|^q + (1-K_w(x,a,\alpha,n)) \cdot \Big|f^{(n)}(x)\Big|^q\Big]^{1/q} \\ +\ & w_{2,n+1}(x)^{1-1/q} \cdot \Big[M_w(b,x,\alpha,n) \cdot \Big|f^{(n)}(x)\Big|^q + (1-M_w(b,x,\alpha,n)) \cdot \Big|f^{(n)}(b)\Big|^q\Big]^{1/q}, \end{aligned} \tag{17}$$

(iv) for $m = \alpha = q = 1$, i.e., $\Big|f^{(n)}\Big|$ is convex on $[0,b]$, we have

$$\begin{aligned} &\Big|\int_a^b w(t)f(t)dt - \sum_{j=1}^n A_{j,w}(x)f^{(j-1)}(x)\Big| \\ \leq\ & w_{1,n+1}(x) \cdot \Big[K_w(x,a,1,n) \cdot \Big|f^{(n)}(a)\Big| + (1-K_w(x,a,1,n)) \cdot \Big|f^{(n)}(x)\Big|\Big] \\ +\ & w_{2,n+1}(x) \cdot \Big[M_w(b,x,1,n) \cdot \Big|f^{(n)}(x)\Big| + (1-M_w(b,x,1,n)) \cdot \Big|f^{(n)}(b)\Big|\Big]. \end{aligned} \tag{18}$$

The following special case of Theorem 3 is the result obtained in [10].

Corollary 4. *Under the conditions of Theorem 3 for $w(t) = \frac{1}{b-a}$ we have*

$$\begin{aligned} &\Big|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{b-a}\sum_{j=1}^n \frac{(b-x)^j-(a-x)^j}{j!} f^{(j-1)}(x)\Big| \\ \leq\ & \frac{(x-a)^{n+1}}{(b-a)(n+1)!} \cdot \Big[\alpha \cdot B(n+2,\alpha) \cdot \Big|f^{(n)}(a)\Big|^q + m \cdot (1-\alpha \cdot B(n+2,\alpha)) \cdot \Big|f^{(n)}(\tfrac{x}{m})\Big|^q\Big]^{1/q} \\ +\ & \frac{(x-b)^{n+1}}{(b-a)(n+1)!} \cdot \Big[\alpha \cdot B(n+2,\alpha) \cdot \Big|f^{(n)}(x)\Big|^q + m \cdot (1-\alpha \cdot B(n+2,\alpha)) \cdot \Big|f^{(n)}(\tfrac{b}{m})\Big|^q\Big]^{1/q}. \end{aligned} \tag{19}$$

Proof. We apply Theorem 2 with $w(t) = \frac{1}{b-a}$. It is easy to compute that

$$w_{1n}(t) = \frac{(t-a)^n}{(b-a)n!} \text{ and } w_{2n}(t) = \frac{(t-b)^n}{(b-a)n!}.$$

After some computation we have

$$K_w(a,x,\alpha,n) = M_w(b,x,\alpha,n) = \alpha \cdot B(n+2,\alpha)$$

for $a < x < b$ and

$$K_w(a,b,\alpha,n) = M_w(b,b,\alpha,n) = 0.$$

□

3. Applications to Special Means

Let us consider the means for arbitrary real numbers:

(i) $A(a,b) = \frac{a+b}{2}$, $a,b \in \mathbb{R}$ arithmetic mean

(ii) $L(a,b) = \frac{b-a}{\ln b - \ln a}$, $a \neq b$, $a,b > 0$ logarithmic mean

(iii) $L_n(a,b) = \frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}$, $n \in \mathbb{Z} \setminus \{-1,0\}$, $a,b \in \mathbb{R}$, $a \neq b$ generalized logarithmic men

(iv) $H(a,b) = \frac{2ab}{a+b}$, $a,b > 0$ harmonic mean

(v) $I(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } t \in a \neq b \end{cases}$, $a,b > 0$ identric mean

(vi) $G(a,b) + \sqrt{ab}$, $a,b > 0$ geometric mean

Now, using the results of Section 2, we give some applications to uniform weight function $w(t) = \frac{1}{b-a}$:

Proposition 1. *Let* $0 \le a < b < \infty$, $\alpha, m \in \langle 0,1]$ *and* $n \in \mathbb{Z} \setminus \{-1,0\}$. *Then for all* $q \ge 1$ *we have:*

$$\left| L_k^k(a,b) - A(a^k,b^k) \right| \tag{20}$$

$$\le \left(\frac{b-a}{2}\right)^{1-\frac{1}{q}} \cdot k \cdot \left[\frac{2 \cdot a^{(k-1)q}}{(\alpha+1)(\alpha+2)} + m \cdot \frac{\alpha \cdot (\alpha+3) \cdot b^{(k-1)q}}{(\alpha+1)(\alpha+2)}\right]^{\frac{1}{q}} \tag{21}$$

If we choose $\alpha = 1$, *we obtain*

$$\left| L_k^k(a,b) - A(a^k,b^k) \right| \le \left(\frac{b-a}{2}\right)^{1-\frac{1}{q}} \cdot k \cdot \left[\frac{a^{(k-1)q}}{3} + m \cdot \frac{2 \cdot b^{(k-1)q}}{3}\right]^{\frac{1}{q}} \tag{22}$$

Proof. The assertion follows from Theorem 3 and Remark 2 applied to the (α,m)-convex mapping $f(x) = x^k$, $k \in \mathbb{Z} \setminus \{-1,0\}$ and $n = 1$. □

Proposition 2. *If* $0 \le a < b < \infty$, *then for all* $q \ge 1$ *we have:*

$$\left| L^{-1}(a,b) - H^{-1}(a,b) \right| \le \left(\frac{b-a}{2}\right)^{1-\frac{1}{q}} \cdot \left[\frac{a^{-2q}}{3} + \frac{2 \cdot b^{-2q}}{3}\right]^{\frac{1}{q}}. \tag{23}$$

Proof. The assertion follows from Theorem 3 and Remark 2 applied to the mapping $f(x) = \frac{1}{x}$ and $m = \alpha = n = 1$. □

Proposition 3. *If $0 \leq a < b < \infty$, then for all $q \geq 1$ we have:*

$$|\ln I(a,b) - \ln G(a,b)| \leq \left(\frac{b-a}{2}\right)^{1-\frac{1}{q}} \cdot \left[\frac{1}{3a^q} + \frac{2}{3b^q}\right]^{\frac{1}{q}}. \quad (24)$$

Proof. The assertion follows from Theorem 3 and Remark 2 applied to the mapping $f(x) = -\ln x$ and $m = \alpha = n = 1$. □

4. Conclusions

In this work, we establish a generalization of the Hermite–Hadamard–Fejér type inequalities for different classes of convex functions. As a special case, the famous midpoint and trapezoidal error bounds were derived.

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