axioms

Article

On the Truncated Multidimensional Moment Problems in $\mathbb{C}^n$

Sergey Zagorodnyuk

School of Mathematics and Computer Sciences, V. N. Karazin Kharkiv National University, Svobody Sq., 4, 61022 Kharkiv, Ukraine; Sergey.M.Zagorodnyuk@gmail.com

Abstract: We consider the problem of finding a (non-negative) measure μ on $\mathfrak{B}(\mathbb{C}^n)$ such that $\int_{\mathbb{C}^n} \mathbf{z}^{\mathbf{k}} d\mu(\mathbf{z}) = s_{\mathbf{k}}$, $\forall \mathbf{k} \in \mathcal{K}$. Here, $\mathcal{K}$ is an arbitrary finite subset of $\mathbb{Z}^n_+$, which contains $(0, ..., 0)$, and $s_{\mathbf{k}}$ are prescribed complex numbers (we use the usual notations for multi-indices). There are two possible interpretations of this problem. Firstly, one may consider this problem as an extension of the truncated multidimensional moment problem on $\mathbb{R}^n$, where the support of the measure μ is allowed to lie in $\mathbb{C}^n$. Secondly, the moment problem is a particular case of the truncated moment problem in $\mathbb{C}^n$, with special truncations. We give simple conditions for the solvability of the above moment problem. As a corollary, we have an integral representation with a non-negative measure for linear functionals on some linear subspaces of polynomials.

Keywords: moment problem; linear functional on polynomials; dilation

MSC: 44A60

1. Introduction

Throughout the whole paper, n means a fixed positive integer. Let us introduce some notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. By $\mathbb{Z}^n_+$ we mean $\mathbb{Z}_+ \times \ldots \times \mathbb{Z}_+$, and $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$, $\mathbb{C}^n = \mathbb{C} \times \ldots \times \mathbb{C}$, where the Cartesian products are taken with n copies. Let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$, $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then, $\mathbf{z}^{\mathbf{k}}$ means the monomial $z_1^{k_1} \ldots z_n^{k_n}$, and $|\mathbf{k}| = k_1 + \ldots + k_n$. By $\mathfrak{B}(M)$, we denote the set of all Borel subsets of a set $M \subseteq \mathbb{C}^n$.

Let $\mathcal{K}$ be an arbitrary finite subset of $\mathbb{Z}^n_+$, which contains $\mathbf{0} := (0, ..., 0)$. Let $\mathcal{S} = (s_{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}}$ be an arbitrary set of complex numbers. We shall consider the problem of finding a (non-negative) measure μ on $\mathfrak{B}(\mathbb{C}^n)$ such that

$$\int_{\mathbb{C}^n} \mathbf{z}^{\mathbf{k}} d\mu(\mathbf{z}) = s_{\mathbf{k}}, \qquad \forall \mathbf{k} \in \mathcal{K}. \tag{1}$$

There are two possible interpretations of this problem. Firstly, one may consider this problem as an extension of the truncated multidimensional moment problem on $\mathbb{R}^n$, where the support of the measure μ is allowed to lie in $\mathbb{C}^n$. A similar situation is known in the cases of the classical Stieltjes and Hamburger moment problems, where the support of the measure lies in $[0, +\infty)$ and in $\mathbb{R}$, respectively. Secondly, and more directly, the moment problem (1) is a particular case of the truncated moment problem in $\mathbb{C}^n$ (see ([1], Chapter 7), [2,3]), with special truncations. These truncations do not include conjugate terms.

It is well-known that the multidimensional moment problems are much more complicated than their one-dimensional prototypes [1,4–8]. An operator-theoretical interpretation of the full multidimensional moment problem was given by Fuglede in [9]. Important ideas in the operator approach to moment problems go back to the works of Naimark in 1940–1943, and then, they were developed by many authors, see historical notes in [10]. In [11], we presented the operator approach to the truncated multidimensional moment problem in

Citation: Zagorodnyuk, S. On the Truncated Multidimensional Moment Problems in $\mathbb{C}^n$. *Axioms* **2022**, *11*, 20. https://doi.org/10.3390/axioms11010020

Academic Editor: Gradimir V. Milovanović

Received: 26 November 2021
Accepted: 4 January 2022
Published: 5 January 2022

$\mathbb{R}^n$. Other approaches to truncated moment problems can be found in [1–3,6,12,13]. A detailed exposition of the theory of (full and truncated) multidimensional moment problems is given in a recent Schmüdgen's book [8]. Recent results can be also found in [14,15].

In the case of the moment problem (1), we shall need a modification of the operator approach, since we have no positive definite kernels here. However, this problem can be passed and we shall come to some commuting bounded operators. At first, we shall provide an auxiliary commuting extension for this tuple inside the original space. After this, we shall not use extensions of operators, but *dilations*. We shall apply the dilation theory for commuting contractions to obtain the required measure. Consequently and surprisingly, we have very simple conditions for the solvability of the moment problem (1) (Theorem 1). As a corollary, we have an integral representation with a *non-negative* measure for linear functionals L on some linear subspaces of polynomials (Corollary 1).

Notations. Besides the given notations above, we shall use the following conventions. If H is a Hilbert space, then $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H, respectively. Indices may be omitted in obvious cases. For a linear operator A in H, we denote by $D(A)$ its domain, by $R(A)$ its range, and A^* means the adjoint operator if it exists. If A is invertible, then A^{-1} means its inverse. $\overline{A}$ means the closure of the operator, if the operator is closable. If A is bounded, then $\|A\|$ denotes its norm. For a set $M \subseteq H$, we denote by $\overline{M}$ the closure of M in the norm of H. By $\mathrm{Lin}\, M$, we mean the set of all linear combinations of elements from M, and $\mathrm{span}\, M := \overline{\mathrm{Lin}\, M}$. By E_H, we denote the identity operator in H, i.e., $E_H x = x$, $x \in H$. In obvious cases, we may omit the index H. If H_1 is a subspace of H, then $P_{H_1} = P_{H_1}^H$ denotes the orthogonal projection of H onto H_1.

2. Truncated Moment Problems on $\mathbb{C}^n$

A solution to the moment problem (1) is given by the following theorem.

Theorem 1. *Let the moment problem (1) with some prescribed $\mathcal{S} = (s_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}}$ be given. The moment problem (1) has a solution if and only if one of the following conditions holds:*

(a) $s_{(0,\dots,0)} > 0$;
(b) $s_{\mathbf{k}} = 0, \forall \mathbf{k} \in \mathcal{K}$.

If one of conditions (a), (b) is satisfied, then there exists a solution μ with a compact support.

Proof. The necessity part of the theorem is obvious. Let moment problem (1) be given and one of conditions (a),(b) holds. If (b) holds, then $\mu \equiv 0$ is a solution of the moment problem. Suppose in what follows that $s_{(0,\dots,0)} > 0$. Observe that we can include the set $\mathcal{K}$ into the following set:

$$K_d := \{\mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}_+^n : k_j \le d, \quad j = 1,2,...,n\},$$

for some large $d \ge 1$. Namely, d may be chosen greater than the maximum value of all possible indices k_j in $\mathcal{K}$. We now set $s_{\mathbf{k}} := 0$, for $\mathbf{k} \in K_d \backslash \mathcal{K}$. Consider another moment problem of type (1), having a new set of indices $\widetilde{\mathcal{K}} = K_d$. We are going to construct a solution to this moment problem, which, of course, will be a solution to the original problem.

Consider the usual Hilbert space l^2 of square summable complex sequences $\vec{c} = (c_0, c_1, c_2, ...)$, $\|\vec{c}\|_{l^2}^2 = \sum_{j=0}^\infty |c_j|^2$. We intend to construct a sequence $\{x_{\mathbf{k}}\}_{\mathbf{k}\in\widetilde{\mathcal{K}}}$ of elements of l^2, such that

$$(x_{\mathbf{k}}, x_{\mathbf{0}})_{l^2} = s_{\mathbf{k}}, \qquad \mathbf{k} \in \widetilde{\mathcal{K}}. \tag{2}$$

The elements of the finite set $\widetilde{\mathcal{K}}$ can be indexed by a single index, i.e., we assume

$$\widetilde{\mathcal{K}} = \{\mathbf{k}_0, \mathbf{k}_1, \dots, \mathbf{k}_\rho\}, \tag{3}$$

with $\rho+1=|\widetilde{\mathcal{K}}|$, and $\mathbf{k}_0=(0,...,0)$. Denote $a:=\sqrt{s_{(0,...,0)}}(>0)$. Set

$$x_{\mathbf{0}}:=a\vec{e}_0,\qquad x_{\mathbf{k}_j}:=\vec{e}_j+\frac{s_{\mathbf{k}_j}}{a}\vec{e}_0,\qquad j=1,2,...,\rho. \tag{4}$$

Here, $\vec{e}_j$ means the vector $\vec{c}=(c_0,c_1,c_2,...)$ from l^2, with $c_j=1$, and 0's in other places. Observe that for this choice of elements $x_{\mathbf{k}}$, conditions (2) hold true. Moreover, it is important for our future purposes that these elements $x_{\mathbf{k}}$ are linearly independent.

Consider a finite-dimensional Hilbert space $H:=\mathrm{Lin}\{x_{\mathbf{k}}\}_{\mathbf{k}\in\widetilde{\mathcal{K}}}$. Set

$$K_{d;l}:=\{\mathbf{k}=(k_1,\ldots,k_n)\in K_d: k_l\le d-1\},\qquad l=1,2,...,n.$$

Consider the following operator W_j on $\mathbf{Z}_+^n$:

$$W_j(k_1,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_n)=(k_1,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_n), \tag{5}$$

for $j=1,\ldots,n$. Thus, the operator W_j increases the j-th coordinate. We introduce the following operators M_j, $j=1,...,n$, in H:

$$M_j\sum_{\mathbf{k}\in K_{d;j}}\alpha_{\mathbf{k}}x_{\mathbf{k}}=\sum_{\mathbf{k}\in K_{d;j}}\alpha_{\mathbf{k}}x_{W_j\mathbf{k}},\qquad \alpha_{\mathbf{k}}\in\mathbb{C}, \tag{6}$$

with $D(M_j)=\mathrm{Lin}\{x_{\mathbf{k}}\}_{\mathbf{k}\in K_{d;j}}$. Since elements $x_{\mathbf{k}}$ are linearly independent, we conclude that M_j are well-defined operators. Operators M_j can be extended to a commuting tuple of bounded operators on H. In fact, consider the following operators $A_j\supseteq M_j$, $j=1,...,n$:

$$A_j\sum_{\mathbf{k}\in K_d}\alpha_{\mathbf{k}}x_{\mathbf{k}}=\sum_{\mathbf{k}\in K_{d;j}}\alpha_{\mathbf{k}}x_{W_j\mathbf{k}},\qquad \alpha_{\mathbf{k}}\in\mathbb{C}. \tag{7}$$

Operators A_j are well-defined linear operators on the whole H. It can be directly verified that they commute pairwisely. Notice that

$$A_1^{k_1}A_2^{k_2}...A_n^{k_n}x_{\mathbf{0}}=x_{(k_1,k_2,...,k_n)},\qquad \mathbf{k}=(k_1,...,k_n)\in K_d. \tag{8}$$

Relation (8) can be verified using the induction argument. Since H is finite-dimensional, then

$$\|A_j\|\le R,\qquad j=1,2,...,n;$$

for some $R>0$. Set

$$B_j:=\frac{1}{C}A_j,\qquad j=1,...,n, \tag{9}$$

where C is an arbitrary number greater than $\sqrt{n}R$. Then,

$$\sum_{j=1}^n\|B_j\|^2<1. \tag{10}$$

In this case, there exists a commuting unitary dilation $\mathcal{U}=(U_1,...,U_n)$ of $(B_1,...,B_n)$, in a Hilbert space $\widetilde{H}\supseteq H$, see Proposition 9.2 in [16] (p. 37). Namely, we have:

$$\left(P_H^{\widetilde{H}}U_1^{k_1}U_2^{k_2}...U_n^{k_n}\right)\Big|_H=B_1^{k_1}B_2^{k_2}...B_n^{k_n},\qquad k_1,...,k_n\in\mathbb{Z}_+. \tag{11}$$

Moreover, we can choose $\mathcal{U}$ to be minimal, that is, the subspaces $U_1^{k_1}...U_n^{k_n}H$ will span the space $\widetilde{H}$ (see Theorem 9.1 in [16] (p. 36)):

$$\widetilde{H}=\mathrm{span}\left\{U_1^{k_1}...U_n^{k_n}H,\ k_1,...,k_n\in\mathbb{Z}\right\}.$$

Then, the Hilbert space $\widetilde{H}$ will be separable. By (2), (8), (9) and (11), we may write for an arbitrary $\mathbf{k} = (k_1, ..., k_n) \in \widetilde{\mathcal{K}}$:

$$s_{\mathbf{k}} = (x_{\mathbf{k}}, x_{\mathbf{0}})_{l^2} = (A_1^{k_1} A_2^{k_2} ... A_n^{k_n} x_{\mathbf{0}}, x_{\mathbf{0}})_{l^2} = C^{|\mathbf{k}|}(B_1^{k_1} B_2^{k_2} ... B_n^{k_n} x_{\mathbf{0}}, x_{\mathbf{0}})_{l^2} =$$
$$= C^{|\mathbf{k}|}(U_1^{k_1} U_2^{k_2} ... U_n^{k_n} x_{\mathbf{0}}, x_{\mathbf{0}})_{l^2} = ((CU_1)^{k_1}(CU_2)^{k_2}...(CU_n)^{k_n} x_{\mathbf{0}}, x_{\mathbf{0}})_{l^2} =$$
$$= (N_1^{k_1} N_2^{k_2} ... N_n^{k_n} x_{\mathbf{0}}, x_{\mathbf{0}})_{l^2}, \tag{12}$$

where $N_j := CU_j$, $j = 1, ..., n$. Applying the spectral theorem for commuting bounded normal operators N_j (or, equivalently, to their commuting real and imaginary parts), we obtain that

$$N_j = \int_{\mathbb{C}^n} z_j dF(z_1, ..., z_n), \qquad j = 1, ..., n,$$

where $F(z_1, ..., z_n)$ is some spectral measure on $\mathfrak{B}(\mathbb{C}^n)$. Then,

$$s_{\mathbf{k}} = \int_{\mathbb{C}^n} z_1^{k_1} ... z_n^{k_n} d(F(z_1, ..., z_n)x_{\mathbf{0}}, x_{\mathbf{0}})_{l^2}, \qquad \mathbf{k} = (k_1, ..., k_n) \in \widetilde{\mathcal{K}}.$$

This means that $\mu = (F(z_1, ..., z_n)x_{\mathbf{0}}, x_{\mathbf{0}})_{l^2}$ is a solution of the moment problem. Since N_j were bounded, μ has compact support. □

Corollary 1. *Let $\mathcal{K}$ be an arbitrary finite subset of $\mathbb{Z}_+^n$, which contains $\mathbf{0}$. Let L be a complex-valued linear functional on*

$$M = M(\mathcal{K}) := \mathrm{Lin}\{z_1^{k_1} ... z_n^{k_n}\}_{\mathbf{k} = (k_1, ..., k_n) \in \mathcal{K}},$$

such that $L(1) > 0$. Then, L has the following integral representation:

$$L(p) = \int_{\mathbb{C}^n} p(z_1, ..., z_n) d\mu, \qquad \forall p \in M, \tag{13}$$

where μ is a (non-negative) measure μ on $\mathfrak{B}(\mathbb{C}^n)$, having compact support.

Proof. It follows directly from Theorem 1. □

3. Conclusions

Let $\mathcal{K}$ be an arbitrary finite subset of $\mathbb{Z}_+^n$, and $\mathcal{S} = (s_{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}}$ an arbitrary set of real numbers. Recall that the truncated multidimensional moment problem consists of finding a (non-negative) measure μ on $\mathfrak{B}(\mathbb{R}^n)$ such that [17]

$$\int \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}) = s_{\mathbf{k}}, \qquad \forall \mathbf{k} \in \mathcal{K}. \tag{14}$$

Assume additionally that $\mathcal{K}$ contains $\mathbf{0} := (0, ..., 0)$. Theorem 1 shows that if we extend the admissible support for the sought-for measure to lie in $\mathbb{C}^n$, then this helps essentially. Such a "complexification" of the problem leads to simple conditions of the solvability.

Corollary 1 can be compared with a well-known theorem of Boas, which gives a representation for functionals (see [18] (p. 74)). In our case, we have a **non-negative** measure in the representation. This provides a Hilbert space structure and corresponding tools for further investigations.

It is of interest to consider similar problems with infinite truncations and full moment problems. This will be studied elsewhere.

Funding: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Curto, R.; Fialkow, L. *Solution of the Truncated Complex Moment Problem for Flat Data*; American Mathematical Society: Providence, RI, USA, 1996; Volume 119.
2. Kimsey, D.P.; Putinar, M. Complex orthogonal polynomials and numerical quadrature via hyponormality. *Comput. Methods Funct. Theory* **2018**, *18*, 495–510. [CrossRef]
3. Kimsey, D.P.; Woerdeman, H. J. The truncated matrix-valued K-moment problem on $\mathbb{R}^d$, $\mathbb{C}^d$, and $\mathbb{T}^d$. *Trans. Am. Math. Soc.* **2013**, *365*, 5393–5430. [CrossRef]
4. Berezans'kiĭ, Ju.M. *Expansions in Eigenfunctions of Selfadjoint Operators*; American Mathematical Society: Providence, RI, USA, 1968.
5. Berg, C.C.; Christensen, J.P.R.; Ressel, P. *Harmonic Analysis on Semigroups*; Springer: New York, NY, USA, 1984.
6. Curto, R.; Fialkow L. Flat extensions of positive moment matrices: Recursively generated relations. *Mem. Am. Math. Soc.* **1998**, *136*, 254. [CrossRef]
7. Marshall, M. *Positive Polynomials and Sums of Squares*; American Mathematical Society: Providence, RI, USA, 2008; Volume 146.
8. Schmüdgen, K. *The Moment Problem*; Graduate Texts in Mathematics; Springer: Cham, Switerland, 2017.
9. Fuglede, B. The multidimensional moment problem. *Expo. Math.* **1983**, *1*, 47–65.
10. Zagorodnyuk, S.M. The Nevanlinna-type parametrization for the operator Hamburger moment problem. *J. Adv. Math. Stud.* **2017**, *10*, 183–199.
11. Zagorodnyuk, S.M. The operator approach to the truncated multidimensional moment problem. *Concr. Oper.* **2019**, *6*, 1–19. [CrossRef]
12. Vasilescu, F.-H. Moment problems in hereditary function spaces. *Concr. Oper.* **2019**, *6*, 64–75. [CrossRef]
13. Zagorodnyuk, S. On the truncated two-dimensional moment problem. *Adv. Oper. Theory* **2018**, *3*, 63–74. [CrossRef]
14. Idrissi, K.; Zerouali, E.H. Complex moment problem and recursive relations. *Methods Funct. Anal. Topol.* **2019**, *25*, 15–34.
15. Yoo, S. Sextic moment problems on 3 parallel lines. *Bull. Korean Math. Soc.* **2017**, *54*, 299–318. [CrossRef]
16. Nagy, B.S.; Foias, C.; Bercovici, H.; Kérchy, V. *Harmonic Analysis of Operators on Hilbert Space*, 2nd ed.; Springer: New York, NY, USA, 2010.
17. Lasserre, J.-B. *Moments, Positive Polynomials and Their Applications*; Imperial College Press Optimization Series, 1; Imperial College Press: London, UK, 2010.
18. Chihara, T.S. *An Introduction to Orthogonal Polynomials*; Mathematics and its Applications; Gordon and Breach Science Publishers: New York, NY, USA; London, UK; Paris, France, 1978; Volume 13.