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Reconstruction of Differential Operators with Frozen Argument

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Abstract: We study spectral properties of a wide class of differential operators with frozen arguments by putting them into a general framework of rank-one perturbation theory. In particular, we give a complete characterization of possible eigenvalues for these operators and solve the inverse spectral problem of reconstructing the perturbation from the resulting spectrum. This approach provides a unified treatment of several recent studies and gives a clear explanation and interpretation of the obtained results.

Keywords: differential operators; Sturm–Liouville-type operators; inverse problems; rank-one perturbations; frozen argument

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1. Introduction

In the recent few years, there has been an increased interest in the inverse spectral problems for the so-called Sturm–Liouville operators with frozen arguments that are given by the differential expression

$$\ell(y) = -y''(x) + y(a)q(x), \qquad x \in (0, \pi), \tag{1}$$

and subject to some boundary conditions. Here $a \in (0, \pi)$ is a fixed (or "frozen") value of the argument x and q is a function in $\mathcal{H} = L_2(0, \pi)$. The corresponding differential operator is non-self-adjoint, unbounded, and has compact resolvent in $\mathcal{H}$. A natural question arises, what spectra such operators may have and whether their eigenvalues completely characterize potentials q, i.e., whether q can be reconstructed from the known freezing point a and the corresponding eigenvalues.

A thorough study of these questions was made in several recent articles. Buterin and Vasiliev [1] studied the problem of reconstructing the potential q from the spectrum of the Sturm–Liouville-type operator (1) in case when $q \in L_2(0, \pi)$ is a complex-valued function and a/π is a rational number. They used the so-called transformation operators to get an integral representation of the solutions of the equation $\ell(y) = \lambda y$, derived an integral representation of the characteristic function, and then obtained asymptotics of the eigenvalues and eigenfunctions. That allowed the authors to study the inverse spectral problem and identify the iso-spectral sets of potentials q sharing the same spectrum and thus causing non-uniqueness of reconstruction; uniqueness was established for the class of potentials possessing some symmetry. Bondarenko et al. [2] extended the above research to the case of various boundary conditions. A more difficult case when a/π is irrational was recently discussed in [3]; uniqueness of q was proved and the reconstruction algorithm suggested. In [4], the authors considered the periodic boundary conditions for the differential expression (1); then the unperturbed operator with $q \equiv 0$ has eigenvalues of multiplicity 2, and the analysis of both direct and inverse spectral problems becomes technically more involved. Trace formulas and the inverse nodal problems were discussed in [5].

We also mention the paper of Nizhnik et al. [6], in which Sturm–Liouville eigenvalue problems (1) on $(0,1)$ were considered with $a = 1$ and non-local boundary conditions $y(0) = y'(1) + (y,q)_{L_2} = 0$. Here, q is an arbitrary complex-valued function in the Hilbert space $L_2(0,1)$ and $(y,q)_{L_2(0,1)}$ denotes the scalar product therein. The corresponding operator is self-adjoint in $L_2(0,1)$, and the authors studied the inverse spectral problem of reconstructing the potential q from its eigenvalues. In [7,8], Nizhnik extended the results of [6] to some other differential expressions and boundary conditions. The inverse problem for a class of self-adjoint perturbations that are integral operators with degenerate kernels was recently studied by Zolotarev [9].

The purpose of this note is to suggest a different approach to the spectral study of the Sturm–Liouville-type operators with frozen arguments. Namely, the operator generated by (1) can be viewed as a rank-one perturbation of the reference operator A corresponding to $q \equiv 0$; indeed, the term $y(a)q(x)$ can be represented as $\langle y, \varphi \rangle \psi$ with $\varphi = \delta_a$ being the Dirac delta-function, $\psi = q$, and $\langle \cdot, \cdot \rangle$ denoting the pairing in the Hilbert space scale generated by A; see details in the next section. We observe that this perturbation is unbounded as φ does not belong to $\mathcal{H}$; however, it is bounded (and even compact) relative to A, and that allows a generalization of the preliminary results obtained for the bounded case with $\varphi, \psi \in \mathcal{H}$.

Generic bounded rank one perturbations of self-adjoint operators are studied quite well, see e.g., the reference lists of [10–13]. In our resent work [12,13], we gave a complete characterization of possible spectra $\sigma(B)$ of bounded non-self-adjoint rank-one perturbations

$$B = A + \langle \cdot, \varphi \rangle \psi, \qquad \varphi, \psi \in \mathcal{H}, \tag{2}$$

of self-adjoint operators A with simple discrete spectrum. In particular, we proved therein that geometric multiplicities are at most 2 while the algebraic ones can be arbitrary; the only essential restriction on the eigenvalues of B comes from their asymptotics.

In this paper, we shall first show that even when φ or ψ are allowed to be singular, the statements on the multiplicity of eigenvalues of B remain valid while their asymptotic distribution should accordingly be modified. The effective tool for proving the main results of the paper is the characteristic function F of the operator B; namely, the eigenvalues of B that are not in the spectrum of A are zeros of F of required multiplicity. By studying the latter, we completely characterize eigenvalue asymptotics as stated in Theorems 1 and 2. We stress that this asymptotics differs from the one derived in [13] for the bounded case $\varphi, \psi \in \mathcal{H}$, and its derivation requires essential changes in the proofs. Next, zeros of F allow for a unique reconstruction of F, thus specifying ψ up to an iso-spectral set, see Corollary 5. Since F is given by explicit formulae, this approach suggests a constructive algorithm of determining ψ from the spectrum of the perturbation B. After establishing these abstract results, we specialize them to a wide variety of differential operators with frozen arguments including those of Sturm–Liouville type (1) thus providing a unified treatment of the questions discussed in the papers [1–4].

The paper is organized as follows. In the next section, we introduce necessary definitions and specify the setting of the paper, recall auxiliary results, derive the characteristic function and explain why the results of [12,13] are applicable here. In Section 3, we establish the asymptotics of the eigenvalues of B and in Section 4, we solve the inverse problem of reconstructing the vector ψ given φ. Several examples are given in Section 5.

2. Preliminaries

In this section, we collect some properties of rank-one perturbations of self-adjoint operators A acting in a fixed separable (infinite-dimensional) Hilbert space $\mathcal{H}$ established in [12,13] that will be used to prove the main results of this work. The reader can find further references and examples of applications in the monographs [10,11].

Throughout the paper, we shall assume that

(A1) the operator A is self-adjoint and has simple discrete spectrum.

Without loss of generality, we assume that A is either bounded below or else is unbounded from both below and above. In both cases, we list the eigenvalues in increasing order as λ_n, $n \in I$, where $I = \mathbb{N}$ in the former case and $I = \mathbb{Z}$ in the latter case. Since the motivation for this work stems from differential operators, we make an additional assumption that

(A2) the eigenvalues of A are d-separated, i.e.,

$$d = \inf_{n \in I}(\lambda_{n+1} - \lambda_n) > 0. \tag{3}$$

To treat rank-one perturbations (2) with φ not a regular function as in (1), we introduce the scale of Hilbert spaces $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$ [14]. Without loss of generality, we assume that A is invertible, adding to it εI otherwise. Then for $\alpha > 0$, $\mathcal{H}_\alpha$ coincides with $\operatorname{dom}(|A|^{\alpha/2})$ and is equipped with the scalar product $\langle f, g \rangle_\alpha = \langle A^{\alpha/2} f, A^{\alpha/2} g \rangle$. For negative α, $\mathcal{H}_\alpha$ is the completion of $\mathcal{H}$ in the norm generated by the above scalar product. The standard scalar product $\langle f, g \rangle$ extends by continuity to $f \in \mathcal{H}_\alpha$ and $g \in \mathcal{H}_{-\alpha}$ via

$$\langle f, g \rangle = \langle |A|^{\alpha/2} f, |A|^{-\alpha/2} g \rangle$$

and is called the *pairing* between $\mathcal{H}_\alpha$ and $\mathcal{H}_{-\alpha}$.

In what follows, we assume that $\psi \in \mathcal{H}$ but $\varphi \in \mathcal{H}_{-\alpha}$ with some $\alpha < 2$. Then the rank-one operator $A_0 f = \langle f, \varphi \rangle \psi$ is compact relative to A ([15], Ch. IV.1): indeed, in view of the relation

$$A_0 A^{-\alpha/2} f = \langle A^{-\alpha/2} f, \varphi \rangle \psi = \langle f, A^{-\alpha/2} \varphi \rangle \psi$$

we find that

$$\|A_0 A^{-\alpha/2} f\| \le \|A^{-\alpha/2} \varphi\| \|\psi\| \|f\|,$$

so that $A_0 A^{-\alpha/2}$ is bounded and $A_0 A^{-1}$ is compact. It follows that the operator $B = A + A_0$ is well defined and closed on the domain $\operatorname{dom}(A)$ of A and has compact resolvent ([15], Ch. IV.1).

Next, for $\lambda \in \rho(A)$, we introduce the *characteristic function*

$$F(\lambda) = \langle \psi, (A - \overline{\lambda})^{-1} \varphi \rangle + 1; \tag{4}$$

this function (in a slightly different form) appears in the Krein resolvent formula for B [11,12], and its zeros characterise the spectrum of B. The standard form of F as discussed in the previous work was

$$F(\lambda) = \langle (A - \lambda)^{-1} \psi, \varphi \rangle + 1;$$

this formula also makes sense in the current setting if we interpret the scalar product as the pairing between $\mathcal{H}_2$ and $\mathcal{H}_{-2}$ as explained above.

To see that both interpretations of F coincide, we suggest yet another representation of F using the spectral theorem for the operator A. Namely, let v_n be a normalized eigenvector of A corresponding to the eigenvalue λ_n (so that the set $\{v_n\}_{n \in I}$ is an orthonormal basis of $\mathcal{H}$), and let a_n and b_n denote the corresponding Fourier coefficients of the vectors φ and ψ, so that

$$\varphi = \sum_{k \in I} a_k v_k, \qquad \psi = \sum_{k \in I} b_k v_k. \tag{5}$$

We point out that the Fourier coefficients a_n of φ are well defined since the formula

$$a_n = \langle \varphi, v_n \rangle = \langle A^{-1} \varphi, A v_n \rangle$$

makes sense as a pairing between $\mathcal{H}_{-2}$ and $\mathcal{H}_2$. Set also

$$I_0 = \{ n \in I \mid a_n b_n = 0 \}, \qquad I_1 = \{ n \in I \mid a_n b_n \neq 0 \};$$

then the characteristic function F of (4) can be written as

$$F(z) = \sum_{k \in I_1} \frac{\overline{a_k} b_k}{\lambda_k - z} + 1 \tag{6}$$

and thus can be analytically extended to $\sigma_0(A) = \{\lambda_n \mid n \in I_0\}$; we keep the notation F for this extension. (Note that in (6) and in what follows, the summations and products over the index sets that are not bounded from below and above are understood in the principal value sense.) It is known [12] that $\sigma_0(A) = \sigma_0(B) = \sigma(A) \cap \sigma(B)$ is the common part of the spectra of A and B, while the spectrum of B in $\mathbb{C} \setminus \sigma_0(A)$ coincides with the set of zeros of F. For convenience, we set $\sigma_1(A) = \{\lambda_n \mid n \in I_1\}$.

It turns out that the function F also characterizes eigenvalue multiplicities of the operator B. We recall that the *geometric* multiplicity of an eigenvalue λ of B is the dimension of the null-space of the operator $B - \lambda$, while its *algebraic* multiplicity is the dimension of the corresponding root subspace, i.e., of the set of all $y \in \operatorname{dom}(B)$ such that $(B - \lambda)^k y = 0$ for some $k \in \mathbb{N}$. As proved in [12], the geometric multiplicity of every eigenvalue μ of B is at most 2; multiplicity 2 is only possible when $\mu \in \sigma_0(A)$, i.e., $\mu = \lambda_n$ for some $n \in I_0$ and, in addition, $a_n = b_n = F(\lambda_n) = 0$. It should be pointed out that the equality $a_n = b_n = 0$ implies that the subspace $\operatorname{ls}\{v_n\}$ is invariant under both B and B^* and thus is reducing for B. Denoting by $\mathcal{H}_0$ the closed linear span of all such subspaces, we conclude that $\mathcal{H}_0$ and $\mathcal{H} \ominus \mathcal{H}_0$ are reducing for B and the operators A and B coincide on $\mathcal{H}_0$. As a result, only the part of B in $\mathcal{H} \ominus \mathcal{H}_0$ is of interest, and we may assume that $\mathcal{H}_0 = \{0\}$ without loss of generality.

Under such an assumption, every eigenvalue μ of B is geometrically simple and the main results of [12] can be summarised as follows:

(a) the algebraic multiplicity m of an eigenvalue $\mu \in \sigma(B) \setminus \sigma_0(B)$ coincides with the multiplicity l of $z = \mu$ as a zero of F;
(b) if $\mu \in \sigma_0(B)$, then the above multiplicities m and l satisfy the relation $m = l + 1$;
(c) for any n-tuple $z_1, z_2, \ldots, z_n$ of pairwise distinct complex numbers and any n-tuple $m_1, m_2, \ldots, m_n$ of natural numbers, there exists a rank-one perturbation B of A such that every z_j is an eigenvalue of B of algebraic multiplicity m_j;
(d) the eigenvalues of B can be enumerated as μ_n, $n \in I$, in such a way that $\mu_n - \lambda_n \to 0$ as $|n| \to \infty$; in particular, B has at most finitely many non-simple eigenvalues.

Property (c) means that locally the spectrum of B can be arbitrary, while (d) describes the asymptotic behavior of the eigenvalues of B at infinity. One of the main aims of this note is to provide a complete characterization of the possible spectra of A under rank-one perturbations by refining the asymptotics of μ_n, cf. Theorems 1 and 2. In view of (a) and (b), this task amounts to the study of zero distribution of the characteristic function F of (4), which will be done in Sections 3 and 4.

3. Eigenvalue Distribution of the Operator B

As we mentioned in the previous section, the eigenvalues of the operator B are determined by the characteristic function F which, in turn, is completely determined by the Fourier coefficients a_n and b_n of the vectors φ and ψ through their products $c_n = \overline{a_n} b_n$, $n \in I$. For typical applications we have in mind (with A a differential operator and φ the Dirac delta-function), the eigenfunctions v_n are uniformly bounded and so are the Fourier coefficients a_n. Therefore, we assume throughout the rest of the paper that

(A3) the Fourier coefficients a_n of φ are uniformly bounded.

Under (A3), the sequence $c_n = \overline{a}_n b_n$ belongs to $\ell_2(I)$; however, we need a more precise characterization of c_n.

Definition 1. *For φ as above, we denote by $\ell_2(\varphi)$ the set of sequences $(c_n)_{n \in I}$ of the form $c_n = \overline{a_n} d_n$ with $(d_n) \in \ell_2(I)$ and a_n the Fourier coefficients of φ in the basis $(v_n)_{n \in \mathbb{N}}$.*

Under assumption (A3), we have $\ell_2(\varphi) \subset \ell_2(I)$; moreover, the inclusion is strict if some of a_n are zero or if $\liminf_{|n|\to\infty} |a_n| = 0$. The main result of this section establishes the asymptotic distribution of the eigenvalues of B in the following form.

Theorem 1. *Under the above assumptions, the eigenvalues of the operator B can be labelled as μ_n, $n \in I$, in such a way that the sequence $(\mu_n - \lambda_n)_{n\in I}$ belongs to $\ell_2(\varphi)$; in particular,*

$$\sum_{n\in I} |\mu_n - \lambda_n|^2 < \infty \tag{7}$$

and all but finitely many eigenvalues of B are simple.

We should point out the effect on the asymptotic distribution of eigenvalues that singularity of φ makes: for regular $\varphi \in \mathcal{H}$, the sequence of $\mu_n - \lambda_n$ was absolutely summable ([13], Theorem 3.1), while here it is only square summable.

As explained in the previous section, the spectrum of B is the union of two parts, $\sigma_0(B)$ and $\sigma_1(B)$; $\sigma_0(B) = \sigma(A) \cap \sigma(B)$ is the common part of the spectra of A and B, while $\sigma_1(B)$ is the set of zeros of the characteristic function

$$F(z) = \sum_{n\in I_1}^{\infty} \frac{c_n}{\lambda_n - z} + 1$$

in the domain $\mathbb{C} \setminus \sigma_1(A)$; moreover, the algebraic multiplicity of an eigenvalue $\mu \in \sigma(B)$ is determined by its multiplicity as a zero of the characteristic function F.

First, we shall show that large enough elements of $\sigma_1(B)$ are located near $\sigma_1(A)$, which will enable their proper enumeration. To begin with, for $k \in I$ we define the functions G_k and H_k by the formulae (here and hereafter, the symbol $\sum^{(1)}$ will denote summation over the index set I_1)

$$G_k(z) = \frac{\overline{a_k} b_k}{\lambda_k - z} + 1, \qquad H_k(z) = \sum_{|n|\le k}^{(1)} \frac{\overline{a_n} b_n}{\lambda_n - z} + 1$$

and introduce the sets

$$\begin{aligned} Q_k &= \{z \in \mathbb{C} \mid \operatorname{Re}(z), \operatorname{Im}(z) \in [\lambda_{-|k|} - \tfrac{d}{2}, \lambda_{|k|} + \tfrac{d}{2}]\}, \\ R_k &= \{z \in \mathbb{C} \mid |z - \lambda_k| < \tfrac{d}{2}\}, \end{aligned} \tag{8}$$

where we replace $\lambda_{-|k|}$ with $-\lambda_{|k|}$ if $I = \mathbb{N}$. Due to the assumption (A2), the sets R_k are pairwise disjoint and also $R_k \cap Q_n = \varnothing$ if $|k| > |n|$.

Lemma 1. *The series*

$$\sum_{n\in I_1} \frac{1}{|\lambda_n - z|^2}$$

converges locally uniformly in $z \in \mathbb{C} \setminus \left(\bigcup_{k\in I_1} R_k\right)$ and its sum is uniformly bounded therein.

Proof. If $I = \mathbb{Z}$, then we choose $k \in \mathbb{Z}$ such that

$$\frac{\lambda_{k-1} + \lambda_k}{2} \le \operatorname{Re} z < \frac{\lambda_k + \lambda_{k+1}}{2};$$

if $I = \mathbb{N}$, then we set $\lambda_0 = 0$ for convenience and define $k \in \mathbb{N}$ as above provided $\operatorname{Re} z > \lambda_1/2$; otherwise, set $k = 0$. Observing that $|\lambda_n - \lambda_k| \ge d|n - k|$, we conclude that $|\lambda_n - z| > d|n - k|/2$ if $n \ne k$; as a result, we derive the uniform bound

$$\sum_{n\in I_1} |\lambda_n - z|^{-2} \le \frac{4}{d^2} + \frac{8}{d^2} \sum_{j=1}^{\infty} j^{-2} = \frac{4(3 + \pi^2)}{3d^2}.$$

The proof is complete. □

Remark 1. *We observe that the same statements on uniform convergence and uniform boundedness hold when the radius $d/2$ of the circles R_k is replaced by an arbitrary ε or when z is taken within the union of R_k but $n = k$ is omitted in the sum; these modifications will be used below.*

Lemma 2. *For every $\varepsilon > 0$, there exist integers $K_\varepsilon > 0$ and $K'_\varepsilon > K_\varepsilon$ such that the following holds:*

(a) for every k with $|k| > K_\varepsilon$ and every $z \in \overline{R_k} = \partial R_k \cup R_k$

$$\sum_{\substack{|n|>K_\varepsilon \\ n\neq k}}^{(1)} \left|\frac{c_n}{\lambda_n - z}\right| < \varepsilon; \tag{9}$$

(b) for every $z \in \mathbb{C} \setminus Q_{K'_\varepsilon}$

$$\sum_{|n|\leq K_\varepsilon}^{(1)} \left|\frac{c_n}{\lambda_n - z}\right| < \varepsilon. \tag{10}$$

Proof. Since the sequence $(c_n)_{n\in I_1}$ belongs to $\ell_2(I_1)$, for every $\varepsilon' > 0$ there exists a K such that

$$\sum_{|n|>K}^{(1)} |c_n|^2 < \varepsilon'.$$

In view of the Cauchy–Bunyakowsky–Schwarz inequality and Lemma 1, we find that

$$\sum_{\substack{|n|>K \\ n\neq k}}^{(1)} \left|\frac{c_n}{\lambda_n - z}\right| \leq \Bigg(\sum_{|n|>K}^{(1)} |c_n|^2\Bigg)^{1/2} \Bigg(\sum_{\substack{|n|>K \\ n\neq k}}^{(1)} |\lambda_n - z|^{-2}\Bigg)^{1/2} \leq \left(\varepsilon' \frac{4(3+\pi^2)}{3d^2}\right)^{1/2}.$$

Part (a) follows by choosing $\varepsilon' > 0$ so that the above value is less than ε and denoting by K_ε the corresponding integer K.

For part (b), note that $|\lambda_n - z| > (K'_\varepsilon - K_\varepsilon)d$ if $|n| \leq K_\varepsilon$ and $z \in \mathbb{C} \setminus Q_k$ with $|k| \geq K'_\varepsilon > K_\varepsilon$; therefore, by choosing K'_ε large enough, we arrive at (10). □

Corollary 1. *Take an arbitrary $\varepsilon \in (0, \min\{1/2, d/2\})$ and fix K'_ε as in the above lemma; then*

$$\sigma(B) \subset Q_{K'_\varepsilon} \cup \Big(\bigcup\nolimits_{n\in I} R_n\Big).$$

Indeed, it suffices to note that if z does not belong to the above set, then

$$\sum_{|n|>K_\varepsilon}^{(1)} \left|\frac{c_n}{\lambda_n - z}\right| < \varepsilon,$$

which together with part (b) of Lemma 2 shows that

$$|F(z)| \geq 1 - \sum_{|n|\leq K_\varepsilon}^{(1)} \left|\frac{c_n}{\lambda_n - z}\right| - \sum_{|n|>K_\varepsilon}^{(1)} \left|\frac{c_n}{\lambda_n - z}\right| > 1 - 2\varepsilon > 0$$

so that such z cannot be an eigenvalue of B.

Lemma 3. *There exists a $K > 0$ such that for all $k \in I_1$ with $|k| > K$ the following holds:*

(a) the function F has exactly one zero in R_k;

(b) the functions H_k and F have the same number of zeros in Q_k.

Proof. Fix an $\varepsilon \in (0, \min\{1/3, d/2\})$; we shall show that (a) and (b) hold for $K = K'_\varepsilon$ of Lemma 2.

If k satisfies $|k| > K$, then by Lemma 2 for every $z \in \partial R_k$ we get

$$|F(z) - G_k(z)| \leq \sum_{|n| \leq K_\varepsilon}^{(1)} \left| \frac{c_n}{\lambda_n - z} \right| + \sum_{\substack{|n| > K_\varepsilon \\ n \neq k}}^{(1)} \left| \frac{c_n}{\lambda_n - z} \right| < 2\varepsilon.$$

On the other hand, $|c_k|/|\lambda_k - z| < \varepsilon$ if $k \in I_1$ satisfies $|k| > K > K_\varepsilon$ and $z \in \partial R_k$, and then

$$|G_k(z)| \geq 1 - \left| \frac{c_k}{\lambda_k - z} \right| > 1 - \varepsilon$$

for all $z \in \partial R_k$. By the choice of ε we conclude that then

$$|G_k(z)| > |F(z) - G_k(z)| \tag{11}$$

for all such z. As the functions G_k and F both have the same number of poles in R_k (namely, a simple pole at λ_k), by estimate (11) and Rouché's theorem [16] they have the same number of zeros in the set R_k. Since $|c_k| < d/2$ for large enough $|k|$, the unique zero $z = \lambda_k + c_k$ of the function G_k belongs to the circle R_k for all $k \in I_1$ with $|k| > K$, and thus the function F has exactly one zero in R_k for such k as well. This completes the proof of part (a).

Next, by the definition of the set Q_k, we see that ∂Q_k belongs to the set $\mathbb{C} \setminus \left(\bigcup_{n \in I} R_n\right)$; repeating the arguments used in the proof of part (a) of Lemma 2, we conclude that

$$|F(z) - H_k(z)| \leq \sum_{|n| > |k|}^{(1)} \left| \frac{c_n}{\lambda_n - z} \right| < \varepsilon$$

and

$$\sum_{K_\varepsilon < |n| \leq |k|}^{(1)} \left| \frac{c_n}{\lambda_n - z} \right| < \varepsilon \tag{12}$$

if $|k| > K_\varepsilon$ and $z \in \partial Q_k$. Also, by part (b) of Lemma 2 we have

$$\sum_{|n| \leq K_\varepsilon}^{(1)} \left| \frac{c_n}{\lambda_n - z} \right| < \varepsilon \tag{13}$$

as soon as $|k| > K$ and $z \in \mathbb{C} \setminus Q_k$. Combining estimates (12) and (13), we conclude that

$$|H_k(z)| \geq 1 - \sum_{|n| \leq k}^{(1)} \left| \frac{c_n}{\lambda_n - z} \right| > 1 - 2\varepsilon \tag{14}$$

for all k with $|k| > K$ and all $z \in \mathbb{C} \setminus Q_k$. It follows that for k with $|k| > K$ and for all $z \in \partial Q_k$

$$|H_k(z)| > |F(z) - H_k(z)|.$$

Since the functions H_k and F have the same poles in Q_k (namely, simple poles λ_n for $n \in I_1$ with $|n| \leq k$), we conclude by Rouché's theorem that they have the same number of zeros in Q_k for all $k > K$. The proof is complete. □

Remark 2. *Take k larger than K of the above lemma and denote by N_k the cardinality of the set $\sigma_1(A) \cap Q_k$. The function H_k is a ratio of two polynomials of degree N_k and due to (14) all its zeros are in Q_k. Therefore, the function F has precisely N_k zeros in Q_k counting with multiplicities.*

Corollary 2. *The zeros of F in $\mathbb{C} \setminus \sigma_0(A)$ can be labelled (counting with multiplicities) as μ_k with $k \in I_1$ in such a way that $|\mu_k - \lambda_k| < \frac{d}{2}$ for all $k \in I_1$ with $|k| > K$.*

Recalling the results of the previous section on the relation between the eigenvalues of B and zeros of the function F in $\mathbb{C} \setminus \sigma_1(A)$, we arrive at the following conclusion.

Corollary 3. *Eigenvalues of the operator B can be labelled (counting with multiplicities) as μ_k with $k \in I$ in such a way that $|\mu_k - \lambda_k| < \frac{d}{2}$ when $|k| > K$, K being the constant of Lemma 3.*

Combining the above corollary with Lemma 4.3 of [12], we conclude that $|\mu_k - \lambda_k| \to 0$ as $|k|$ goes to infinity, cf. Theorem 4.7(ii) of [12]. However, the estimates established above will enable us to prove a stronger statement of Theorem 1 on the asymptotics of $|\mu_k - \lambda_k|$.

Proof of Theorem 1. We fix an enumeration of μ_k as in Corollary 3. Then $\mu_k = \lambda_k$ for all $k \in I_0$ with large enough $|k|$, whence it suffices to prove that the differences $\mu_k - \lambda_k$ for $k \in I_1$ of sufficiently large absolute value are square summable.

We take $\varepsilon \in (0, \min\{1/4, d/2\})$ and $K = K'_\varepsilon$ as in Lemma 2; then, according to Corollary 3, for every $k \in I_1$ with $|k| > K$ the eigenvalue $\mu_k \in R_k$ is a zero of F, so that

$$F(\mu_k) = \sum_{|n|\le K_\varepsilon}^{(1)} \frac{c_n}{\mu_k - \lambda_n} + \sum_{\substack{|n|>K_\varepsilon \\ n\ne k}}^{(1)} \frac{c_n}{\mu_k - \lambda_n} + \frac{c_k}{\mu_k - \lambda_k} + 1 = 0$$

and

$$\left|\frac{c_k}{\mu_k - \lambda_k}\right| > 1 - \sum_{|n|\le K_\varepsilon}^{(1)} \left|\frac{c_n}{\lambda_n - \mu_k}\right| - \sum_{\substack{|n|>K_\varepsilon \\ n\ne k}}^{(1)} \left|\frac{c_n}{\lambda_n - \mu_k}\right|.$$

By virtue of Lemma 2 we conclude that

$$\left|\frac{c_k}{\mu_k - \lambda_k}\right| > 1 - 2\varepsilon > \tfrac{1}{2},$$

so that

$$|\mu_k - \lambda_k| < 2|c_k| \tag{15}$$

for all $k \in I_1$ with $|k| > K$. Since the sequence $(c_k)_{k\in I}$ belongs to $\ell_2(\varphi)$, the proof is complete. □

4. Inverse Spectral Problem

The purpose of this section is two-fold. Firstly, we show that Theorem 1 gives not only necessary but also sufficient conditions on the eigenvalues μ_n of B. Secondly, we study the inverse spectral problem of reconstructing the operator B from its spectrum $(\mu_n)_{n\in I}$ assuming that the operator A and the vector φ are known. As a by-product, we come up with the constructive algorithm of determining the vector ψ in the rank-one perturbation $\langle\,\cdot\,, \varphi\rangle\psi$ from the given data—the operator A, the vector φ, and the spectrum of the rank-one perturbation B.

Thus we fix an operator A satisfying the standing assumptions (A1) and (A2), i.e., is self-adjoint and has a simple discrete spectrum $(\lambda_n)_{n\in I}$ that is d-separated as in (3). Assume further that φ is a vector in the Hilbert space $\mathcal{H}_{-\alpha}$ for some $\alpha < 2$ such that (A3) holds. Then the following statement holds true.

Theorem 2. *Assume that a sequence $\boldsymbol{\nu}$ of complex numbers can be enumerated as ν_n, $n \in I$, in such a way that the differences $\nu_n - \lambda_n$ form an element of $\ell_2(\varphi)$. Then there exists a vector $\psi \in \mathcal{H}$ such that the spectrum of the operator B coincides with $\boldsymbol{\nu}$ counting with multiplicities.*

Observe that the assumptions of the theorem imply that the series

$$\sum_{n\in I} |\nu_n - \lambda_n|^2 \tag{16}$$

converges and that $\nu_n = \lambda_n$ for every $n \in I$ such that $a_n = 0$. More generally, denote by I_0 the set of indices $n \in I$ for which λ_n appears in $\boldsymbol{\nu}$ and set $\Lambda_0 = \{\lambda_n \mid n \in I_0\}$; then every $n \in I$ for which $a_n = 0$ belongs to I_0. By virtue of (16), for every $\varepsilon \in (0, d/2)$ there exists a

$K > 0$ such that $|\nu_n - \lambda_n| < \varepsilon$ for all $n \in I$ with $|n| > K$. Therefore, if $n \in I_0$ and $|n| > K$, then $\nu_n = \lambda_n$, and without loss of generality we may assume that $\nu_n = \lambda_n$ for all $n \in I_0$.

We also set $I_1 = I \setminus I_0$, $\Lambda_1 = \{\lambda_n \mid n \in I_1\}$, and introduce the function

$$\tilde{F}(z) = \prod_{n \in I_1} \frac{\nu_n - z}{\lambda_n - z}. \tag{17}$$

We first show that $\tilde{F}$ is well defined; to this end, we take an arbitrary $\varepsilon \in (0, d/2)$, introduce the sets

$$R_n(\varepsilon) = \{z \in \mathbb{C} \mid |z - \lambda_n| < \varepsilon\}, \qquad R(\varepsilon) = \mathbb{C} \setminus \big(\cup_{n \in I_1} R_n(\varepsilon)\big)$$

and prove the following result.

Lemma 4. *For each $\varepsilon \in (0, d/2)$, the product in (17) converges locally uniformly in $R(\varepsilon)$ to a function that is uniformly bounded in $R(\varepsilon)$.*

Proof. It is enough to show that the series

$$\sum_{n \in I_1} \log\left(1 + \left|\frac{\nu_n - \lambda_n}{\lambda_n - z}\right|\right)$$

converges locally uniformly on the set $R(\varepsilon)$. By the standard bound on $\log(1 + |z|)$ and the Cauchy–Bunyakovski–Schwarz inequality, we get

$$\sum_{n \in I_1} \log\left(1 + \left|\frac{\nu_n - \lambda_n}{\lambda_n - z}\right|\right) \leq \sum_{n \in I_1} \left|\frac{\nu_n - \lambda_n}{\lambda_n - z}\right| \leq \Big(\sum_{n \in I_1} |\nu_n - \lambda_n|^2\Big)^{1/2} \Big(\sum_{n \in I_1} |\lambda_n - z|^{-2}\Big)^{1/2}. \tag{18}$$

Convergence of the series (16) along with Lemma 1 results in the locally uniform convergence of the product for $\tilde{F}$ on the set $R(\varepsilon)$ as well as in the uniform boundedness of $\tilde{F}$ therein. □

Similar arguments based on the Lebesgue dominated convergence theorem justify passage to the limit in

$$\lim_{u \to +\infty} \sum_{n \in I_1} \left|\frac{\nu_n - \lambda_n}{\lambda_n - iu}\right| = 0;$$

as a result, we get

Corollary 4. *The function $\tilde{F}$ tends to 1 along the imaginary line, i.e.,*

$$\lim_{u \to \pm\infty} \tilde{F}(iu) = 1.$$

We next develop $\tilde{F}$ in the sum of simple fractions. The function $\tilde{F}$ is meromorphic in $\mathbb{C}$, and its residue at the point $\lambda_n \in \Lambda_1$ is

$$-c_n = \lim_{z \to \lambda_n} (z - \lambda_n)\tilde{F}(z) = (\lambda_n - \nu_n) \prod_{\substack{m \in I_1 \\ m \neq n}} \frac{\nu_m - \lambda_n}{\lambda_m - \lambda_n}; \tag{19}$$

the minus sign is introduced here for convenience. We also set $c_n = 0$ for $n \in I_0$.

Lemma 5. *Under the assumptions of Theorem 2, the sequence (c_n) belongs to $\ell_2(\varphi)$.*

Proof. In view of (19) and the assumption of Theorem 2, it suffices to prove that the sequence

$$\prod_{\substack{m \in I_1 \\ m \neq n}} \left|\frac{\nu_m - \lambda_n}{\lambda_m - \lambda_n}\right| \tag{20}$$

is uniformly bounded in $n \in I_1$.

Applying the same reasoning as in the proof of Lemma 1 (see also Remark 1), we conclude that the sum of the series

$$\sum_{m\neq n}^{(1)} \log\left|\frac{\nu_m-\lambda_n}{\lambda_m-\lambda_n}\right| \le \sum_{m\neq n}^{(1)} \log\left(1+\left|\frac{\nu_m-\lambda_m}{\lambda_m-\lambda_n}\right|\right)$$
$$\le \sum_{m\neq n}^{(1)} \left|\frac{\nu_m-\lambda_m}{\lambda_m-\lambda_n}\right| \le \Big(\sum_{m\in I_1} |\nu_m-\lambda_m|^2\Big)^{1/2}\Big(\sum_{m\neq n} |\lambda_m-\lambda_n|^{-2}\Big)$$

has an n-independent bound, which implies that the sequence (20) is uniformly bounded. The proof is complete. □

Given the above lemma and the uniform bound established in the proof of Lemma 1, the series

$$\sum_{n\in I_1} \frac{c_n}{\lambda_n - z}$$

converges locally uniformly in $R(\varepsilon)$ for every $\varepsilon \in (0, d/2)$. It follows that the function

$$F(z) = 1 + \sum_{n\in I_1} \frac{c_n}{\lambda_n - z} \tag{21}$$

is well defined and analytic in the set $\mathbb{C} \setminus \Lambda_1$ and has simple poles at the points $z \in \Lambda_1$.

Lemma 6. *For every $\varepsilon > 0$, the function F is uniformly bounded in the set $R(\varepsilon)$ and, moreover,*

$$\lim_{u\to\pm\infty} F(iu) = 1.$$

Proof. To prove the first statement, it suffices to apply the Cauchy–Bunyakowski–Schwarz inequality to the sum in (21) and use Lemmas 1 and 5. Since for real u we have

$$\left|\frac{c_n}{\lambda_n - iu}\right| \le \frac{|c_n|}{|\lambda_n|}$$

and since the series $\sum_{n\in I_1} |c_n|/|\lambda_n|$ converges, the Lebesgue dominated convergence theorem justifies term-wise passage to the limit in the series of (21) and thus produces the required limit. □

Now we are ready to show that the functions F and $\tilde{F}$ coincide.

Lemma 7. *The function $F - \tilde{F}$ is equal to zero identically in $\mathbb{C}$.*

Proof. The function $G = F - \tilde{F}$ is entire: indeed, it is meromorphic in $\mathbb{C}$ with possible single poles at the points of Λ_1, and since the residua of F and $\tilde{F}$ at the point $\lambda_n \in \Lambda_1$ are equal to $-c_n$, each such a singularity is removable. Being uniformly bounded over $\mathbb{C}$ by virtue of Lemmas 4 and 5, G is constant by the Liouville theorem. Since

$$\lim_{u\to+\infty} G(iu) = \lim_{u\to+\infty} F(iu) - \lim_{u\to+\infty} \tilde{F}(iu) = 0$$

according to Corollary 4 and Lemma 6, this constant is zero. The proof is complete. □

Proof of Theorem 2. Given any sequence $\boldsymbol{\nu}$ of complex numbers satisfying the assumption of the theorem, we construct the meromorphic function $\tilde{F}$ via (17). Next, we calculate the residua $-c_n$ of $\tilde{F}$ at the points $\lambda_n \in \Lambda_1$ via (19) and define the sequence $(b_n)_{n\in I}$ via

$$b_n = c_n/\overline{a_n}, \qquad n \in I_1, \tag{22}$$

and

$$b_n = 0, \qquad n \in I_0. \tag{23}$$

Since the sequence $(c_n)_{n\in I}$ belongs to $\ell_2(\varphi)$, it follows that the sequence $(b_n)_{n\in I}$ defined this way belongs to $\ell_2(I)$. Therefore, there exists a vector ψ in the Hilbert space $\mathcal{H}$ whose Fourier coefficients in the basis (v_n) are equal to b_n.

We now consider the operator B of the form (2) with the given vector φ and the vector ψ constructed above and conclude by virtue of Lemma 7 that the corresponding characteristic function F of (6) coincides with $\tilde{F}$. Therefore, zeros of F are precisely the elements of the subsequence $\boldsymbol{\nu}_1 = (\nu_n)_{n\in I_1}$, both counting multiplicity; namely, if a number ν occurs k times in $\boldsymbol{\nu}_1$, it is a zero of F of multiplicity k. The analysis of the paper [12] summarised in Section 2 shows that each element ν of $\boldsymbol{\nu}$ is an eigenvalue of B and its multiplicity is equal to the number of times ν is repeated in the sequence $\boldsymbol{\nu}$. The proof is complete. □

The above proof also suggests an algorithm of constructing a particular operator B whose spectrum corresponds to a sequence $\boldsymbol{\nu}$ of complex numbers satisfying (16). Namely, given such a sequence $\boldsymbol{\nu}$, we

1. construct the product $\tilde{F}$ of (17);
2. then calculate the residua $-c_n$ of $\tilde{F}$ at the points λ_n;
3. construct the sequence $(b_n)_{n\in I}$ via (22) and (23);
4. determine the function ψ from its Fourier coefficients b_n via (5).

Corollary 5. *The above analysis allows to completely describe the* isospectral set $\mathrm{Iso}(\varphi, \boldsymbol{\nu})$ *of all vectors $\psi \in \mathcal{H}$ for which the corresponding rank-one perturbations B of (2) have spectrum $\boldsymbol{\nu}$ counting with multiplicity. Namely, all such ψ have fixed Fourier coefficients b_n for $n \in \Lambda_1$, which are given by (22), while b_n must be zero for those $n \in \Lambda_0$ for which $a_n \neq 0$. Therefore, the "degree of freedom" in* $\mathrm{Iso}(\varphi, \boldsymbol{\nu})$ *coincides with the number of zero Fourier coefficients of the vector φ; in particular, ψ is uniquely determined by $\boldsymbol{\nu}$ if and only if all Fourier coefficients of φ are nonzero.*

5. Examples and Discussions

We give here two examples illustrating the results of the paper.

Example 1. *In the first example, the unperturbed operator A is defined in the Hilbert space $L_2(0, 2\pi)$ via*

$$A = \frac{1}{i}\frac{d}{dx}$$

subject to the periodic boundary condition $y(0) = y(2\pi)$. The spectrum of A coincides with the set $\mathbb{Z}$, and a normalized eigenfunction v_n corresponding to the eigenvalue $\lambda_n = n$ is equal to $e^{inx}/\sqrt{2\pi}$.

Take an arbitrary $a \in (0, 2\pi)$ and denote by φ the Dirac delta-function δ_a centred at a. Since the Hilbert space scale $\mathcal{H}_\alpha$ coincides with the Sobolev scale $W^2_{\alpha,\mathrm{per}}(0, 2\pi)$ of periodic functions ([17], Ch. 3), the Sobolev embedding theorem ([17], Ch. 4) shows that δ_a belongs to $\mathcal{H}_{-\alpha}$ for all $\alpha > \frac{1}{2}$. Next, the Fourier coefficient a_n of φ in the basis of v_n is equal to $\langle \varphi, v_n\rangle = e^{ina}/\sqrt{2\pi}$ and satisfies $|a_n| = 1/\sqrt{2\pi}$; therefore, none of a_n vanishes and, moreover, $\ell_2(\varphi)$ coincides with $\ell_2(\mathbb{Z})$.

As a result, for every $\psi \in L_2(0, 2\pi)$ the eigenvalues μ_n of the rank-one perturbation B of the operator A of the form (2), i.e., of the operator given by the differential expression

$$\ell(y) = -iy'(x) + y(a)\psi(x) \tag{24}$$

and the periodic boundary condition $y(0) = y(2\pi)$, can be enumerated in such a way that $\varepsilon_n = \mu_n - n$ form a sequence in $\ell_2(\mathbb{Z})$.

Vice versa, for every sequence $(\varepsilon_n) \in \ell_2(\mathbb{Z})$, there is a unique $\psi \in L_2(0, 2\pi)$ such that the eigenvalues of the operator B given by (24) and the periodic boundary conditions coincide with *the sequence* $(n+\varepsilon_n)_{n\in\mathbb{Z}}$ *counted with multiplicities. This* ψ *is given by the trigonometric series* $\sum b_n v_n$ *with* b_n *constructed in the proof of Theorem 2.*

Example 2. *Consider now the Sturm–Liouville operator B with frozen argument given by* (1) *with* $q \in L_2(0,\pi)$ *and some* $a \in (0,\pi)$, *and, say, the Dirichlet boundary conditions* $y(0) = y(\pi) = 0$; *other separated boundary conditions are treated in the same way. The unperturbed operator A with* $q \equiv 0$ *is self-adjoint and has simple discrete spectrum* $(n^2)_{n\in\mathbb{N}}$, *the corresponding normalized eigenfunctions being*

$$v_n(x) = \sqrt{\tfrac{2}{\pi}}\sin(nx).$$

Taking φ *to be the Dirac delta-function* δ_a *centred at a, we find that the corresponding Fourier coefficients* a_n *are* $\sqrt{2/\pi}\sin(na)$; *they are bounded but some of them may vanish or approach zero over subsequences depending on whether or not* a/π *is rational.*

In the former case, $a = \pi k/m$ *with relatively prime natural k and m, and the sequence* a_n *is periodic of period* $2m$ *and* $a_n = 0$ *if and only if* $n = jm$ *with a natural j. Then the eigenvalues of the operator B do not determine the potential q uniquely as its Fourier coefficients* $b_n = \langle q, v_n\rangle$ *with* $n = jm$, $j \in \mathbb{N}$, *are then not determined.*

If a and π *are rationally incommensurable, then none of* a_n *vanishes, which by Corollary 5 implies that the potential q of* (1) *is uniquely determined by the eigenvalues of the operator B.*

The above results for Sturm–Liouville-type operators with frozen argument are proved in several recent papers [1–3] using quite an involved technique that is based on the integral representation of the eigenfunctions v_n and crucially depends on the special properties of the Sturm–Liouville differential expressions. In the context of the present research, these results are obtained as special cases of more general abstract statements proved in Sections 3 and 4, which demonstrates the efficiency of the proposed approach.

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