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A Family of Generalized Legendre-Based Apostol-Type Polynomials

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Abstract: Numerous polynomials, their extensions, and variations have been thoroughly explored, owing to their potential applications in a wide variety of research fields. The purpose of this work is to provide a unified family of Legendre-based generalized Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials, with appropriate constraints for the Maclaurin series. Then we look at the formulae and identities that are involved, including an integral formula, differential formulas, addition formulas, implicit summation formulas, and general symmetry identities. We also provide an explicit representation for these new polynomials. Due to the generality of the findings given here, various formulae and identities for relatively simple polynomials and numbers, such as generalized Bernoulli, Euler, and Genocchi numbers and polynomials, are indicated to be deducible. Furthermore, we employ the umbral calculus theory to offer some additional formulae for these new polynomials.

Keywords: Legendre-based Apostol Bernoulli polynomials; Legendre-based Apostol Euler polynomials; Legendre-based Apostol Genocchi polynomials; summation formulae; symmetric identities; umbral calculus

MSC: primary 11B68; 11B83; secondary 33C05; 33C45; 33E20; 05A40

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1. Introduction and Preliminaries

Certain multivariate special polynomials enable the study of various classes of partial differential equations that are often encountered in physical problems. It arises often in physics or applied mathematics that the solution of a specific problem needs the evaluation of infinite sums linked with special functions. This sort of problem occurs, for example, in the computing of a distribution's higher-order moments or in the evaluation of transition matrix elements in quantum mechanics. Dattoli [1] demonstrated that summation formulae for special functions, which are often encountered in applications ranging from electromagnetic processes to combinatorics, may be stated in terms of multivariate Hermite polynomials. For example, the 2-variable Hermite Kampé de Fériet polynomials $H_n(x,y)$ (see [2]) defined by the generating function:

$$\exp\left(xt + yt^2\right) = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \tag{1}$$

are a solution of the following heat equation:

$$\frac{\partial}{\partial y}H_n(x,y) = \frac{\partial^2}{\partial x^2}H_n(x,y) \quad \text{and} \quad H_n(x,0) = x^n. \tag{2}$$

Higher-order Hermite polynomials $H_n^{(m)}(x,y)$, sometimes called Kampé de Fériet polynomials of order m or the Gould–Hopper polynomials, defined by the generating function (see, e.g., [3] (p. 58, Equation (6.3))):

$$\exp(xt+yt^m)=\sum_{n=0}^{\infty}H_n^{(m)}(x,y)\,\frac{t^n}{n!} \tag{3}$$

are the solutions to the generalized heat equation (see, e.g., [4]):

$$\frac{\partial}{\partial y}f(x,y)=\frac{\partial^m}{\partial x^m}f(x,y) \quad \text{and} \quad f(x,0)=x^n. \tag{4}$$

Note:

$$H_n^{(2)}(x,y)=H_n(x,y) \tag{5}$$

and

$$H_n(2x,-1)=H_n(x), \tag{6}$$

where $H_n(x)$ are the classical Hermite polynomials (see, e.g., [5,6]).

The 2-variable Legendre polynomials $S_n(x,y)$ and $R_n(x,y)$ are given by (see [7]):

$$S_n(x,y)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{x^k y^{n-2k}}{(n-2k)!\,(k!)^2} \tag{7}$$

and

$$R_n(x,y)=(n!)^2\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^{n-k}x^{n-k}y^k}{[(n-2k)!]^2(k!)^2}, \tag{8}$$

respectively, and related with the ordinary Legendre polynomials $P_n(x)$ as (see, e.g., [6]):

$$P_n(x)=S_n\left(-\frac{1-x^2}{4},x\right)=R_n\left(\frac{1-x}{2},\frac{1+x}{2}\right). \tag{9}$$

From (7) and (8), we have:

$$S_n(x,0)=n!\frac{x^{[\frac{n}{2}]}}{[(\frac{n}{2})!]^2}, \quad S_n(0,y)=y^n \tag{10}$$

and

$$R_n(x,0)=(-x)^n, \quad R_n(0,y)=y^n. \tag{11}$$

The generating functions for two variable Legendre polynomials $S_n(x,y)$ and $R_n(x,y)$ are given by (see [7]):

$$e^{yt}C_0(-xt^2)=\sum_{n=0}^{\infty}S_n(x,y)\frac{t^n}{n!} \tag{12}$$

and

$$C_0(xt)C_0(-yt)=\sum_{n=0}^{\infty}R_n(x,y)\frac{t^n}{(n!)^2}, \tag{13}$$

where $C_0(x)$ is the 0-th order Tricomi function (see, e.g., [6]):

$$C_0(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{(r!)^2}. \tag{14}$$

It is noted that:

$$C_0(x)=J_0\left(2x^{\frac{1}{2}}\right), \tag{15}$$

where $J_\nu(z)$ is the Bessel function of the first kind (see, e.g., [8] (Chapter 10)). It is found that $C_0(x)$ satisfies the following second-order differential equation:

$$x\,\frac{d^2}{dx^2}C_0(x)+\frac{d}{dx}C_0(x)=-C_0(x). \tag{16}$$

Ozden [9] (see also [10]) defined and investigated the following generalized Apostol-type polynomials:

$$\frac{2^{1-k}t^k}{\beta^b\,e^t-a^b}\,e^{xt}=\sum_{n=0}^{\infty}\mathscr{P}_{n,\beta}(x;k,a,b)\frac{t^n}{n!} \tag{17}$$

$$\begin{aligned}&\Big(k\in\mathbb{N},\ a,\ \beta\in\mathbb{C}\setminus\{0\},\ x,\ b\in\mathbb{C};\\ &a=\beta,\ 1^b:=1\ \Rightarrow\ |t|<2\pi;\ a\neq\beta,\ b\in\mathbb{C}\setminus\{0\}\ \Rightarrow\ |t|<\left|b\log\left(\frac{a}{\beta}\right)\right|\Big).\end{aligned} \tag{18}$$

Here and elsewhere, let log$z$ denote the principal branch of $\log z$, and let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, integers, and positive integers, respectively, and let $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

Remark 1. *For constraints of* (17) *and some important observations, one may be referred to [11] (pp. 99–100). Since the generating functions in* (22), (23), (25), *and* (27) *are expanded as a Maclaurin series, their respective convergence regions, which include similar expressions elsewhere, seem to be more appropriate to be provided as origin-centered as those of the second line of* (18). *Allow us to describe the procedure for determining the convergence regions in the second line of* (18). *To do this, denote by $\mathcal{G}(t)$ the generating function in the left-side of* (17). *We also assume the first line* (18).

(i) *The case $a=\beta$, 1^b: = 1. We write:*

$$\mathcal{G}(t)=\frac{2^{1-k}t^{k-1}}{\beta^b}\cdot\frac{t}{e^t-1}\cdot e^{xt}.$$

Note that $e^t-1=0\ \Leftrightarrow\ t=i\,2\nu\pi\ \ (\nu\in\mathbb{Z})$. We observe that $\mathcal{G}(t)$ has isolated singular points at:

$$t=i\,2\nu\pi\quad(\nu\in\mathbb{Z}), \tag{19}$$

in particular, at $t=0$. Yet, since t^{k-1} is analytic at $t=0$, we obtain:

$$\lim_{t\to0}\mathcal{G}(t)=\frac{2^{1-k}}{\beta^b}\cdot\lim_{t\to0}t^{k-1}\cdot\lim_{t\to0}\frac{t}{e^t-1}=\begin{cases}\dfrac{2^{1-k}}{\beta^b}\quad(k=1),\\ 0\quad(k\in\mathbb{N}\setminus\{0\}).\end{cases} \tag{20}$$

Equation (20) *implies that $t=0$ is a removable singular point of $\mathcal{G}(t)$. So, if $\mathcal{G}(0)$ is defined as the limit values in* (20), *$\mathcal{G}(t)$ is considered to be analytic at $t=0$. Hence, $\mathcal{G}(t)$ can be expanded in the Maclaurin series as the right member of* (17), *whose radius of convergence, in view of* (19), *is 2π.*

(ii) *The case $a\neq\beta$, $b\in\mathbb{C}\setminus\{0\}$. Then $\left(\frac{a}{\beta}\right)^b\neq1$. We write:*

$$\mathcal{G}(t)=\frac{2^{1-k}t^k}{\beta^b}\cdot\frac{1}{e^t-\left(\frac{a}{\beta}\right)^b}\cdot e^{xt}.$$

Then $e^t-\left(\frac{a}{\beta}\right)^b=0$ gives:

$$t=b\,\log\left(\frac{a}{\beta}\right)=b\left\{\ln\left|\frac{a}{\beta}\right|+i(\theta_0+2\nu\pi)\right\}\quad(\nu\in\mathbb{Z}), \tag{21}$$

θ_0 being the principal argument of $\frac{a}{\beta}$ $(-\pi < \theta_0 \leq \pi)$, all of which are found to be non-zero isolated singular points of $\mathcal{G}(t)$. Since $e^0 - \left(\frac{a}{\beta}\right)^b = 1 - \left(\frac{a}{\beta}\right)^b \neq 0$, we find that $1/\left\{e^t - \left(\frac{a}{\beta}\right)^b\right\}$ itself is analytic at $t = 0$. Therefore, also t^k being analytic at $t = 0$, $\mathcal{G}(t)$ is analytic at $t = 0$. Hence $\mathcal{G}(t)$ can be expanded in the Maclaurin series as the right member of (17), whose radius of convergence, in view of (21), is $\left|b \log\left(\frac{a}{\beta}\right)\right|$.

Özarslan [12] proposed the following unification of Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials, which has a generating function given by:

$$\left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathscr{P}_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!} \tag{22}$$

$$\left(\left|t + b\,\log\left(\frac{\beta}{a}\right)\right| < 2\pi;\ k \in \mathbb{N}_0;\ x \in \mathbb{R};\right.$$
$$\left. a \in \mathbb{R} \setminus \{0\},\ b \in \mathbb{R}^+;\ \alpha \in \mathbb{C},\ \beta \in \mathbb{C} \setminus \{0\};\ 1^{\alpha} := 1\right).$$

Ozden et al. [10] explored the polynomials in (22) to give many properties and formulas. Here we consider some specific special cases of (22).

1. By setting $a = b = k = 1$ and $\beta = \lambda$ into (22), we have the Apostol-Bernoulli polynomials $\mathscr{P}_{n,\lambda}^{(1)}(x;1,1,1) = B_n^{(\alpha)}(x;\lambda)$, which are defined by means of the following generating function:

$$\left(\frac{t}{\lambda\, e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad (|t + \log\lambda| < 2\pi) \tag{23}$$

(see, for details, Refs. [13–20] and see also the references cited therein). Furthermore, the case $\lambda = \alpha = 1$ of (23) reduces to the classical Bernoulli polynomials $B_n(x)$ generated by:

$$\frac{t}{e^t - 1}\, e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi) \tag{24}$$

(see, e.g., [21–38] and the reference cited therein).

2. By setting $b = 1$, $k = 0$, $a = -1$, and $\beta = \lambda$ in (22), we have the Apostol-Euler polynomials $\mathscr{P}_{n,\lambda}^{(\alpha)}(x;0,-1,1) = E_n^{(\alpha)}(x;\lambda)$, which are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad (|t + \log\lambda| < \pi) \tag{25}$$

(see, for details, Refs. [11,13,14,16–20] and the references cited therein). The case $\lambda = \alpha = 1$ of (25) reduces to yield the classical Euler polynomials $E_n(x)$ generated by:

$$\frac{2}{e^t + 1}\, e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi) \tag{26}$$

(see, e.g., [11,13–15,21,22,27,31,33,34] and the references cited therein).

3. By setting $b = 1$, $k = 1$, $a = -1$, and $\beta = \lambda$ in (22), we obtain the Apostol-Genocchi polynomials $\mathscr{P}_{n,\lambda}^{(1)}(x;1,-1,1) = \frac{1}{2}G_n^{(1)}(x;\lambda)$, which are defined by means of the following generating function:

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad (|t + \log\lambda| < \pi) \tag{27}$$

(see, for details, [13–15,17,18,24,32] and the references cited therein). The case $\lambda = \alpha = 1$ of (27) reduces to yield the classical Genocchi polynomials $G_n(x)$ generated by:

$$\frac{2t}{e^t+1}\,e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \quad (|t| < \pi) \tag{28}$$

(see, e.g., [10,13,21,22,24,32] and the reference cited therein).

4. By setting $x = 0$ in the generating function (22), along with the conditions in (24), (26) and (28), we obtain corresponding unifications of the generating functions function of Bernoulli, Euler, and Genocchi numbers of higher order. In this regard, we call generalized Apostol-type numbers $\mathscr{P}^{(\alpha)}_{n,\beta}(0;k,a,b) := \mathscr{P}^{(\alpha)}_{n,\beta}(k,a,b)$ generated by:

$$\left(\frac{2^{1-k}t^k}{\beta^b\,e^t - a^b}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathscr{P}^{(\alpha)}_{n,\beta}(k,a,b)\frac{t^n}{n!}. \tag{29}$$

The purpose of this paper is to introduce generalized Legendre-based Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials ${}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)$ in (30) and to investigate the formulas and identities associated with them, including an integral formula, differential formula, addition formula, implicit summation formula, and general symmetry identities. Also, we provide an explicit expression for the polynomials in (30). Owing to the generality of the results given here, various formulae and identities for relatively simple polynomials and numbers, such as generalized Bernoulli, Euler, and Genocchi numbers and polynomials, are pointed out to be obtainable. Furthermore, we apply umbral calculus theory to offer some additional formulae for the polynomials in (30).

2. Generalized Legendre-Based Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi Polynomials

In Definition 1, we introduce generalized Legendre-based Apostol-type polynomials and discuss several of their particular cases.

Definition 1. *Let $k \in \mathbb{N}$, a, $\beta \in \mathbb{C}\setminus\{0\}$, x, y, z, $b \in \mathbb{C}$. Also let $\alpha \in \mathbb{C}$ be such that $k\alpha \in \mathbb{N}_0$ and $(k-1)\alpha \in \mathbb{N}_0$. Furthermore, let $1^\alpha := 1$. Then generalized Legendre-based Apostol-type polynomials ${}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)$ are defined by:*

$$\left(\frac{2^{1-k}t^k}{\beta^b\,e^t-a^b}\right)^{\alpha} e^{yt+zt^2}C_0(-xt^2) = \sum_{n=0}^{\infty} {}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)\frac{t^n}{n!} \tag{30}$$

$$\left(a=\beta,\ 1^b := 1 \ \Rightarrow\ |t| < 2\pi;\ a \neq \beta,\ b \in \mathbb{C}\setminus\{0\} \ \Rightarrow\ |t| < \left|b\log\left(\frac{a}{\beta}\right)\right|\right).$$

Here and in the following, each exponential with base a complex (or real) number and logarithm are assumed to take their principal values.

Remark 2.

(i) *Let $\mathcal{G}^{(\alpha)}(t)$ be the generating function given in the left-hand side of (30). As in the analysis of Remark 1, we consider the following two cases.*

(a) *The case $a = \beta$, $1^b := 1$. Then we write:*

$$\mathcal{G}^{(\alpha)}(t) = \frac{2^{(1-k)\alpha}t^{(k-1)\alpha}}{\beta^{b\alpha}}\cdot\left(\frac{t}{e^t-1}\right)^{\alpha}\cdot e^{yt+zt^2}C_0(-xt^2).$$

If $(k-1)\alpha \in \mathbb{C}\setminus\mathbb{N}_0$, then the factor $t^{(k-1)\alpha}$ is not analytic at $t=0$. Accordingly, the Maclaurin series of $\mathcal{G}^{(\alpha)}(t)$ in the right-hand side of (30) cannot be achieved.

(b) *The case $a \neq \beta$, $b \in \mathbb{C} \setminus \{0\}$. Then we write:*

$$\mathcal{G}^{(\alpha)}(t) = \frac{2^{(1-k)\alpha} t^{k\alpha}}{\beta^{b\alpha}} \cdot \left(\frac{1}{e^t - \left(\frac{a}{\beta}\right)^b} \right)^{\alpha} \cdot e^{yt+zt^2} C_0(-xt^2).$$

If $k\alpha \in \mathbb{C} \setminus \mathbb{N}_0$, then the factor $t^{k\alpha}$ is not analytic at $t = 0$. Hence the Maclaurin series of $\mathcal{G}^{(\alpha)}(t)$ cannot be expanded as the Maclaurin series in the right member of (30).

(ii) *Considering the Maclaurin series expansion of e^{zt^2}, we find that:* ${}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)$ $(n = 0, 1)$ *do not include the variable z, that is, and are constant with respect to the variable z. In view of (14),* ${}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)$ $(n = 0, 1)$ *are constant with respect to the variable x.*
The newly-introduced polynomials in Definition 1 are very general. We consider some interesting special cases.

(iii) *Obviously,*

$$ {}_S\mathscr{P}^{(\alpha)}_{n,\beta}(0,y,0;k,a,b) = \mathscr{P}^{(\alpha)}_{n,\beta}(y;k,a,b), \tag{31}$$

which are the polynomials in (22).

(iv) *We define generalized Legendre-based Apostol-Bernoulli polynomials by:*

$$ {}_S B^{(\alpha)}_n(x,y,z;\lambda) := {}_S\mathscr{P}^{(\alpha)}_{n,\lambda}(x,y,z;1,1,1),$$

which are generated by:

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{yt+zt^2} C_0(-xt^2) = \sum_{n=0}^{\infty} {}_S B^{(\alpha)}_n(x,y,z;\lambda) \frac{t^n}{n!} \tag{32}$$

$$\big(\alpha,\ x,\ y,\ z \in \mathbb{C},\ 1^{\alpha} := 1,\ \lambda = 1 \ \Rightarrow\ |t| < 2\pi;$$
$$x,\ y,\ z \in \mathbb{C},\ \alpha \in \mathbb{N}_0,\ \lambda \in \mathbb{C} \setminus \{1\} \ \Rightarrow\ |t| < |\log\lambda|\big).$$

(v) *We define generalized Legendre-based Apostol-Euler polynomials by*

$$ {}_S E^{(\alpha)}_n(x,y,z;\lambda) := {}_S\mathscr{P}^{(\alpha)}_{n,\lambda}(x,y,z;0,-1,1),$$

which are generated by

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{yt+zt^2} C_0(-xt^2) = \sum_{n=0}^{\infty} {}_S E^{(\alpha)}_n(x,y,z;\lambda) \frac{t^n}{n!} \tag{33}$$

$$\big(\alpha,\ x,\ y,\ z \in \mathbb{C},\ 1^{\alpha} := 1,\ \lambda = 1 \ \Rightarrow\ |t| < \pi;$$
$$\alpha,\ x,\ y,\ z \in \mathbb{C},\ 1^{\alpha} := 1,\ \lambda \in \mathbb{C} \setminus \{1\} \ \Rightarrow\ |t| < |\log(-\lambda)|\big).$$

(vi) *We define generalized Legendre-based Apostol-Genocchi polynomials by:*

$$ {}_S G^{(\alpha)}_n(x,y,z;\lambda) := {}_S\mathscr{P}^{(\alpha)}_{n,\frac{\lambda}{2}}\left(x,y,z;1,-\frac{1}{2},1\right),$$

which are generated by:

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{yt+zt^2} C_0(-xt^2) = \sum_{n=0}^{\infty} {}_S G^{(\alpha)}_n(x,y,z;\lambda) \frac{t^n}{n!} \tag{34}$$

$$\big(\alpha,\ x,\ y,\ z \in \mathbb{C},\ 1^{\alpha} := 1,\ \lambda = 1 \ \Rightarrow\ |t| < \pi;$$
$$x,\ y,\ z \in \mathbb{C},\ \alpha \in \mathbb{N}_0,\ \lambda \in \mathbb{C} \setminus \{1\} \ \Rightarrow\ |t| < |\log(-\lambda)|\big).$$

(vii) *We define generalized Hermite-based Apostol-type polynomials by:*

$$ {}_H\mathscr{P}_{n,\beta}^{(\alpha)}(y,z;k,a,b) := {}_S\mathscr{P}_{n,\beta}^{(\alpha)}(0,y,z;k,a,b), $$

which are generated by:

$$ \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{yt+ut^2} = \sum_{n=0}^{\infty} {}_H\mathscr{P}_{n,\beta}^{(\alpha)}(y,u;k,a,b)\frac{t^n}{n!}, \tag{35} $$

whose restrictions can be easily modified by setting $x \mapsto 0$ *and* $z \mapsto u$ *in those given in Definition 1.*

3. An Explicit Expression

In this section we afford an explicit expression for the generalized Legendre-based Apostol-type polynomials ${}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)$ in Definition 1. Here we consider only the case $a \neq \beta$, $b \in \mathbb{C} \setminus \{0\}$.

(i) Ozden's generalized numbers (17) (see [18]) are $\mathscr{P}_{n,\beta}(k,a,b) = {}_S\mathscr{P}_{n,\beta}^{(1)}(0,0,0;k,a,b)$ which are generated by:

$$ \frac{2^{1-k}t^k}{\beta^b} \cdot f(t) = \frac{2^{1-k}t^k}{\beta^b\, e^t - a^b} = \sum_{n=0}^{\infty} \mathscr{P}_{n,\beta}(k,a,b)\,\frac{t^n}{n!}, \tag{36} $$

where

$$ f(t) := \frac{1}{e^t - (a/\beta)^b} = \sum_{n=0}^{\infty} f^{(n)}(0)\,\frac{t^n}{n!}. \tag{37} $$

Here, the restrictions are given as in (18). Applying Faà di Bruno's formula (see, e.g., [8] (p. 5)) to $f(t)$, we obtain an explicit expression for $\mathscr{P}_{n,\beta}(k,a,b)$:

$$ \mathscr{P}_{n,\beta}(k,a,b) = \begin{cases} 0 & (n = 0,\, 1,\, \ldots,\, k-1), \\ \frac{n!\,2^{1-k}}{(n-k)!\,\beta^b}\, f^{(n-k)}(0) & (n \geq k). \end{cases} \tag{38} $$

Here,

$$ f^{(n)}(0) = \sum \frac{n!}{m_1!m_2!\cdots m_n!} \cdot \frac{(-1)^{\ell}\,\ell!}{\left(1-(a/\beta)^b\right)^{\ell+1}} \cdot \prod_{\nu=1}^{n}\left(\frac{1}{\nu!}\right)^{m_\nu} \quad (n \in \mathbb{N}_0), \tag{39} $$

where the sum is taken over all n-tuples $(m_1, m_2, \ldots, m_n) \in (\mathbb{N}_0)^n$ that satisfy:

$$ \begin{cases} m_1 + 2\,m_2 + \cdots + n\,m_n = n, \\ m_1 + m_2 + \cdots + m_n = \ell. \end{cases} \tag{40} $$

The first few of $f^{(n)}(0)$ are:

$$ f(0) = \frac{1}{1-(a/\beta)^b}, \quad f^{(1)}(0) = -\frac{1}{\left(1-(a/\beta)^b\right)^2}, \quad f^{(2)}(0) = \frac{1+(a/\beta)^b}{\left(1-(a/\beta)^b\right)^3}, $$
$$ f^{(3)}(0) = -\frac{1+4(a/\beta)^b+(a/\beta)^{2b}}{\left(1-(a/\beta)^b\right)^4}. $$

(ii) Consider Özarslan's generalized Apostol-type numbers $\mathscr{P}^{(\alpha)}_{n,\beta}(k,a,b)$ (29) (see [17]). Then we write the generating function in (29) as follows:

$$\mathcal{H}_\alpha(t) := \frac{2^{(1-k)\alpha}t^{k\alpha}}{\beta^{b\alpha}} \cdot \mathcal{F}_\alpha(t) = \sum_{n=0}^{\infty} \mathscr{P}^{(\alpha)}_{n,\beta}(k,a,b)\frac{t^n}{n!}, \tag{41}$$

where $\mathcal{F}_\alpha(t) := (f(t))^\alpha$, $f(t)$ is the same as given in (37) and whose restrictions are the same as those given in Definition 1 with replaced x, y, and z by 0, in particular, $k\alpha \in \mathbb{N}_0$. By applying Faà di Bruno's formula (see, e.g., [8] (p. 5)) to $\mathcal{F}_\alpha(t)$, we obtain:

$$\mathcal{F}^{(n)}_\alpha(0) = \sum \frac{n!}{m_1!m_2!\cdots m_n!} \cdot \frac{\ell!\binom{\alpha}{\ell}}{\left(1-(a/\beta)^b\right)^{\alpha-\ell}} \cdot \prod_{j=1}^{n}\left(\frac{f^{(j)}(0)}{j!}\right)^{m_j}, \tag{42}$$

where $f^{(j)}(0)$ are the same as in (39) and the sum is taken over all n-tuples $(m_1, m_2, \ldots, m_n) \in (\mathbb{N}_0)^n$ that satisfy (40). We find from (41) that:

$$\mathcal{H}_\alpha(t) = \frac{2^{(1-k)\alpha}}{\beta^{b\alpha}} \cdot \sum_{n=0}^{\infty} \mathcal{F}^{(n)}_\alpha(0)\,\frac{t^{n+k\alpha}}{n!} = \frac{2^{(1-k)\alpha}}{\beta^{b\alpha}} \sum_{n=k\alpha}^{\infty} \mathcal{F}^{(n-k\alpha)}_\alpha(0)\,\frac{t^n}{(n-k\alpha)!}. \tag{43}$$

Comparing the coefficients of t^n on the right-most sides of (41) and (43), we get:

$$\mathscr{P}^{(\alpha)}_{n,\beta}(k,a,b) = \begin{cases} 0 & (n = 0,\,1,\,\ldots,\,k\alpha-1), \\ \dfrac{2^{(1-k)\alpha}}{\beta^{b\alpha}} \cdot \dfrac{n!}{(n-k\alpha)!} \cdot \mathcal{F}^{(n-k\alpha)}_\alpha(0) & (n \geq k\alpha). \end{cases} \tag{44}$$

The first few of $\mathcal{F}^{(n)}_\alpha(0)$ are:

$$\mathcal{F}^{(0)}_\alpha(0) = \frac{1}{\left(1-(a/\beta)^b\right)^\alpha}, \quad \mathcal{F}^{(1)}_\alpha(0) = -\frac{\alpha}{\left(1-(a/\beta)^b\right)^{\alpha+1}},$$

$$\mathcal{F}^{(2)}_\alpha(0) = \frac{\alpha\left(\alpha-1+(a/\beta)^b\right)}{\left(1-(a/\beta)^b\right)^{\alpha+2}}, \quad \mathcal{F}^{(3)}_\alpha(0) = -\frac{\alpha\left(\alpha^2+(3\alpha+1)(a/\beta)^b+(a/\beta)^{2b}\right)}{\left(1-(a/\beta)^b\right)^{\alpha+3}}.$$

(iii) The polynomials ${}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)$ in Definition 1. Write the generating function in (30) as follows:

$$\mathcal{G}^{(\alpha)}(t) = \left(\frac{2^{1-k}t^k}{\beta^b \cdot e^t - a^b}\right)^\alpha \cdot e^{yt}C_0(-xt^2) \cdot e^{zt^2}.$$

Then, use (12) and (29) in the factors in the right member to get the triple series which, by employing series manipulation techniques two times, gives a single series. Finally, equating the coefficients of t^n on the resulting single series and the right-hand sided series of (30), we obtain the following explicit expression:

$${}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b) = \sum_{m=0}^{n}\sum_{r=0}^{[m/2]} \frac{n!\,\mathscr{P}^{(\alpha)}_{n-m,\beta}(k,a,b)\,S_{m-2r}(x,y)\,z^r}{(n-m)!\,(m-2r)!\,r!} \tag{45}$$

$$\left(k,\,k\alpha \in \mathbb{N},\; n \in \mathbb{N}_0;\, a \neq \beta,\, b \in \mathbb{C}\setminus\{0\} \;\Rightarrow\; |t| < \left|b\log\left(\frac{a}{\beta}\right)\right|\right),$$

where $\mathscr{P}^{(\alpha)}_{n,\beta}(k,a,b)$ and $S_m(x,y)$ are given in (7) and (44), respectively.

The first few of ${}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)$ when $k\alpha = 1$ are:

$$
\begin{aligned}
&{}_S\mathscr{P}_{0,\beta}^{(\alpha)}(x,y,z;k,a,b) = 0, \quad {}_S\mathscr{P}_{1,\beta}^{(\alpha)}(x,y,z;k,a,b) = \left(\frac{2}{\beta^b - a^b}\right)^{\alpha}, \\
&{}_S\mathscr{P}_{2,\beta}^{(\alpha)}(x,y,z;k,a,b) = \left(\frac{2}{\beta^b - a^b}\right)^{\alpha} \left(y - \frac{\alpha}{1-(a/\beta)^b}\right), \\
&{}_S\mathscr{P}_{3,\beta}^{(\alpha)}(x,y,z;k,a,b) = 3\left(\frac{2}{\beta^b - a^b}\right)^{\alpha} \\
&\qquad \times \left\{ x + \frac{y^2}{2} + z - \frac{2\alpha y}{1-(a/\beta)^b} + \frac{\alpha\left(\alpha - 1 + (a/\beta)^b\right)}{2\left(1-(a/\beta)^b\right)^2} \right\}.
\end{aligned}
$$

It may be worthwhile to mention in passing that Özarslan [12] (Theorem 2.1) presented an explicit representation of the unified family (22) in terms of a terminating Gauss hypergeometric function ${}_2F_1(z)$ with the argument z depending on the inner summation index.

4. Integral Formula

The following theorem establishes an integral formula involving the generalized Legendre-based Apostol-type polynomials (30).

Theorem 1. *The following integral formula holds:*

$$
\begin{aligned}
&\int_0^u {}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\, dy \\
&\quad = \begin{cases} \frac{{}_S\mathscr{P}_{n+1,\beta}^{(\alpha)}(x,u,z;k,a,b) - {}_S\mathscr{P}_{n+1,\beta}^{(\alpha)}(x,0,z;k,a,b)}{n+1} & (n \in \mathbb{N}) \\ 0 & (n = 0). \end{cases}
\end{aligned} \tag{46}
$$

Proof. By integrating both sides of (30) with respect to y from $y = 0$ to $y = u$, we get:

$$
\left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\alpha} e^{zt^2} C_0(-xt^2)\, \frac{e^{ut}-1}{t} = \sum_{n=0}^{\infty} \int_0^u {}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\, dy\, \frac{t^n}{n!},
$$

which, in view of the generating function (30) itself, yields:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \int_0^u {}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\, dy\, \frac{t^{n+1}}{n!} \\
&\quad = \sum_{n=0}^{\infty} \frac{{}_S\mathscr{P}_{n+1,\beta}^{(\alpha)}(x,u,z;k,a,b) - {}_S\mathscr{P}_{n+1,\beta}^{(\alpha)}(x,0,z;k,a,b)}{n+1}\, \frac{t^{n+1}}{n!}.
\end{aligned} \tag{47}
$$

Equating the coefficients of t^{n+1} on both sides of (47), we obtain (46). □

5. Differential Formulas

The following theorem offers some differential formulae for the polynomials in Definition 1.

Theorem 2. *The following differential formulas hold:*

$$
\begin{aligned}
&x\, \frac{\partial^2}{\partial x^2}\, {}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b) + \frac{\partial}{\partial x}\, {}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b) \\
&\quad = \frac{n!}{(n-2)!}\, {}_S\mathscr{P}_{n-2,\beta}^{(\alpha)}(x,y,z;k,a,b) \quad (n \in \mathbb{N} \setminus \{1\});
\end{aligned} \tag{48}
$$

$$\frac{\partial^m}{\partial y^m}\,{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)=\frac{n!}{(n-m)!}\,{}_S\mathscr{P}^{(\alpha)}_{n-m,\beta}(x,y,z;k,a,b) \tag{49}$$

$$(n,\, m\in\mathbb{N}_0,\, m\leq n);$$

$$\frac{\partial}{\partial z}{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)=\frac{n!}{(n-2)!}\,{}_S\mathscr{P}^{(\alpha)}_{n-2,\beta}(x,y,z;k,a,b)\quad (n\in\mathbb{N}\setminus\{1\}); \tag{50}$$

$$\frac{\partial^{m+1}}{\partial y^m\partial z}{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)=\frac{n!}{(n-2-m)!}\,{}_S\mathscr{P}^{(\alpha)}_{n-2-m,\beta}(x,y,z;k,a,b) \tag{51}$$

$$(n\in\mathbb{N}\setminus\{1\},\, m\in\mathbb{N}_0,\, m\leq n-2).$$

Proof. From (14), we find:

$$\frac{\partial}{\partial x}C_0\left(-xt^2\right)=\frac{\partial}{\partial x}\sum_{r=0}^{\infty}\frac{x^r t^{2r}}{(r!)^2}=\sum_{r=1}^{\infty}\frac{x^{r-1}t^{2r}}{r!\,(r-1)!}.$$

Then we have:

$$x\,\frac{\partial}{\partial x}C_0\left(-xt^2\right)=\sum_{r=1}^{\infty}\frac{x^r t^{2r}}{r!\,(r-1)!}. \tag{52}$$

Differentiating both sides of (52) with respect to the variable x, we obtain:

$$x\frac{\partial^2}{\partial x^2}C_0\left(-xt^2\right)+\frac{\partial}{\partial x}C_0\left(-xt^2\right)=t^2\sum_{r=0}^{\infty}\frac{x^r t^{2r}}{(r!)^2}=t^2\,C_0\left(-xt^2\right). \tag{53}$$

Using (53) in (30), we obtain:

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)\frac{t^n}{n!}+x\sum_{n=0}^{\infty}\frac{\partial^2}{\partial x^2}{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)\frac{t^n}{n!}$$
$$=\sum_{n=2}^{\infty}{}_S\mathscr{P}^{(\alpha)}_{n-2,\beta}(x,y,z;k,a,b)\frac{t^n}{(n-2)!},$$

which, in view of (i) in Remark 2, leads to:

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\partial}{\partial x}{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)\frac{t^n}{n!}+x\sum_{n=2}^{\infty}\frac{\partial^2}{\partial x^2}{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)\frac{t^n}{n!}\\&=\sum_{n=2}^{\infty}{}_S\mathscr{P}^{(\alpha)}_{n-2,\beta}(x,y,z;k,a,b)\frac{t^n}{(n-2)!}.\end{aligned} \tag{54}$$

Equating the coefficients of t^n on both sides of (54), we get (48).
Differentiating both sides of (30) with respect the variable y, m times, we find:

$$\begin{aligned}&\sum_{n=m}^{\infty}\frac{\partial^m}{\partial y^m}\,{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)\frac{t^n}{n!}=t^m\cdot\left(\frac{2^{1-k}t^k}{\beta^b\,e^t-a^b}\right)^{\alpha}e^{yt+zt^2}C_0(-xt^2)\\&=\sum_{n=0}^{\infty}{}_S\mathscr{P}^{(\alpha)}_{n,\beta}(x,y,z;k,a,b)\frac{t^{n+m}}{n!}\\&=\sum_{n=m}^{\infty}{}_S\mathscr{P}^{(\alpha)}_{n-m,\beta}(x,y,z;k,a,b)\frac{t^n}{(n-m)!}.\end{aligned} \tag{55}$$

Equating the coefficients of t^n on both sides of (55), we get (49).

Differentiating both sides of (30) with respect the variable z, in view of (i) in Remark 2, we have:

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\partial}{\partial z} {}_{S}\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!} \\
&\quad = t^2 \cdot \left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\alpha} e^{yt+zt^2} C_0(-xt^2) \\
&\quad = \sum_{n=0}^{\infty} {}_{S}\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^{n+2}}{n!} \\
&\quad = \sum_{n=2}^{\infty} {}_{S}\mathscr{P}_{n-2,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{(n-2)!}.
\end{aligned} \tag{56}$$

Equating the coefficients of t^n on the first and last summations of (56), we obtain (50).

Differentiating both sides of (50), m times, with respect the variable y, and using (49), we get (51). □

6. Addition Formulas

Theorem 3 establishes some addition formulae for the generalized Legendre-based Apostol-type polynomials (30).

To facilitate application, we will first review certain formal manipulations of double series, as shown in the following lemma (see, e.g., [6] (pp. 56–57), [24,25,39], and [40] (p. 52)).

Lemma 1. *The following identities hold:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathcal{B}_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \mathcal{B}_{k,n-pk} \quad (p \in \mathbb{N}) \tag{57}$$

and

$$\sum_{N=0}^{\infty} g(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} g(m+n)\frac{x^n}{n!}\frac{y^m}{m!}. \tag{58}$$

Here, the $\mathcal{B}_{k,n}$ and $g(N)$ $(k,\ n,\ N \in \mathbb{N}_0)$ are real or complex valued functions of the k, n, and N, respectively, and x and y are real or complex numbers. Also, in order to verify rearrangements of the involved series, each associated series should be absolutely convergent.

Theorem 3. *Let $a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{R}^+$ and $\alpha,\ \gamma \in \mathbb{C}$, $\beta \in \mathbb{C}\setminus\{0\}$. Also let $k,\ n \in \mathbb{N}_0$. Further let $u,\ v,\ x,\ y,\ z \in \mathbb{C}$. Then,*

$$\begin{aligned}
&{}_{S}\mathscr{P}_{n,\beta}^{(\alpha+\gamma)}(x,y+z,u+v;k,a,b) \\
&\quad = \sum_{j=0}^{n}\binom{n}{j} {}_{S}\mathscr{P}_{n-j,\beta}^{(\alpha)}(x,y,u;k,a,b)\, {}_{H}\mathscr{P}_{j,\beta}^{(\gamma)}(z,v;k,a,b) \\
&\quad = \sum_{j=0}^{[n/2]}\sum_{k=0}^{n-2j} \frac{n!\, v^j}{(n-2j-k)!k!j!}\, {}_{S}\mathscr{P}_{n-2j-k,\beta}^{(\alpha)}(x,y,u;k,a,b)\, \mathscr{P}_{k,\beta}^{(\gamma)}(z;k,a,b) \\
&\quad = \sum_{j=0}^{n}\sum_{k=0}^{n-j}\binom{n}{j+k}\binom{j+k}{k} {}_{S}\mathscr{P}_{n-j-k,\beta}^{(\alpha)}(x,y,u;k,a,b)\, \mathscr{P}_{k,\beta}^{(\gamma)}(k,a,b)\, H_j(z,v).
\end{aligned} \tag{59}$$

Proof. From (30), we have:

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{S}\mathscr{P}_{n,\beta}^{(\alpha+\gamma)}(x,y+z,u+v;k,a,b)\frac{t^n}{n!} \\ = \left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\alpha+\gamma} e^{(y+z)t+(u+v)t^2}\, C_0(-xt^2).\end{aligned} \tag{60}$$

Let $\mathcal{R}$ be the right side of (60). Using (30) and (35), we find:

$$\begin{aligned}\mathcal{R} &= \left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\alpha} e^{yt+ut^2}\, C_0(-xt^2)\cdot\left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\gamma} e^{zt+vt^2} \\ &= \sum_{n=0}^{\infty} {}_{S}\mathscr{P}_{n,\beta}^{(\alpha)}(x,y,u;k,a,b)\frac{t^n}{n!}\cdot \sum_{j=0}^{\infty} {}_{H}\mathscr{P}_{j,\beta}^{(\gamma)}(z,v;k,a,b)\frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} {}_{S}\mathscr{P}_{n-j,\beta}^{(\alpha)}(x,y,u;k,a,b)\, {}_{H}\mathscr{P}_{j,\beta}^{(\gamma)}(z,v;k,a,b)\, \frac{t^n}{(n-j)!j!},\end{aligned} \tag{61}$$

where (57) with $p=1$ is used. Now, equating the first expression of (60) the last one of (61), and identifying the coefficients of t^n of the resulting identity, we get the first equality of (59).

We write:

$$\mathcal{R} = \left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\alpha} e^{yt+ut^2}\, C_0(-xt^2)\cdot\left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\gamma} e^{zt}\cdot e^{vt^2}. \tag{62}$$

Use (30) and (22), respectively, in the first and second factors in (62), and the Maclaurin series expansion of e^{vt^2} to get a product of three series expansions. Then, applying (57) with $p=1$ to combine the first two series to get a single series and, with the aid of (57) with $p=2$, combining the resulting single series and the third remaining series, we get a single series for the product of three series expansions. Similarly, as in getting the first equality, we can obtain the second equality of (60).

We arrange:

$$\mathcal{R} = \left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\alpha} e^{yt+ut^2}\, C_0(-xt^2)\cdot\left(\frac{2^{1-k}t^k}{\beta^b\, e^t - a^b}\right)^{\gamma}\cdot e^{zt+vt^2}. \tag{63}$$

Using (30), (29), and (1) in the first, second, and third factors in (63), similarly as in proving the second one, we obtain the third equality. □

7. Implicit Summation Formula Involving Generalized Legendre-Based Apostol-Type Polynomials

Implicit summation formulas for a number of relatively simple polynomials in comparison with the polynomials in (30), for example, ordinary Hermite and related polynomials and Hermite-based polynomials, have been investigated (see, e.g., [17,24]). Theorem 4 asserts the existence of an implicit summation formula involving generalized Legendre-based Apostol-type polynomials (30).

Theorem 4. *Let $a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{R}^+$, and $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}\setminus\{0\}$. Also let k, m, $n \in \mathbb{N}_0$. Furthermore, let v, x, y, $z \in \mathbb{C}$. Then:*

$$\begin{aligned}{}_{S}\mathscr{P}_{m+n,\beta}^{(\alpha)}(x,v,z;k,a,b) \\ = \sum_{p=0}^{n}\sum_{q=0}^{m}\binom{n}{p}\binom{m}{q}(v-y)^{p+q}\, {}_{S}\mathscr{P}_{m+n-p-q,\beta}^{(\alpha)}(x,y,z;k,a,b).\end{aligned} \tag{64}$$

Proof. Replacing t by $t+u$ in (30) and using (58), we obtain:

$$\begin{aligned}&\left(\frac{2^{1-k}(t+u)^k}{\beta^b \mathrm{e}^{(t+u)} - a^b}\right)^{\alpha} \mathrm{e}^{z(t+u)^2} C_0(-x(t+u)^2) \\ &\quad = \mathrm{e}^{-y(t+u)} \sum_{m,n=0}^{\infty} {}_S\mathscr{P}_{m+n,\beta}^{(\alpha)}(x,y,z;k,a,b)\, \frac{t^n}{n!}\, \frac{u^m}{m!}.\end{aligned} \tag{65}$$

We find that the left side of (65) is independent of the variable y. Replacing y by any other variable, say v, in the right side of (65), and equating the right sides of (65) and the resulting identity, we get:

$$\begin{aligned}&\sum_{m,n=0}^{\infty} {}_S\mathscr{P}_{m+n,\beta}^{(\alpha)}(x,v,z;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!} \\ &\quad = \mathrm{e}^{(v-y)(t+u)} \sum_{m,n=0}^{\infty} {}_S\mathscr{P}_{m+n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!}.\end{aligned} \tag{66}$$

Using (58), we have:

$$\mathrm{e}^{(v-y)(t+u)} = \sum_{N=0}^{\infty} (v-y)^N \frac{(t+u)^N}{N!} = \sum_{p,\,q=0}^{\infty} (v-y)^{p+q}\, \frac{t^p}{p!}\, \frac{u^q}{q!}. \tag{67}$$

Let $\mathcal{R}_1$ be the right side of (66). Setting (67) and using (57), we find:

$$\begin{aligned}\mathcal{R}_1 &= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{\infty} {}_S\mathscr{P}_{m+n,\beta}^{(\alpha)}(x,y,z;k,a,b)\,(v-y)^{p+q}\,\frac{t^n}{n!}\frac{t^p}{p!}\frac{u^m}{m!}\frac{u^q}{q!} \\ &= \sum_{m,\,n=0}^{\infty}\sum_{p=0}^{n}\sum_{q=0}^{m} {}_S\mathscr{P}_{m+n-p-q,\beta}^{(\alpha)}(x,y,z;k,a,b)\,(v-y)^{p+q}\,\frac{t^n}{(n-p)!p!}\,\frac{u^m}{(m-q)!q!}.\end{aligned} \tag{68}$$

Equating the left side of (66) and the second one of (68) and identifying the coefficients of t^n and u^m of the resulting identity, we obtain (64). □

Since the generalized Legendre-based Apostol-type polynomials (30) is very general, a number of special cases of the result in Theorem 4 can be considered. Here, we demonstrate only implicit summation formulas for the generalized Legendre-based Apostol-Bernoulli polynomials (32), the generalized Legendre-based Apostol-Euler polynomials (33), and the generalized Legendre-based Apostol-Genocchi polynomials (34), which are, respectively, given in Corollaries 1–3.

Corollary 1. *Let $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C}\setminus\{0\}$. Also let $m,\, n \in \mathbb{N}_0$. Furthermore, let $v,\, x,\, y,\, z \in \mathbb{C}$. Then,*

$$\begin{aligned}&{}_SB_{m+n}^{(\alpha)}(x,v,z;\lambda) \\ &\quad = \sum_{p=0}^{n}\sum_{q=0}^{m}\binom{n}{p}\binom{m}{q}(v-y)^{p+q}\,{}_SB_{m+n-p-q}^{(\alpha)}(x,y,z;\lambda).\end{aligned} \tag{69}$$

Corollary 2. *Let $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C}\setminus\{0\}$. Also let $m,\, n \in \mathbb{N}_0$. Furthermore, let $v,\, x,\, y,\, z \in \mathbb{C}$. Then,*

$$\begin{aligned}&{}_SE_{m+n}^{(\alpha)}(x,v,z;\lambda) \\ &\quad = \sum_{p=0}^{n}\sum_{q=0}^{m}\binom{n}{p}\binom{m}{q}(v-y)^{p+q}\,{}_SE_{m+n-p-q}^{(\alpha)}(x,y,z;\lambda).\end{aligned} \tag{70}$$

Corollary 3. *Let $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$. Also let m, $n \in \mathbb{N}_0$. Furthermore, let v, x, y, $z \in \mathbb{C}$. Then,*

$$
\begin{aligned}
&{}_S G_{m+n}^{(\alpha)}(x,v,z;\lambda) \\
&\quad = \sum_{p=0}^{n} \sum_{q=0}^{m} \binom{n}{p} \binom{m}{q} (v-y)^{p+q} \, {}_S G_{m+n-p-q}^{(\alpha)}(x,y,z;\lambda).
\end{aligned} \tag{71}
$$

8. Symmetry Identities for Generalized Legendre-Based Apostol-Type Polynomials

Khan et al. [24,25], Özarslan [17], Yang [35], and Zhang and Yang [38] have presented symmetry identities for various simpler polynomials than the polynomials (30). Here we establish some symmetry identities for the generalized Legendre-based Apostol-type polynomials (30), which are stated in Theorem 5.

Theorem 5. *Let $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^+$ and α_1, $\alpha_2 \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$. Also let k, $n \in \mathbb{N}_0$. Furthermore, let x_1, x_2, y_1, y_2, z_1, $z_2 \in \mathbb{C}$. Then,*

$$
\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \, {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_1,y_1,z_1;k,a,b) \, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_2,y_2,z_2;k,a,b) \\
&= \sum_{k=0}^{n} \binom{n}{k} \, {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_1,y_1,z_2;k,a,b) \, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_2,y_2,z_1;k,a,b) \\
&= \sum_{k=0}^{n} \binom{n}{k} \, {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_1,y_2,z_1;k,a,b) \, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_2,y_1,z_2;k,a,b) \\
&= \sum_{k=0}^{n} \binom{n}{k} \, {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_1,y_2,z_2;k,a,b) \, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_2,y_1,z_1;k,a,b) \\
&= \sum_{k=0}^{n} \binom{n}{k} \, {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_2,y_1,z_1;k,a,b) \, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_1,y_2,z_2;k,a,b) \\
&= \sum_{k=0}^{n} \binom{n}{k} \, {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_2,y_1,z_2;k,a,b) \, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_1,y_2,z_1;k,a,b) \\
&= \sum_{k=0}^{n} \binom{n}{k} \, {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_2,y_2,z_1;k,a,b) \, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_1,y_1,z_2;k,a,b) \\
&= \sum_{k=0}^{n} \binom{n}{k} \, {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_2,y_2,z_2;k,a,b) \, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_1,y_1,z_1;k,a,b).
\end{aligned} \tag{72}
$$

Proof. Let,

$$
g(t) := \left(\frac{2^{1-k} \, t^k}{\beta^b \, e^t - a^b} \right)^{\alpha_1+\alpha_2} e^{(y_1+y_2)\,t+(z_1+z_2)\,t^2} C_0\big(-x_1 t^2\big) C_0\big(-x_2 t^2\big). \tag{73}
$$

We find that $g(t)$ is symmetric with respect to the two variables in each of the following four sets $\{\alpha_1, \alpha_2\}$, $\{x_1, x_2\}$, $\{y_1, y_2\}$, and $\{z_1, z_2\}$. In order to define the generalized Legendre-based Apostol-type polynomials (30), we need $\{\alpha_i, x_j, y_k, z_l\}$ $(i,\ j,\ k,\ l = 1,\, 2)$. Then we observe that $g(t)$ has the following eight pairs to define the polynomials (30):

(i) $\{\alpha_1, x_1, y_1, z_1\} \longleftrightarrow \{\alpha_2, x_2, y_2, z_2\}$;
(ii) $\{\alpha_1, x_1, y_1, z_2\} \longleftrightarrow \{\alpha_2, x_2, y_2, z_1\}$;
(iii) $\{\alpha_1, x_1, y_2, z_1\} \longleftrightarrow \{\alpha_2, x_2, y_1, z_2\}$;
(iv) $\{\alpha_1, x_1, y_2, z_2\} \longleftrightarrow \{\alpha_2, x_2, y_1, z_1\}$;
(v) $\{\alpha_1, x_2, y_1, z_1\} \longleftrightarrow \{\alpha_2, x_1, y_2, z_2\}$;
(vi) $\{\alpha_1, x_2, y_1, z_2\} \longleftrightarrow \{\alpha_2, x_1, y_2, z_1\}$;
(vii) $\{\alpha_1, x_2, y_2, z_1\} \longleftrightarrow \{\alpha_2, x_1, y_1, z_2\}$;

(viii) $\{\alpha_1, x_2, y_2, z_2\} \longleftrightarrow \{\alpha_1, x_1, y_1, z_1\}$.

Using the combination (i), with the aid of (57) with $p = 1$, we have:

$$g(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} {}_S\mathscr{P}_{n-k,\beta}^{(\alpha_1)}(x_1, y_1, z_1; k, a, b)\, {}_S\mathscr{P}_{k,\beta}^{(\alpha_2)}(x_2, y_2, z_2; k, a, b)\, \frac{t^n}{(n-k)!\, k!}. \tag{74}$$

We also can get a similar expression of $g(t)$ as in (74). Then, equating two expressions from the eight ones, we obtain the symmetry identities in (72). □

As the identity in Theorem 4, the result in Theorem 4 can reduce to yield symmetry identities for a number of relatively simple polynomials. Here, only those for the generalized Legendre-based Apostol-Bernoulli polynomials (32) are considered in Corollary 4.

Corollary 4. *Let $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$. Also let $n \in \mathbb{N}_0$.*
Furthermore, let x_1, x_2, y_1, y_2, z_1, $z_2 \in \mathbb{C}$. Then,

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k}\, {}_SB_{n-k}^{(\alpha_1)}(x_1, y_1, z_1; \lambda)\, {}_SB_{k}^{(\alpha_2)}(x_2, y_2, z_2; \lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k}\, {}_SB_{n-k}^{(\alpha_1)}(x_1, y_1, z_2; \lambda)\, {}_SB_{k}^{(\alpha_2)}(x_2, y_2, z_1; \lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k}\, {}_SB_{n-k}^{(\alpha_1)}(x_1, y_2, z_1; \lambda)\, {}_SB_{k}^{(\alpha_2)}(x_2, y_1, z_2; \lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k}\, {}_SB_{n-k}^{(\alpha_1)}(x_1, y_2, z_2; \lambda)\, {}_SB_{k}^{(\alpha_2)}(x_2, y_1, z_1; \lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k}\, {}_SB_{n-k}^{(\alpha_1)}(x_2, y_1, z_1; \lambda)\, {}_SB_{k}^{(\alpha_2)}(x_1, y_2, z_2; \lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k}\, {}_SB_{n-k}^{(\alpha_1)}(x_2, y_1, z_2; \lambda)\, {}_SB_{k}^{(\alpha_2)}(x_1, y_2, z_1; \lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k}\, {}_SB_{n-k}^{(\alpha_1)}(x_2, y_2, z_1; \lambda)\, {}_SB_{k}^{(\alpha_2)}(x_1, y_1, z_2; \lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k}\, {}_SB_{n-k}^{(\alpha_1)}(x_2, y_2, z_2; \lambda)\, {}_SB_{k}^{(\alpha_2)}(x_1, y_1, z_1; \lambda).
\end{aligned} \tag{75}$$

9. Certain Formulas Deducible from Umbral Calculus

In this section, we try to use the theory of umbral calculus (see, e.g., [41]; see also [23,36,42]) to offer some new formulas which are not presented in the previous sections. To do this, some notations with modified ones and certain known facts are recalled and introduced (see [41]). Let $\mathscr{P}$ indicate the algebra of formal power series in the variable t over the field C of characteristic zero, say, $\mathbb{C}$ the field of complex numbers. If $\eta(t)$ and $\xi(t)$ in $\mathscr{P}$ gratify:

$$\eta(t) = \sum_{k=0}^{\infty} a_k t^k \quad (a_0 = 0,\ a_1 \neq 0) \quad \text{and} \quad \xi(t) = \sum_{k=0}^{\infty} b_k t^k \quad (b_0 \neq 0),$$

then $\eta(t)$ and $\xi(t)$ are called a delta series and an invertible series, respectively. With each pair of a delta series $\eta(t)$ and an invertible series $\xi(t)$, there exists a unique sequence $s_n(u)$ of polynomials satisfying the orthogonality conditions (see [41] (p. 17, Theorem 2.3.1)):

$$\left\langle \xi(t)\, \eta(t)^k \middle| s_n(u) \right\rangle = n!\, \delta_{n,k} \quad (n,\ k \in \mathbb{N}_0), \tag{76}$$

$\delta_{n,k}$ being the Kronecker delta function defined by $\delta_{n,k} = 1$ $(n = k)$ and $\delta_{n,k} = 0$ $(n \neq k)$, if and only if the sequence $s_n(u)$ is generated by means of the following function:

$$\frac{1}{\xi(\eta^{-1}(t))}\, e^{v\,\eta^{-1}(t)} = \sum_{k=0}^{\infty} \frac{s_k(v)}{k!}\, t^k \tag{77}$$

for all v in $\mathbb{C}$, where $\eta^{-1}(t)$ is the compositional inverse of $\eta(t)$ (see [41] (p. 18, Theorem 2.3.4)). Here the operator $\langle \cdot \mid \cdot \rangle$ remains the same as in [41] (Chapter 2). That is,

$$\left\langle \xi(t)\,\eta(t)^k \middle| s_n(u) \right\rangle := \xi(D_u)\,\eta(D_u)^k \{s_n(u)\}\big|_{u=0},$$

where $D_u := \frac{\partial}{\partial u}$. The sequence $s_n(u)$ satisfying (76) or (77) is called the Sheffer sequence for $(\xi(t), \eta(t))$, or $s_n(u)$ is Sheffer for $(\xi(t), \eta(t))$. In particular, if $s_n(u)$ is Sheffer for $(1, \eta(t))$, then $s_n(u)$ is called the associated sequence for $\eta(t)$, or $s_n(u)$ is associated to $\eta(t)$; if $s_n(u)$ is Sheffer for $(\xi(t), t)$, then $s_n(u)$ is called the Appell sequence for $\xi(t)$, or $s_n(u)$ is Appell for $\xi(t)$ (see [41] (p. 17)).

Setting $a = \beta$ and $k = 1$ in (30), we obtain:

$$\xi(t; x, z; \alpha, b) \cdot e^{yt} = \sum_{n=0}^{\infty} {}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x, y, z; 1, \beta, b)\frac{t^n}{n!}, \tag{78}$$

where,

$$\xi(t) := \xi(t; x, z; \alpha, b) = \frac{1}{\beta^{b\alpha}} \left(\frac{t}{e^t - 1}\right)^{\alpha} e^{zt^2} C_0(-xt^2).$$

Since

$$\xi(0; x, z; \alpha, b) = \lim_{t \to 0} \xi(t; x, z; \alpha, b) = \frac{1}{\beta^{b\alpha}} \neq 0,$$

$\xi(t; x, z; \alpha, b)$ is an invertible series. Therefore the sequence ${}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x, y, z; 1, \beta, b)$ $(n \in \mathbb{N}_0)$ is Appell for $\xi(t; x, z; \alpha, b)$. There are many equivalent statements for the Appell sequence (see [41] (pp. 26–28)). In this regard, without their proofs, some of equivalent statements for the Appell sequence ${}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x, y, z; 1, \beta, b)$ $(n \in \mathbb{N}_0)$ are selected to be offered in the following theorems.

Theorem 6 (The Polynomial Expansion Theorem)**.** *Let the modified restrictions in Definition 1 with $a = \beta$ and $k = 1$ be assumed. Then for any polynomial $p(u)$ we obtain:*

$$p(u) = \sum_{k=0}^{\infty} \frac{\left[\xi(D_u)\left\{p^{(k)}(u)\right\}\right]_{u=0}}{k!}\, {}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x, y, z; 1, \beta, b), \tag{79}$$

where $p^{(k)}(u)$ is the kth derivative of $p(u)$.

Theorem 7 (The conjugate representation)**.** *Let the modified restrictions in Definition 1 with $a = \beta$ and $k = 1$ be assumed. Then,*

$${}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x, y, z; 1, \beta, b) = \sum_{k=0}^{n} \binom{n}{k} \left[\xi(D_u)^{-1}\{u^{n-k}\}\right]_{u=0} u^k. \tag{80}$$

Theorem 8. *Let the modified restrictions in Definition 1 with $a = \beta$ and $k = 1$ be assumed. Then,*

$${}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x, y, z; 1, \beta, b) = \xi(D_u)^{-1}\, u^n. \tag{81}$$

Theorem 9. *Let the modified restrictions in* Definition 1 *with $a = \beta$ and $k = 1$ be assumed. Then for any $h(t)$ in $\mathscr{P}$:*

$$h(D_u)\{{}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,u,z;1,\beta,b)\} = \sum_{k=0}^{n}\binom{n}{k}\left[h(D_u)\{{}_S\mathscr{P}_{k,\beta}^{(\alpha)}(x,u,z;1,\beta,b)\}\right]_{u=0} u^{n-k}. \tag{82}$$

Theorem 10 (The Multiplication Theorem)**.** *Let the modified restrictions in* Definition 1 *with $a = \beta$ and $k = 1$ be assumed. Then for any constant $\kappa \in \mathbb{C} \setminus \{0\}$:*

$${}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,\kappa u,z;1,\beta,b) = \kappa^n \frac{\xi(D_u)}{\xi(D_u/\kappa)}\{{}_S\mathscr{P}_{n,\beta}^{(\alpha)}(x,u,z;1,\beta,b)\} \quad (n \in \mathbb{N}_0). \tag{83}$$

10. Concluding Remarks

Numerous polynomials, their extensions, and variants have been intensively investigated due to their potential applications in a wide variety of research fields (see, e.g., [29,30,37]). In this paper, we introduced a unification of various polynomials which are called generalized Legendre-based Apostol-type polynomials. The constraints for these newly-introduced polynomials were specified in detail, in accordance with the Maclaurin series (see also [11] (pp. 99–100)). We provided their diverse identities in a systematic manner, including addition formulae, an implicit summation formula, and symmetry identities. Due to the generality of the discoveries reported here, they may be reduced to formulae and identities for a number of relatively simple polynomials, several of which are explicitly demonstrated. Furthermore, utilizing umbral calculus theory, we proposed several new identities involving these new polynomials that are distinct from those in the preceding sections.

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