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Article

Riemann–Liouville Fractional Sobolev and Bounded Variation Spaces †

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† Dedicated to Delfim F. M. Torres on the Occasion of His 50th Birthday.

Abstract: We establish some properties of the bilateral Riemann–Liouville fractional derivative D^s. We set the notation, and study the associated Sobolev spaces of fractional order s, denoted by $W^{s,1}(a,b)$, and the fractional bounded variation spaces of fractional order s, denoted by $BV^s(a,b)$. Examples, embeddings and compactness properties related to these spaces are addressed, aiming to set a functional framework suitable for fractional variational models for image analysis.

Keywords: fractional derivatives; distributional derivatives; Sobolev spaces; bounded variation functions; embeddings; compactness; calculus of variations; Abel equation

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1. Introduction

Among several different available definitions for fractional derivatives and corresponding functional spaces, this paper focuses the analysis on some classical pointwise defined or distributional fractional derivatives connected to integral-convolution operators. Precisely, we refer to bilateral definitions of Riemann–Liouville fractional derivatives and related Sobolev and bounded variation spaces that we introduced in [1]: here, we show some compactness and embedding properties of these spaces.

First, we recall the classical Riemann–Liouville left and right fractional derivatives $(d/dx)^s_+$ and $(d/dx)^s_-$ and introduce the distributional Riemann–Liouville left and right fractional derivatives D^s_+, D^s_- together with their bilateral even and odd versions, respectively D^s_e, D^s_o, all of them defined for non-integer orders s, $0<s<1$ (see Definition 4).

Second, we provide the definitions of the fractional Sobolev spaces $W^{s,1}$ and fractional bounded variation spaces BV^s, associated to these bilateral derivatives (see Definitions 9 and 10). These function spaces are studied here (see Theorem 6, Examples 2–5, 6 and 8) in comparison with their non-bilateral counterpart ([2–8]).

The spaces $W^{s,1}$ and BV^s turn out to be the natural setting for data of Abel integral equations in order to make them well-posed problems in the distributional framework too: see Propositions 2 and 3 showing that if $f \in BV^s(a,b)$ with $-\infty < a \le b \le +\infty$, then the distributional Abel integral equation $I^s_{a+}[u] = f$ admits a unique solution and provides an explicit resolvent formula. Corollaries 1 and 2 state analogous results for backward equations. This approach provides an alternative formulation of classical L^1 representability (see [9]); precisely, this approach leads to a straightforward extension of solvability for the Abel integral equation under conditions weaker than L^1 representability, namely with data possibly belonging to $BV^s(a,b)$.

Basic properties of the functional spaces introduced in present article (weak compactness property stated by Theorems 3 and 11 together with comparison embeddings and strict embeddings stated in Theorems 6 and 8 and by (92) and (93)), namely

$$BV(a,b) \underset{\neq}{\subsetneq} \bigcap_{\sigma\in(0,1)} W^{\sigma,1}(a,b) \underset{\neq}{\subsetneq} W^{s,1}(a,b) \underset{\neq}{\subsetneq} BV^s_+(a,b) \qquad \forall s \in (0,1),$$

$$W^{s,1}(a,b) \subsetneq BV^s_+(a,b)\,, \qquad W^{s,1}(a,b) \subsetneq BV^s_-(a,b)\,, \qquad \forall s \in (0,1)\,.$$

are studied with the aim of providing a functional framework suitable to fractional variational models for image analysis ([10–17]), which are the object of a forthcoming paper [18]. The present preliminary study deals with the one-dimensional case only.

We thank an anonymous referee for useful remarks and pointing us to the recent article [19] containing a different approach to the Sonin–Abel equation in weighted Lebesgue spaces.

2. Bilateral Fractional Integral and Derivative

In this paper $(a,b) \subset \mathbb{R}$ is a nonempty (possibly unbounded) open interval, u is a real function of one variable and $0 < s < 1$. The support of a function u is denoted by $\operatorname{spt} u$. The notation d/dx stands for the classical pointwise derivative; D_x, or shortly D, denotes the distributional derivative with respect to the variable x. For every open interval $A \subset \mathbb{R}$, we denote by $AC(A)$ the set of absolutely continuous functions with the domain in the interval A, which coincides ([20]) with the space the Gagliardo–Sobolev space $W^{1,1}_G(A) = \{u \in L^1(A) \mid Du \in L^1(A)\}$ when they are both endowed with the standard norm $\|u\|_{L^1(A)} + \|Du\|_{L^1(A)}$. Moreover, we set $L^1_{loc}(A)$ as the set of measurable functions which are Lebesgue integrable on every compact subset of A, and $AC_{loc}(A) = W^{1,1}_{G,loc}(A) = \{u \in L^1_{loc}(A) \mid Du \in L^1_{loc}(A)\}$ and $BV(A) = \{u \in L^1(A) \mid Du \in \mathcal{M}(A)\}$, where $\mathcal{M}(A)$ denotes the measures whose total variation on A is bounded. We denote by $\mathcal{D}'(A)$ and $\mathcal{S}'(A)$ respectively the space of distributions and the space of tempered distributions on the open set A. We denote by $C^{0,\alpha}(K)$ the space of of Hölder continuous functions on the set K.

For the reader's convenience, we recall the definition of Gagliardo's fractional Sobolev spaces $W^{s,1}_G$ ([21,22]). For any $s \in (0,1)$, we set

$$W^{s,1}_G = \left\{ u \in L^1(a,b) \;:\; \frac{|u(x)-u(y)|}{|x-y|^{1+s}} \in L^1([a,b]\times[a,b]) \right\}, \tag{1}$$

which is a Banach space endowed with the norm

$$\|u\|_{W^{s,1}_G} = \left[\int_{[a,b]} |u(x)|dx \;+\; \int_{[a,b]}\int_{[a,b]} \frac{|u(x)-u(y)|}{|x-y|^{1+s}} dx\,dy \right],$$

and we recall also the definition of the Riemann–Liouville fractional integral and derivative of order s for L^1-functions, whose standard references can be found in the book by Samko et al. [9].

In the sequel, H denotes the Heaviside function $H(x)=1$ if $x\geq 0$, $H(x)=0$ if $x<0$, while sign denotes the sign function $\operatorname{sign}(x)=1$ if $x>0$, $\operatorname{sign}(x)=-1$ if $x<0$, $\operatorname{sign}(0)=0$.

Definition 1. ***(Riemann–Liouville fractional integral)***

Assume $u \in L^1(a,b)$ and $s > 0$.

The left-side and right-side Riemann–Liouville fractional integrals ${}_{RL}I^s_{a+}$ and ${}_{RL}I^s_{b-}$ are defined by setting, respectively,

$${}_{RL}I^s_{a+}[u](x) \;=\; \frac{1}{\Gamma(s)}\int_a^x \frac{u(t)}{(x-t)^{1-s}}dt\,, \qquad x \in [a,b]\,, \tag{2}$$

$${}_{RL}I^s_{b-}[u](x) \;=\; \frac{1}{\Gamma(s)}\int_x^b \frac{u(t)}{(t-x)^{1-s}}dt\,, \qquad x \in [a,b]\,, \tag{3}$$

Here, Γ denotes the Euler gamma function [23].

Notice that ${}_{RL}I^1_{a+}[u](x) = \int_a^x u(t)\,dt$ and in general, for every strictly positive integer value, $s = n \in \mathbb{N}$, ${}_{RL}I^n_{a+}[u]$ coincides with the n-th order primitive, vanishing at $x = a$ together with all derivatives up to order $n-1$.

Both ${}_{RL}I^s_{a+}[u]$ and ${}_{RL}I^s_{b-}[u]$ are absolutely continuous functions if $s \geq 1$ since they are primitives of L^1 functions, whereas we can only say that they are L^q functions if $0<s<1$,

$1 \le q < 1/(1-s)$ (see [9]): indeed, jump discontinuities are allowed if $0 < s < 1$, as shown by the next example.

Example 1. *Set $(a,b) = (-1,1)$. Then for every $0 < s < 1$ there is $u \in L^p(-1,1)$, $1 \le p < 1/s$, s.t. $I^s_{(-1)+}[u]$ is discontinuous. For instance, consider $u(x) = H(x)\,x^{-s}$, thus, exploiting the Euler beta function $B(\nu,\mu) = \int_0^1 y^{\nu-1}(1-y)^{\mu-1}dy = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}$, one gets*

$$_{RL}I^s_{(-1)+}[u](x) = \frac{1}{\Gamma(s)}\int_{-1}^{x} \frac{H(t)}{t^s\,(x-t)^{1-s}}\,dt = H(x)\frac{B(s,1-s)}{\Gamma(s)} = \begin{cases} 0 & \text{if } -1 < x \le 0 \\ \Gamma(1-s) & \text{if } 0 < x < 1\,. \end{cases}$$

Thus, $I^s_{(-1)+}[u]$ is a piecewise constant function on $(-1,+1)$ with a jump at $x = 0$.

Next, we recall the classical definition of left and right Riemann–Liouville fractional derivatives as in [9,24–27].

Definition 2. ***(Classical Riemann–Liouville fractional derivative)***
Assume $u \in L^1(a,b)$ and $0 < s < 1$.
The left Riemann–Liouville derivative of u at $x \in [a,b]$ is defined by

$$_{RL}\left(\frac{d}{dx}\right)^s_{a+}[u](x) \;=\; \frac{d}{dx}\,{}_{RL}I^{1-s}_{a+}[u](x) \;=\; \frac{1}{\Gamma(1-s)}\frac{d}{dx}\int_a^x \frac{u(t)}{(x-t)^s}dt \tag{4}$$

for every value of x such that this derivative exists.
Similarly, we may define the right Riemann–Liouville derivative of u at $x \in [a,b]$ as

$$_{RL}\left(\frac{d}{dx}\right)^s_{b-}[u](x) \;=\; -\frac{d}{dx}\,{}_{RL}I^{1-s}_{b-}[u](x) \;=\; \frac{-1}{\Gamma(1-s)}\frac{d}{dx}\int_x^b \frac{u(t)}{(t-x)^s}dt \tag{5}$$

for every value of x such that this derivative exists.

Then we introduce the distributional Riemann–Liouville fractional derivative as in [25]: a refinement of the Riemann–Liouville fractional derivative, obtained by the plain substitution of the pointwise classical derivative with the distributional derivative

Definition 3. ***(Distributional Riemann–Liouville fractional derivative)***
Assume $u \in L^1(a,b)$ and $0 < s < 1$. The distributional left Riemann–Liouville derivative of u, $_{RL}D^s_{a+}[u] \in \mathcal{D}'(a,b)$, is defined by

$$_{RL}D^s_{a+}[u](x) \;=\; D_x\,{}_{RL}I^{1-s}_{a+}[u](x) \;=\; \frac{1}{\Gamma(1-s)}\,D_x\int_a^x \frac{u(t)}{(x-t)^s}dt\,. \tag{6}$$

Similarly, we may define the distributional right Riemann–Liouville derivative of u, $_{RL}D^s_{b-}[u] \in D'(a,b)$, as

$$_{RL}D^s_{b-}[u](x) \;=\; -D_x\,{}_{RL}I^{1-s}_{b-}[u](x) \;=\; \frac{-1}{\Gamma(1-s)}\,D_x\int_x^b \frac{u(t)}{(t-x)^s}dt\,. \tag{7}$$

Remark 1. *The distributional Riemann–Liouville fractional derivatives $D^s_{\pm}$ provide a suitable refinement of the classical ones $(d/dx)^s_{\pm}$ for the purposes of the present paper. However, we emphasize that they coincide on every L^1 function u such that $I^{1-s}[u]$ is absolutely continuous, as it was always the case in the classical applications of fractional derivatives ([28–30]).*

In Lemma 5 below, we examine the case when the above pointwise-defined derivative d/dx exists a.e. and defines an L^1 function coincident with the distributional derivative D, respectively of $_{RL}I^{1-s}_{a+}[u]$ and $_{RL}I^{1-s}_{b-}[u]$.

In the sequel, we omit the suffix RL of the interval without loss of information, since in this paper, we do not consider any other fractional derivative than the Riemann–Liouville one; we omit also the endpoints a_+ and b_- suffix whenever they are clearly established.

Therefore, we will write shortly $I^s_+[u]$, $I^s_-[u]$, $D^s_+[u]$, $D^s_-[u]$, $(d/dx)^s_+[u]$ and $(d/dx)^s_-[u]$, respectively, in place of ${}_{RL}I^s_{a+}[u]$, ${}_{RL}I^s_{b-}[u]$, ${}_{RL}D^s_{a+}[u]$, ${}_{RL}D^s_{b-}[u]$, ${}_{RL}(d/dx)^s_{a+}[u]$ and ${}_{RL}(d/dx)^s_{b-}[u]$.

One of the disadvantages of the one-side Riemann–Liouville derivative and integral, as defined above, is the fact that only one endpoint of the interval plays a role (see (56) and (57)) since they are "anisotropic" definitions (see [31] and Lemma 6). On the other hand, if we aim to exploit such definitions in a variational context, we have to deal with boundary conditions so that both interval endpoints must play a role ([32]). Therefore, we introduced the bilateral fractional integral and derivative, by keeping separate their "even" and "odd" parts:

Definition 4. *For every $u \in L^1(a,b)$ we set the even and odd versions of bilateral fractional integrals and derivatives:*

$$\begin{aligned} I^s_e[u](x) &:= \frac{1}{2}(I^s_+[u](x) + I^s_-[u](x)) \\ &= \frac{1}{2\,\Gamma(s)}\int_a^b \frac{u(t)}{|x-t|^{1-s}}dt = \frac{(u * 1/|t|^{1-s})(x)}{2\,\Gamma(s)}, \end{aligned} \tag{8}$$

$$\begin{aligned} D^s_e[u](x) &:= D_x\, I^{1-s}_e[u](x) \\ &= \frac{1}{2}(D^s_+[u](x) - D^s_-[u](x)) = D_x\,\frac{(u * 1/|t|^{s})(x)}{2\,\Gamma(1-s)}, \end{aligned} \tag{9}$$

$$\begin{aligned} I^s_o[u](x) &:= \frac{1}{2}(I^s_+[u](x) - I^s_-[u](x)) \\ &= \frac{1}{2\,\Gamma(s)}\int_a^b u(t)\,\frac{\text{sign}(x-t)}{|x-t|^{1-s}}\,dt = \frac{(u * \frac{\text{sign}(t)}{|t|^{1-s}})(x)}{2\,\Gamma(s)}, \end{aligned} \tag{10}$$

$$\begin{aligned} D^s_o[u](x) &:= D_x\, I^{1-s}_o[u](x) \\ &= \frac{1}{2}(D^s_+[u](x) + D^s_-[u](x)) = D_x\,\frac{(u * \text{sign}(t)/|t|^{s})(x)}{2\,\Gamma(1-s)}. \end{aligned} \tag{11}$$

So that

$$I^s_+[u] = I^s_e[u] + I^s_o[u], \qquad D^s_+[u] = D^s_e[u] + D^s_o[u], \tag{12}$$

$$I^s_-[u] = I^s_e[u] - I^s_o[u], \qquad D^s_-[u] = D^s_o[u] - D^s_e[u]. \tag{13}$$

Whenever $(a,b) \neq \mathbb{R}$, the convolution in (8)–(11) has to be understood, without relabeling, as the convolution of the trivial extension of u (still an $L^1(\mathbb{R})$ function with support on $[a,b]$) with either $1/|t|^s$ or $\text{sign}(t)/|t|^s$ (both belonging to $L^1_{loc}(\mathbb{R})$). Also $I^s_\pm[u](x)$, $I^s_e[u](x)$, $I^s_o[u](x)$ have to be understood, without relabeling, as the natural extension for $x \in \mathbb{R}\backslash[a,b]$, provided by the convolution of the trivial extension of u with the corresponding kernels (here, H denotes the Heaviside function):

$$I^s_+[u] = u * \tfrac{H(x)}{\Gamma(s)|x|^{1-s}}, \qquad I^s_-[u] = u * \tfrac{H(-x)}{\Gamma(s)|x|^{1-s}}, \qquad \text{for every } x \in \mathbb{R}, \tag{14}$$

$$I^s_e[u] = u * \tfrac{1}{2\Gamma(s)|x|^{1-s}}, \qquad I^s_o[u] = u * \tfrac{\text{sign}(x)}{2\Gamma(s)|x|^{1-s}}, \qquad \text{for every } x \in \mathbb{R}, \tag{15}$$

namely

$$I^s_+[u](x) = \frac{1}{\Gamma(s)} \int_a^b \frac{u(t)\, H(x-t)}{|x-t|^{1-s}}\, dt \qquad \text{for every } x \in \mathbb{R}\,, \tag{16}$$

$$I^s_-[u](x) = \frac{1}{\Gamma(s)} \int_a^b \frac{u(t)\, H(t-x)}{|x-t|^{1-s}}\, dt \qquad \text{for every } x \in \mathbb{R}\,, \tag{17}$$

$$I^s_e[u](x) = \frac{1}{2\,\Gamma(s)} \int_a^b \frac{u(t)}{|x-t|^{1-s}}\, dt \qquad \text{for every } x \in \mathbb{R}\,, \tag{18}$$

$$I^s_o[u](x) = \frac{1}{2\,\Gamma(s)} \int_a^b \frac{u(t)\, \mathrm{sign}(x-t)}{|x-t|^{1-s}}\, dt \qquad \text{for every } x \in \mathbb{R}\,. \tag{19}$$

Moreover

$$\mathrm{spt}\, I^s_+[u] \subset [a, +\infty) \qquad \mathrm{spt}\, I^s_-[u] \subset (-\infty, b]\,. \tag{20}$$

Remark 2. *In (9) and (11), D_x denotes the distributional derivative in $\mathcal{D}'(\mathbb{R})$, but obviously its restriction as a distribution on the open set (a,b) is understood whenever one works in the bounded interval (a,b).*

Up to a normalization constant (see (25)), $I^s_e[u]$ is called the Riesz potential of u ([1,9]). These fractional integrals $I^s_+[u]$, $I^s_-[u]$, $I^s_e[u]$, $I^s_o[u]$ turn out to be in $L^p_{loc}(\mathbb{R})$ (thus $L^p(I)$ on every bounded interval I) for every $1 \le p < 1/(1-s)$, since they are convolutions of $u \in L^1(\mathbb{R})$ with an $L^p_{loc}(\mathbb{R})$ kernel. Moreover, we have the next result.

Lemma 1. *If $-\infty < a < b < +\infty$, $u \in L^\infty(\mathbb{R})$, $\mathrm{spt}(u) \subset [a,b]$ and $0 < s < 1$ then $I^s_+[u]$, $I^s_-[u]$, $I^s_e[u]$, $I^s_o[u]$ belong to $L^\infty(\mathbb{R}) \cap C^{0,s}$.*

Proof. See Lemmas 2.5 and 3.6 (iii) in [1]. □

The behavior of all the above operators, as $s \to 0_+$ or $s \to 1_-$, is clarified by subsequent Lemmas of the present section, whose proof can be found in [1].

Notice that both $1/|x|^s$ and $\mathrm{sign}(x)/|x|^s$ belong to $L^1_{loc}(\mathbb{R})$, for $0 < s < 1$; hence, the convolution with any L^1 function is well defined and belongs to $L^1_{loc}(\mathbb{R})$; moreover $\mathrm{sign}(x)/|x|^s \to \mathrm{p.v.}\,\frac{1}{x}$ in $\mathcal{S}'$ as $s \to 1_-$, while $1/|x|^s$ has no limit in $\mathcal{S}'$ as $s \to 1_-$, where $\mathcal{S}'$ denotes the space of tempered distributions.

Fractional derivatives degenerate developing singularities as $s \to 1_-$; nevertheless, they can be made convergent to meaningful limits by suitable normalization.

Lemma 2. *Assume $0 < s < 1$, $u \in W^{1,2}_G(\mathbb{R})$ and choose the constants in the Fourier transform such that $\widehat{u}(\xi) = \int_{\mathbb{R}} \exp(-i\xi x)\, u(x)\, dx$. Then*

$$\frac{D^s_e[u]}{\sin(s\,\pi/2)} \longrightarrow \mathcal{F}^{-1}\{\, i\,\xi\, \widehat{u}(\xi)\,\} = Du \quad \text{in } L^2(\mathbb{R}) \quad \text{as } s \to 1_-\,, \tag{21}$$

$$\frac{D^s_o[u]}{\cos(s\,\pi/2)} \longrightarrow \mathcal{F}^{-1}\{\, |\xi|\, \widehat{u}(\xi)\,\} \quad \text{in } L^2(\mathbb{R}) \quad \text{as } s \to 1_-\,, \tag{22}$$

$$D^s_+[u] \longrightarrow Du \quad \text{in } L^2(\mathbb{R}) \quad \text{as } s \to 1_-\,, \tag{23}$$

$$D^s_-[u] \longrightarrow -Du \quad \text{in } L^2(\mathbb{R}) \quad \text{as } s \to 1_-\,. \tag{24}$$

Remark 3. *Notice that relations (21), (23) and (24) tell that, as $s \to 1_-$, both $D^s_+[u]$ (left Riemann–Liouville fractional derivative of order s of u) and $D^s_e[u]$ (even Riemann–Liouville fractional derivative of order s of u) converge in L^2 to the distributional derivative Du, while $D^s_-[u]$ converges in L^2 to $-Du$.*

On the other hand relationship (22) means that $D^s_o[u]$ (odd Riemann–Liouville fractional derivative of order s of u) fades as $s \to 1^-$ but, when suitably normalized as $D^s_o[u]/\cos(s\,\pi/2)$, it con-

verges in L^2 *to the Gagliardo fractional derivative of order 1 of* u, *say* $(-\Delta)^{1/2}u \;:=\; \mathcal{F}^{-1}\,\{\,|\xi|\,\widehat{u}(\xi)\,\}\,.$

Fractional integrals degenerate developing singularities as $s \to 0_+$; indeed the convolution term fulfills $|x|^{s-1}/(2\Gamma(s)\cos(s\pi/2)) \to \delta$ in $\mathcal{S}'$ as $s \to 0_+$; nevertheless, fractional integrals are convergent to meaningful limits by suitable normalization.

Lemma 3. *Assume* $0 < s < 1$, $u \in L^1(\mathbb{R})$ *with* $\widehat{u} \in L^1(\mathbb{R})$ *and set the constants in the Fourier transform such that* $\widehat{u}(\xi) = \int_{\mathbb{R}} \exp(-i\xi x)\, u(x)\, dx$. *Then*

$$\frac{1}{\cos(s\,\pi/2)}\; I^s_e[u](x) \;\longrightarrow\; u(x) \qquad \text{uniformly in } \mathbb{R} \quad \text{as } s \to 0_+\,, \tag{25}$$

$$\frac{\pi}{\sin(s\,\pi/2)}\; I^s_o[u](x) \;\longrightarrow\; (\text{p.v.}\,1/x) * u \quad \text{in } \mathcal{S}'(\mathbb{R}) \quad \text{as } s \to 0_+\,. \tag{26}$$

Lemma 4. *Assume* $0 < s < 1$, $u \in L^1(\mathbb{R})$.

If $I^{1-s}_o[u] \in AC_{loc}(\mathbb{R})$, *then*

$$\frac{1}{(\cos(s\,\pi/2))^2}\; I^s_e[\,D^s_o[u]\,] \;=\; u\,. \tag{27}$$

If $I^{1-s}_e[u] \in AC_{loc}(\mathbb{R})$, *then*

$$\frac{1}{(\sin(s\,\pi/2))^2}\; I^s_o[\,D^s_e[u]\,] \;=\; u\,. \tag{28}$$

If $I^{1-s}_o[I^s_e[u]] \in AC_{loc}(\mathbb{R})$, *then*

$$\frac{1}{(\cos(s\,\pi/2))^2}\; D^s_o[\,I^s_e[u]\,] \;=\; u\,. \tag{29}$$

If $I^{1-s}_e[I^s_o[u]] \in AC_{loc}(\mathbb{R})$, *then*

$$\frac{1}{(\sin(s\,\pi/2))^2}\; D^s_e[\,I^s_o[u]\,] \;=\; u\,. \tag{30}$$

Lemma 5. *Assume* $0 < s < 1$, $u \in L^1(\mathbb{R})$. *Then*

$$D^s_+[\,I^s_+[u]\,] \;=\; u \tag{31}$$

$$D^s_-[\,I^s_-[u]\,] \;=\; u\,. \tag{32}$$

If in addition $I^{1-s}_+[u] \in AC_{loc}(\mathbb{R})$, *then*

$$I^s_+[\,D^s_+[u]\,] \;=\; u\,. \tag{33}$$

If in addition $I^{1-s}_-[u] \in AC_{loc}(\mathbb{R})$, *then*

$$I^s_-[\,D^s_-[u]\,] \;=\; u\,. \tag{34}$$

Remark 4. *Every distributional fractional derivative (left, right, even, and odd) appearing in the statements of Lemmas 3–5, which are proved in [1] with fractional classical derivatives* $(d/dx)^s_\pm$, *still hold true in the present formulation with corresponding distributional derivatives* $(D_x)^s_\pm$ *by exactly the same proof, since the assumptions ensure that all derivatives are evaluated on local absolute continuous functions.*

Remark 5. *Notice that, when $\mathbb{R}$ is replaced by a bounded interval, the identities (27), (28), (33) and (34) require an additional correction term, taking into account of boundary values (see (52) and (53) in Theorem 1), whereas (31) and (32) remain true (see (136), (137)).*

Symmetries of even or odd functions are inherited neither by fractional integrals, nor by fractional derivatives. Nevertheless, the next lemma holds true.

Lemma 6. *For every $s \in (0,1)$, $0 < a \le +\infty$ and $v \in L^1(-a,a)$, by setting*

$$\check{v}(x) = v(-x)\,, \tag{35}$$

we obtain

$$I^s_+[\check{v}](x) \;=\; I^s_-[v](-x) \qquad \text{on } (-a,a)\,, \tag{36}$$

$$D^s_+[\check{v}](x) \;=\; -\,D^s_-[v](-x) \qquad \text{on } (-a,a)\,. \tag{37}$$

For every $s \in (0,1)$, $0 < a \le +\infty$ and every even function $v \in L^1(-a,a)$, we get

$$I^s_+[v](x) \;= I^s_+[\check{v}](x) \;=\; I^s_-[v](-x) \qquad \text{on } (-a,a)\,, \tag{38}$$

$$D^s_+[v](x) \;=\; D^s_+[\check{v}](x) \;=\; -\,D^s_-[v](-x) \qquad \text{on } (-a,a)\,, \tag{39}$$

For every $s \in (0,1)$ and every odd function $v \in L^1(-a,a)$, we obtain

$$I^s_+[v](x) \;=\; -\,I^s_+[\check{v}](x) \;=\; -\,I^s_-[v](-x) \qquad \text{on } (-a,a)\,, \tag{40}$$

$$D^s_+[v](x) \;=\; -D^s_+[\check{v}](x) \;=\; D^s_-[v](-x) \qquad \text{on } (-a,a)\,, \tag{41}$$

Proof.

$$\begin{aligned} I^s_+[\check{v}](x) \;&=\; \frac{1}{\Gamma(s)}\int_{-a}^{x}\frac{v(-t)}{(x-t)^{1-s}}\,dt \;=\; \frac{1}{\Gamma(s)}\int_{-a}^{x}\frac{v(-t)}{\big(-t-(-x)\big)^{1-s}}\,dt \overset{s=-t}{=} \\ &=\; -\,\frac{1}{\Gamma(s)}\int_{a}^{-x}\frac{v(s)}{\big(s-(-x)\big)^{1-s}}\,ds \;=\; \frac{1}{\Gamma(s)}\int_{-x}^{a}\frac{v(s)}{\big(s-(-x)\big)^{1-s}}\,ds \;=\; I^s_-[v](-x). \end{aligned}$$

By inserting $1-s$ in place of s in (36), if v is even we obtain (37) via

$$D^s_+[\check{v}](x) \;=\; D_x\, I^{1-s}_+[\check{v}](x) \;=\; D_x I^{1-s}_-[v](-x) \;=\; -D^s_-[v](-x)\,.$$

Even v entails $v(x) = v(-x)$, $v = \check{v}$, $I^s_+[v](x) = I^s_+[\check{v}](x)$ and $D^s_+[v](x) = D^s_+[\check{v}](x)$; hence, (36) and (37) entail, respectively, (38) and (39).

Odd v entails $v(x) = -v(-x)$, $v = -\check{v}$, $I^s_+[v](x) = -I^s_+[\check{v}](x)$ and $D^s_+[v](x) = -D^s_+[\check{v}](x)$; hence, (36), (37) entail, respectively, (40), (41). □

Results listed above (mainly Lemmas 3 and 4 proved in [1]) lead to the natural definition of the operators representing the bilateral version of Riemann–Liouville fractional derivatives and integrals, as stated below. Results similar to the ones in Lemma 6 can be found also in [33].

Definition 5. ***(Bilateral Riemann–Liouville fractional integral of order s)***

$$I^s[u] \;=\; \frac{1}{\cos(s\,\pi/2)}\,I^s_e[u] \;=\; \frac{1}{2\,\Gamma(s)\,\cos(s\pi/2)}\,\big(I^s_+[u] \,+\, I^s_-[u]\big)\,.$$

Definition 6. ***(Bilateral Riemann–Liouville fractional derivative of order s)***

$$D^s[u] \;=\; \frac{1}{\cos(s\,\pi/2)}\,D^s_o[u] \;=\; \frac{1}{2\,\Gamma(s)\,\cos(s\pi/2)}\,\big(D^s_+[u] \,-\, D^s_-[u]\big)\,.$$

3. The Bilateral Fractional Sobolev Space

From now on, we consider only functions defined on a bounded interval (a,b).

As already mentioned in [25], possible naïve definitions of bilateral fractional Sobolev spaces could be set by $U^{s,1} = U^{s,1}_+ \cap U^{s,1}_-$, where $s \in (0,1)$ and

$$U^{s,1}_+ = \{u \in L^1(a,b) \mid (d/dx)^s_+ \, u \in L^1(a,b)\} ,$$

$$U^{s,1}_- = \{u \in L^1(a,b) \mid (d/dx)^s_- \, u \in L^1(a,b)\} ,$$

for example, a definition which refers to L^1-functions whose classical Riemann–Liouville fractional derivative of prescribed order $s \in (0,1)$ exists finitely almost everywhere and belongs to $L^1(a,b)$.

Actually, if the classical Riemann–Liouville fractional derivative $(d/dx)^s_+ [u](x)$ of u exists a.e. for x for some $s \in (0,1)$, then $I^{1-s}_{a+}[u]$ is differentiable almost everywhere, referring to the same s; nevertheless, such an a.e. derivative does not provide complete information about the distributional derivative of the fractional integral $I^{1-s}_{a+}[u]$, when $I^{1-s}_{a+}[u]$ is not an absolutely continuous function. Thus, the differential properties are not completely described by the pointwise fractional derivative, though existing almost everywhere in (a,b). This shows that the previous definitions $U^{s,1}_+$ and $U^{s,1}_-$ are not suitable to obtain an integration by parts formula, whereas the appropriate ones refer to distributional Riemann–Liouville fractional derivative $D^s_+ [u](x)$ in Definition 3, namely, they are given by

$$\mathfrak{U}^{s,1}_+ = \{u \in L^1(a,b) \mid D^s_+ u \in L^1(a,b)\} , \tag{42}$$

$$\mathfrak{U}^{s,1}_- = \{u \in L^1(a,b) \mid D^s_- u \in L^1(a,b)\} , \tag{43}$$

Therefore, to develop a satisfactory theory of fractional Sobolev spaces, we introduced a more effective function space in [25], by defining the fractional Sobolev spaces related to one-sided fractional derivatives, which are recalled in subsequent Definition 7, where we confine to the case $p = 1$.

Definition 7. *We recall the definitions of Riemann–Liouville fractional Sobolev spaces related to one-sided fractional derivatives, as introduced in [25]:*

$$W^{s,1}_+(a,b) := \{u \in L^1(a,b) \mid I^{1-s}_+[u] \in W^{1,1}_G(a,b) \} = \mathfrak{U}^{s,1}_+ , \tag{44}$$

$$W^{s,1}_-(a,b) := \{u \in L^1(a,b) \mid I^{1-s}_-[u] \in W^{1,1}_G(a,b) \} = \mathfrak{U}^{s,1}_- . \tag{45}$$

Explicitly, the properties $u \in W^{s,1}_\pm(a,b)$ entail, respectively, that the distributional derivatives $D\left[I^{1-s}_\pm[u]\right]$ belong to $L^1(a,b)$, thus $W^{s,1}_+(a,b) = \mathfrak{U}^{s,1}_+$ and $W^{s,1}_-(a,b) = \mathfrak{U}^{s,1}_-$.

Here, we introduce also the "even" and "odd" fractional Sobolev spaces.

Definition 8. *The even/odd Riemann–Liouville fractional Sobolev spaces are*

$$W^{s,1}_e(a,b) := \{u \in L^1(a,b) \mid I^{1-s}_e[u] \in W^{1,1}_G(a,b) \} , \tag{46}$$

$$W^{s,1}_o(a,b) := \{u \in L^1(a,b) \mid I^{1-s}_o[u] \in W^{1,1}_G(a,b) \} . \tag{47}$$

Eventually, we define the bilateral Riemann–Liouville fractional Sobolev spaces, with the aim to achieve a symmetric framework.

Definition 9. ***The (Bilateral) Riemann–Liouville Fractional Sobolev spaces****.*

For every $s \in (0,1)$, we set $W^{s,1}(a,b) = W^{s,1}_+(a,b) \cap W^{s,1}_-(a,b)$, that is,

$$W^{s,1}(a,b) := \{u \in L^1(a,b) \mid I^{1-s}_+[u] \in W^{1,1}_G(a,b) \text{ and } I^{1-s}_-[u] \in W^{1,1}_G(a,b)\}. \tag{48}$$

Notice that, concerning Definition 9, by exploiting (12) and (13), we get also

$$W^{s,1}(a,b) \;=\; W^{s,1}_{+}(a,b) \cap W^{s,1}_{-}(a,b) \;=\; W^{s,1}_{o}(a,b) \cap W^{s,1}_{e}(a,b)\,. \tag{49}$$

Theorem 1. *Assume $0 < s < 1$ and (a,b) is bounded. Then, the (bilateral) Riemann–Liouville fractional Sobolev space $W^{s,1}(a,b)$ (Definition 9) is a normed space when endowed with the natural norm*

$$\|u\|_{W^{s,1}} := \|u\|_{L^1(a,b)} + \|\, D^s_{a+}[u]\,\|_{L^1(a,b)} + \|\, D^s_{b-}[u]\,\|_{L^1(a,b)}\,. \tag{50}$$

The set $W^{s,1}(a,b)$ is a Banach space and, for every $q \in [1, 1/(1-s))$ there is $C = C(s,q,a,b)$ such that

$$\|u\|_{L^q(a,b)} \;\le\; C(s,q,a,b)\,\|u\|_{W^{s,1}(a,b)}\,. \tag{51}$$

Every $u \in W^{s,1}(a,b)$ can be represented by both

$$u(x) \;=\; I^s_{a+}\left[D^s_{a+}[u]\right](x) + \frac{I^{1-s}_{a+}[u](a)}{\Gamma(s)}\,(x-a)^{s-1} \qquad a.e\ x \in (a,b), \tag{52}$$

and

$$u(x) \;=\; I^s_{b-}\left[D^s_{b-}[u]\right](x) + \frac{I^{1-s}_{b-}[u](b)}{\Gamma(s)}\,(b-x)^{s-1} \qquad a.e\ x \in (a,b). \tag{53}$$

Proof. The map $u \mapsto \|u\|_{W^{s,1}}$ is a norm on $W^{s,1}(a,b)$, indeed, $\|u\|_{L^1(a,b)} + \|D^s_{a+}[u]\|_{L^1(a,b)}$ is equivalent to the norm $\|I^{1-s}_{a+}[u]\|_{W^{1,1}_G(a,b)}$ since $I^{1-s}_{a+}[u]$ belongs to $W^{1,1}_G$, $D^s_{a+}[u] = \frac{d}{dx} I^{1-s}_{a+}[u] = D I^{1-s}_{a+}[u]$ and $\|I^{1-s}_{a+}[u]\|_{L^1(a,b)} \le C\|u\|_{L^1(a,b)}$; analogously $\|u\|_{L^1(a,b)} + \|D^s_{b-}[u]\|_{L^1(a,b)}$ is a norm for $I^{1-s}_{b-}[u]$, due to $I^{1-s}_{b-}[u] \in W^{1,1}_G$, $D^s_{b-}[u] = \frac{d}{dx} I^{1-s}_{b-}[u] = D I^{1-s}_{b-}[u]$ and $\|I^{1-s}_{a+}[u]\|_{L^1(a,b)} \le C\|u\|_{L^1(a,b)}$.

The completeness of $W^{s,1}(a,b)$ with respect to such a norm when (a,b) is bounded, follows by the completeness of L^1 and $W^{1,1}_G$ together with the fact that u_k is a Cauchy sequence in the norm $W^{s,1}(a,b)$ if and only if u_k is a Cauchy sequence in $L^1(a,b)$ and $I^{1-s}_{\pm}[u_k]$ are Cauchy sequences in $W^{1,1}_G$.

Estimate (51) and representations (52) and (53) follow by (129), (130) and (135) of Proposition 12, which is shown in Section 5. □

Remark 6. *Thanks to $\|I^{1-s}_{a+}[u]\|_{L^1(a,b)} \le \|u\|_{L^1(a,b)}$, $\|I^{1-s}_{b-}[u]\|_{L^1(a,b)} \le \|u\|_{L^1(a,b)}$, we have replaced the terms $\|I^{1-s}_{\pm}[u]\|_{W^{1,1}(a,b)}$ in the norm (50) by $\|D\left[I^{1-s}_{\pm}[u]\right]\|_{L^1(a,b)}$, where D denotes the distributional derivative.*

Obviously, the terms $\|D^s_{+}[u]\|_{L^1(a,b)}$ and $\|D^s_{-}[u]\|_{L^1(a,b)}$ in the norm (50) could be alternatively replaced by $\|D^s_e[u]\|_{L^1(\mathbb{R})}$ and $\|D^s_o[u]\|_{L^1(\mathbb{R})}$, referring to (8), (16) and (17), still achieving an equivalent norm on $W^{s,1}(a,b)$.

Example 2. *For every $s \in (0,1)$, the constant functions and $v(x) = x(1-x)$ both belong to the space $W^{s,1}(a,b)$. Spaces $C^\infty_0(a,b)$, test functions on (a,b), and $C^1([a,b])$, continuously differentiable functions, are contained in $W^{s,1}(a,b)$.*

Example 3. *For every $0 < s < 1$, the discontinuous piecewise constant $H(x)$ belongs to $W^{s,1}(-1,1) \backslash W^{1,1}_G(-1,1)$.*

Indeed, both $I^{1-s}_{+}[H](x) = H(x)\frac{|x|^{1-s}}{\Gamma(2-s)}$ and $I^{1-s}_{-}[H](x) = \frac{H(x)(1-x)^{1-s} + H(-x)(|1-x|^{1-s} - |x|^{1-s})}{\Gamma(2-s)}$ belong to $W^{1,1}_G(-1,1)$.

Example 4. *For every $s \in (0,1)$ and $\beta \ge 0$, the function x^β belongs to $W^{s,1}(0,1)$. This claim is straightforward for $\beta = 0$ and $\beta \ge 1$; we refer to (97) and (98) for $0 < \beta < 1$.*

Example 5. *Function $x^{-\alpha}$ with $\alpha > 0$ belongs to $W^{s,1}(0,1)$ if and only if $0 < \alpha < 1-s$.*

Indeed, $I_+^{1-s}[x^{-\alpha}](x) = \frac{\Gamma(1-\alpha)}{\Gamma(2-s-\alpha)} x^{1-s-\alpha} \in W_G^{1,1}(0,1)$ if $-s-\alpha > -1$, while $I_+^{1-s}[x^{s-1}](x) = \frac{1}{\Gamma(1-s)} \in W_G^{1,1}(0,1)$, summarizing $x^{-\alpha}$ belongs to $W_+^{s,1}(0,1)$ for $\alpha \le 1-s$; on the other hand, $I_-^{1-s}[x^{-\alpha}]$ belongs to $L^1(a,b)$ for $\alpha < 1$ and is bounded on $(0,1)$ if and only if $\alpha < 1-s$, due to

$$
\begin{aligned}
I_-^{1-s}[\,x^{-\alpha}\,](x) &= \frac{1}{\Gamma(1-s)} \int_x^1 t^{-\alpha}(t-x)^{-s}\,dt \overset{t=xy}{=} \\
&= \frac{x^{1-s-\alpha}}{\Gamma(1-s)} \int_1^{1/x} y^{-\alpha}(y-1)^{-s}\,dy\,.
\end{aligned}
$$

Summarizing, and taking into account Example 4 for $\alpha \le 0$,

$$
x^{-\alpha} \in W_+^{s,1}(0,1) \cap W_-^{s,1}(0,1) \quad \textit{if and only if} \quad \alpha < 1-s\,. \tag{54}
$$

In the particular case $\alpha = 1-s$, we recover

$$
x^{s-1} \in W_+^{s,1}(0,1) \setminus W_-^{s,1}(0,1) \tag{55}
$$

with $I_+^{1-s}[x^{s-1}](x) = \Gamma(s)$ and $I_-^{1-s}[x^{s-1}](x)$ unbounded in a right neighborhood of $x = 0$.

Theorem 2. ***(Integration by parts in $W^{s,1}(a,b)$)***
Next, identities hold true for $0 < s < 1$, $-\infty < a < b < +\infty$:

$$
\begin{cases}
\displaystyle\int_a^b u(x)\, D_+^s[v](x)\,dx \;=\; -\int_a^b \frac{d}{dx}u(x)\, I_+^{1-s}[v](x)\,dx \;+\; u(b)\, I_+^{1-s}[v](b) \\
\hfill \forall\, v \in W_+^{s,1}(a,b),\ \forall\, u \in W_G^{1,1}(a,b)\,,
\end{cases} \tag{56}
$$

$$
\begin{cases}
\displaystyle\int_a^b u(x)\, D_-^s[v](x)\,dx \;=\; +\int_a^b \frac{d}{dx}u(x)\, I_-^{1-s}[v](x)\,dx \;+\; u(a)\, I_-^{1-s}[v](a) \\
\hfill \forall\, v \in W_-^{s,1}(a,b),\ \forall\, u \in W_G^{1,1}(a,b)\,,
\end{cases} \tag{57}
$$

$$
\begin{cases}
\displaystyle\int_a^b u(x)\, D_e^s[v](x)\,dx \;= \\
\displaystyle\quad -\int_a^b \frac{d}{dx}u(x)\, I_e^{1-s}[v](x)\,dx + \frac{1}{2}\Big(u(b)\, I_+^{1-s}[v](b) - u(a)\, I_-^{1-s}[v](a)\Big) \\
\hfill \forall\, v \in W^{s,1}(a,b),\ \forall\, u \in W_G^{1,1}(a,b)\,,
\end{cases} \tag{58}
$$

$$
\begin{cases}
\displaystyle\int_a^b u(x)\, D_o^s[v](x)\,dx \;= \\
\displaystyle\quad -\int_a^b \frac{d}{dx}u(x)\, I_0^{1-s}[v](x)\,dx + \frac{1}{2}\Big(u(b)\, I_+^{1-s}[v](b) + u(a)\, I_-^{1-s}[v](a)\Big) \\
\hfill \forall\, v \in W^{s,1}(a,b),\ \forall\, u \in W_G^{1,1}(a,b)\,,
\end{cases} \tag{59}
$$

Proof. Identity (56) follows by (2), (5) and

$$
\begin{aligned}
&\int_a^b u(x)\, D_+^s[v](x)\,dx \;=\; \frac{1}{\Gamma(1-s)} \int_a^b u(x) \left(\frac{d}{dx} \int_a^x \frac{v(t)}{(x-t)^s}\,dt \right) dx \;= \\
&= -\frac{1}{\Gamma(1-s)} \int_a^b \frac{d}{dx}u(x) \left(\int_a^x \frac{v(t)}{(x-t)^s}\,dt \right) dx \;+\; u(b)\, \frac{1}{\Gamma(1-s)} \int_a^b \frac{v(t)}{(x-t)^s}\,dt\,.
\end{aligned}
$$

Identity (57) follows by (3), (4) and similar computations.
Identities (58) and (59) follow by the subtraction and sum of (56) and (57). $\square$

Remark 7. *Notice that when u is representable, e.g., under a slightly stronger condition, then we find a more symmetric formulation. For instance,* (59) *translates into*

$$\begin{cases} \displaystyle\int_a^b I_o^{1-s}[w](x)\, D_o^s[v](x)\, dx \;= \\ \displaystyle -\int_a^b D_o^s[w](x)\, I_o^{1-s}[v](x)\, dx + \frac{1}{2}\left(I_o^{1-s}[w](b)\, I_+^{1-s}[v](b) + I_o^{1-s}[w](a)\, I_-^{1-s}[v](a)\right) \\ \qquad\qquad\qquad\qquad \forall\, v,\, w \in W^{s,1}(a,b)\,. \end{cases} \tag{60}$$

Lemma 7. *The strict embedding*

$$W_G^{1,1}(a,b) \underset{\neq}{\subset} W^{s,1}(a,b) \qquad 0 < s < 1 \tag{61}$$

holds true with the related uniform estimate: there is a constant $K = K(s,a,b)$ such that

$$\|v\|_{W^{s,1}(a,b)} \;\leq\; K\, \|v\|_{W_G^{1,1}(a,b)} \tag{62}$$

Proof. By computations in Example 3, we know that the Heaviside function belongs to $W^{s,1}(-1,1)\backslash W_G^{1,1}(-1,1)$; thus if the embedding holds true, then it is strict.

Recalling the definition ([34]) of right Caputo fractional derivative ${}_C D_+^s[u]$ and its relationship with the right Riemann–Liouville fractional derivative

$${}_C D_+^s[u](x) \;:=\; I_+^{1-s}[u'](x) \;=\; \frac{1}{\Gamma(1-s)} \int_a^x \frac{u'(t)}{(x-t)^s}\, dt \qquad \forall u \in W_G^{1,1}(a,b)\,,$$

$$\begin{aligned} {}_{RL}D_+^s[u](x) \;&=\; {}_C D^s s_+[u](x) \;+\; \frac{u(a)}{\Gamma(1-s)}\,(x-a)^{-s} = \\ &=\; I_+^{1-s}[u'](x) \;+\; \frac{u(a)}{\Gamma(1-s)}\,(x-a)^{-s} \qquad \forall u \in W_G^{1,1}(a,b)\,, \end{aligned}$$

and taking into account

$$\|I_+^{1-s}[u']\|_{L^1(a,b)} \;\leq\; K_1\, \|u'\|_{L^1(a,b)} \;\leq\; K_1\, \|u\|_{W_G^{1,1}(a,b)}\,, \qquad |u(a)| \;\leq\; K_2\, \|u\|_{W_G^{1,1}(a,b)}\,,$$

we get (7) and (62). □

Theorem 3. [*Compactness in* $W^{s,1}(a,b)$]
Assume that the interval (a,b) is bounded, the parameter s fulfills $0<s<1$, and

$$\|u_n\|_{W^{s,1}(a,b)} \;\leq\; C\,. \tag{63}$$

Then there exist $u \in L^q(a,b)$, $\forall q \in [1, 1/(1-s))$, and a subsequence such that, without relabeling,

$$\begin{cases} (i) & u_n \rightharpoonup u & \text{weakly in } L^q(a,b)) \quad \forall q \in [1,1/(1-s))\,, \\ (ii) & I_+^{1-s}[u_n] \to I_+^{1-s}[u] & \text{strongly in } L^p(a,b)\,,\ \forall p \in [1,+\infty)\,, \\ (iii) & I_-^{1-s}[u_n] \to I_-^{1-s}[u] & \text{strongly in } L^p(a,b)\,,\ \forall p \in [1,+\infty)\,, \end{cases}$$

$$I_+^{1-s}[u_n] \rightharpoonup I_+^{1-s}[u]\,, \qquad I_-^{1-s}[u_n] \rightharpoonup I_-^{1-s}[u] \qquad \text{weakly in } BV(a,b). \tag{64}$$

Proof. Claim (i) follows by (51) and (63) and reflexivity of $L^q(a,b)$ for any fixed $1 < q_k < 1/(1-s)$; thus, by choosing a sequence $q_k \to 1/(1-s)$ and extracting a diagonal sequence, we get the claim for a unique subsequence and unique u valid for every q fulfilling $1 < q < 1/(1-s)$. Moreover, such u belongs to $L^1(a,b)$. Eventually, for $q = 1$

there is a measure μ such that $u_n \rightharpoonup \mu$ in $\mathcal{M}(a,b)$, but such μ must be equal to u, then $u_n \rightharpoonup u$ in $L^1(a,b)$.

The compact embedding $W^{1,1}_G(a,b) \hookrightarrow L^p(a,b)$ valid for any $p \in [1,+\infty)$ (Rellich Theorem) entails the existence of z_+ and z_- in $L^p(a,b)$ fulfilling, up to subsequences,

$$I_+^{1-s}[u_n] \to z_+ \quad \text{strongly in } L^p(a,b)\,,\ \forall p \in [\,1,+\infty\,)\,, \tag{65}$$

$$I_-^{1-s}[u_n] \to z_- \quad \text{strongly in } L^p(a,b)\,,\ \forall p \in [\,1,+\infty\,)\,, \tag{66}$$

$$I_e^{1-s}[u_n] \to \frac{1}{2}(z_+ + z_-) \quad \text{strongly in } L^p(a,b)\,,\ \forall p \in [\,1,+\infty\,)\,, \tag{67}$$

$$I_o^{1-s}[u_n] \to \frac{1}{2}(z_+ - z_-) \quad \text{strongly in } L^p(a,b)\,,\ \forall p \in [\,1,+\infty\,)\,. \tag{68}$$

By (i) and the Mazur Theorem, there is a sequence of convex combinations y_n, which is strongly converging: precisely, $y_n \to u$ strongly in $L^q(a,b)$ for every $q \in [1, 1/(1-s))$ with $y_n = \sum_{j=1}^n c_{n,j} u_j$, $c_{n,j} \geq 0$, $\sum_{j=1}^n c_{n,j} = 1$. Hence, by (63),

$$\|y_n\|_{W^{s,1}} \leq \sum_{j=1}^{n} c_{n,j} \|u_j\|_{W^{s,1}} \leq C\,. \tag{69}$$

I^{1-s} is a continuous map from L^q to L^r, $q \in [1, 1/(1-s))$ and $r \in [1, q/\big(1-(1-s)sq\big)\big)$, hence, we obtain

$$I_\pm^{1-s}[y_n] \to I_\pm^{1-s}[u] \quad \text{strongly in } L^r(a,b),\ r \in [1, q/\big(1-(1-s)sq\big)\big) \tag{70}$$

and hence, in $\mathcal{D}'(a,b)$. Moreover, by (69), $I^{1-s}[y_n]$ is bounded in $W^{1,1}_G(a,b)$; then, there exists $w_\pm \in BV(a,b)$ such that, possibly up to subsequences,

$$I_+^{1-s}[y_n] \rightharpoonup w_+, \quad I_-^{1-s}[y_n] \rightharpoonup w_- \qquad \text{weakly in } BV(a,b). \tag{71}$$

Taking into account (70), (71) and the uniqueness of limit in $\mathcal{D}'(a,b)$, we obtain

$$w_+ = I_+^{1-s}[u] \in BV(a,b),\ \ w_- = I_-^{1-s}[u] \in BV(a,b), \quad I_\pm^{1-s}[y_n] \rightharpoonup I_\pm^{1-s}[u] \ \text{ in } BV(a,b). \tag{72}$$

Taking into account $u_n \in W^{s,1}(a,b)$, we set $f_n = I^{1-s}[u_n]$, so u_n solves Abel integral equation $I^{1-s}[u_n] = f_n$. By the semigroup property, $I_+^s[f_n](x) = I^s\big[I_+^{1-s}[u_n]\big](x) = I_+^1[u_n] = \int_a^x u_n(t)\,dt$; hence, $I_+^s[f_n](a) = 0$. Therefore (by Proposition 2), the Abel integral equations have a unique solution in $L^1(a,b)$, given by $u_n = D_+^{1-s}[f_n] = D_+^{1-s}\big[I_+^{1-s}[u_n]\big]$, $n \in \mathbb{N}$.

Set $f = I^{1-s}[u] \in BV(a,b)$. $u \in L^1(a,b)$. So u solves the Abel equation $I^{1-s}[u] = f$. Moreover, by the semigroup property, $I_+^s[f](x) = I^s\big[I_+^{1-s}[u]\big](x) = I_+^1[u] = \int_a^x u(t)\,dt$; hence, $I_+^s[f](a_+) = 0$. Therefore (Proposition 3), the Abel equation has a unique solution in $L^1(a,b)$, given by $u = D_+^{1-s}[f] = D_+^{1-s}\big[I_+^{1-s}[u]\big]$.

By (i), $u_n \to u$ strongly in L^q, hence,

$$f_n = I_+^{1-s}[u_n] \to I_+^{1-s}[u] = f \qquad \text{strongly in } L^q,\ q \in \big[1, 1/(1-s)\big).$$

Then, by (65), $I^{1-s}[u] = f = z_+$. Hence we have shown claims (ii) and (iii) .

Moreover, the convergence is also in the sense of distributions and the sequence is bounded in $W^{1,1}_G$; therefore, $I_+^{1-s}[u]$ belongs to $BV(a,b)$ and, again up to subsequences,

$$f_n = I_+^{1-s}[u_n] \rightharpoonup I_+^{1-s}[u] = f \qquad \text{weakly in } BV(a,b).$$

We can deal with z_- by the same argument, exploiting Corollary 1 for the backward Abel integral equation $I_-^{1-s}[u_n] = g_n$, leading to $I^{1-s}[u] = g = z_-$. $\square$

Remark 8. *The boundedness of (a,b) is an essential assumption in the previous compactness theorem, not only to exploit the Rellich theorem, but also to avoid slow non-integrable decay at infinity of the fractional integral: indeed, even for an integrable compactly supported u, we may have $I_+^{1-s}[u](x) \sim (x-a)^{-s}$ at $+\infty$, e.g., if $u = \chi_{[a,b]}$.*

Remark 9. *We emphasize that in Theorem 3, we cannot improve (64), since $I_+^{1-s}[u]$ may belong to $BV(a,b) \setminus W_G^{1,1}(a,b)$.*

Indeed, we can choose $f(x) = \mathrm{sign}(x)$ if $-1 < x < 1$, $f_n(x) = -1$ if $-1 < x < -1/n$, $f_n(x) = n\,x$ if $-1/n < x < 1/n$ and $f_n(x) = 1$ if $1/n < x < 1$. Thus, f belongs to $BV(-1,1)\setminus W^{1,1}(-1,1)$ and $\|f_n\|_{W_G^{1,1}(-1,1)}$ is uniformly bounded. Solving the Abel equations $I_+^{1-s}[u_n] = f_n$ and $I_+^{1-s}[u] = f$ with Propositions 2 and 3 provides $u = D_+^s[f] \in \mathcal{M}(-1,1)\setminus L^1(-1,1)$, whereas $u_n = D_+^s[f_n]$ is uniformly bounded in L^1; hence, u_n is uniformly bounded in $W^{s,1}(-1,1)$, due to Lemma 7.

We recall a well-known result [9] (Theorem 2.1) concerning the L^1-representability of functions.

Theorem 4. **[*L^1-representability*]** *Given $f \in L^1(a,b)$, then*

$f \in I_+^s(L^1(a,b))$ for some $s \in (0,1)$ if and only if

$$I_+^{1-s}[f] \in W^{1,1}(a,b) \quad \text{and} \quad I_+^{1-s}[f](a) = 0\,.$$

$f \in I_-^s(L^1(a,b))$ for some $s \in (0,1)$ if and only if

$$I_-^{1-s}[f] \in W^{1,1}(a,b) \quad \text{and} \quad I_-^{1-s}[f](b) = 0\,.$$

Moreover, in the affirmative case, say, when there exists $u \in L^1(a,b)$ such that $f = I_+^s[u]$ (resp. $f = I_-^s[u]$), we obtain

$$u = D_+^s f \quad (\text{respectively } u = D_-^s f)\,. \tag{73}$$

In Section 5, we provide a self-contained proof of the above result together with a discussion of the related forward and backward Abel equation in the distributional framework, even in the cases when $I_+^{1-s}[f](a) \neq 0$ or $I_-^{1-s}[f](b) \neq 0$ (see Propositions 2 and 3 and Corollaries 1 and 2).

Here, we show that the representability result has a natural extension to the bilateral case.

Theorem 5. *Assume $0 < s < 1$. Then*

$$f \in L^1(a,b) \quad \text{and} \quad f \in I_+^s(L^1(a,b)) \cap I_-^s(L^1(a,b))$$

if and only if

$$f \in W^{s,1}(a,b), \qquad 2I^{1-s}[f](a) - I_-^{1-s}[f](a) = 0 = 2I^{1-s}[f](b) - I_+^{1-s}[f](b)\,,$$

if and only if

$$f \in W^{s,1}(a,b), \qquad 2I_+^{1-s}[f](b) - I_-^{1-s}[f](a) = 0 = 2I_-^{1-s}[f](a) - I_+^{1-s}[f](b)\,.$$

Proof. Since

$$I^{1-s}[f](a) = \frac{1}{2\Gamma(1-s)} \int_a^b \frac{f(t)}{(t-a)^s} dt = I_+^{1-s}[f](b)\,,$$

$$I^{1-s}[f](b) = \frac{1}{2\Gamma(1-s)} \int_a^b \frac{f(t)}{(b-t)^s} dt = I_-^{1-s}[f](a)\,,$$

$$I_+^{1-s}[f](a) = 2\,I^{1-s}[f](a) \,-\, I_-^{1-s}[f](a) = 2\,I_+^{1-s}[f](b) \,-\, I_-^{1-s}[f](a)\,,$$

$$I_-^{1-s}[f](b) = 2\,I^{1-s}[f](b) \,-\, I_+^{1-s}[f](b) = 2\,I_-^{1-s}[f](a) \,-\, I_+^{1-s}[f](b)\,,$$

the claim follows by Definition 9, Theorem 4, Proposition 1 (semigroup property of fractional integral), Propositions 2 and 3 and Corollaries 1 and 2. □

Next, we make explicit some embedding relationship between $W_{\pm}^{s,1}$ and $U_{\pm}^{s,1}$.

Theorem 6. *The following strict embeddings hold true:*

$$W^{s,1} \underset{\neq}{\subset} W_+^{s,1} \underset{\neq}{\subset} U_+^{s,1}\,, \qquad W^{s,1} \underset{\neq}{\subset} W_-^{s,1} \underset{\neq}{\subset} U_-^{s,1} \qquad \forall s \in (0,1)\,, \tag{74}$$

where we refer to Definitions 7–9 about $W^{s,1}(a,b)$, shortly denoted $W^{s,1}$ here, versus the naïve definition $U_{\pm}^{s,1}$ at the beginning of the present Section 3.

Proof. Without loss of generality, we assume $(a,b) = (0,1)$.

Strict embeddings of $W^{s,1}$ in $W_+^{s,1}$ and of $W^{s,1}$ in $W_-^{s,1}$ are shown respectively by x^{s-1} and $(1-x)^{s-1}$: see (55) in Example 5.

Therefore, in order to show (74), it is sufficient to show an example for the strict embedding $W_+^{s,1} \underset{\neq}{\subset} U_+^{s,1}$: indeed, the proof of $W_-^{s,1} \underset{\neq}{\subset} U_-^{s,1}$ is achieved by replacing the variable t with $(1-t)$ in the counterexample showing the other strict embedding by exploiting the symmetry with respect to $x = 1/2$, analogous to the one with respect to $x = 0$ in (36) and (37).

We first note that $W_{\pm}^{s,1} \subset U_{\pm}^{s,1}$ follows by definition (4): the existence of a weak derivative in L^1 of $I_{\pm}^{1-s}[u]$ entails the existence of the fractional derivative $D_{\pm}^{s}[u]$, coincident with the almost everywhere defined fractional derivative $(d/dx)_{\pm}^{s}[u]$(x).

The strict embeddings $W_+^{s,1} \underset{\neq}{\subset} U_+^{s,1}$, and $W_-^{s,1} \underset{\neq}{\subset} U_-^{s,1}$ follows by the subsequent argument, which, for any fixed $s \in (0,1)$, provides the existence of a function in $U_+^{s,1} \backslash W_+^{s,1}$ and a function in $U_-^{s,1} \backslash W_-^{s,1}$.

Given $s \in (0, \ln 2/\ln 3)$, we show a function z in $U_+^{s,1}$ such that $z \notin W_+^{s,1}$.

Precisely, by denoting V the Cantor–Vitali function on $[0,1]$ ([21]), we claim that

$$z \,:=\, D_+^{1-s}[V] \,\in\, U_+^{s,1} \backslash W_+^{s,1} \qquad s \in (\alpha, 1)\,.$$

Indeed V is α-Hölder continuous with $\alpha := \ln 2/\ln 3$. So $I_+^{s}[V]$ belongs to $\mathcal{C}^1(a,b)$ for every $s \in (1-\alpha, 1)$ by Theorem 3.1 in [9] and the fact that $V(0) = 0$. Therefore $I_+^{s}[V] \in W_G^{1,1}$ (hence $V \in W_+^{1-s,1}(0,1)$) for $s \in (1-\alpha, 1)$. Moreover, $I_+^{s}[V](0) = 0$ for $s \in (0,1)$: indeed, due to continuity of V in $[0,1]$, $I_+^{s}[V]$ is continuous in $[0,1]$ and we obtain

$$\Gamma(s) I_+^{s}[V](0) = \lim_{x\to 0} \int_0^x \frac{V(t)}{(x-t)^{1-s}}dt = \lim_{x\to 0}\left(\int_0^x \frac{V(t)-V(x)}{(x-t)^{1-s}}dt + \frac{x^s V(x)}{s}\right).$$

By Hölder continuity ($V \in \mathcal{C}^{0,\alpha}(a,b)$), we obtain $|V(t) - V(x)| \le C|x-t|^{\alpha}$. Then

$$\left|\int_0^x \frac{V(t)}{(x-t)^{1-s}}dt\right| \le C\int_0^x (x-t)^{\alpha+s-1}dt + \frac{x^s V(x)}{s} = \frac{Cx^{s+\alpha}}{s+\alpha} + \frac{x^s V(x)}{s}.$$

Therefore, the limit above is equal to 0, as $x \to 0^+$, thus proving the claim $I_+^{s}[V](0) = 0$. Summarizing, $V \in BV(0,1)$, $I_+^{s}[V](0) = 0$ and $I_+^{s}[V] \in W^{1,1}(0,1)$, for $s \in (1-\alpha, 1)$. Therefore we can consider the Abel integral equation in the distributional setting

$$\text{find } z \in L^1(0,1): \qquad I_+^{1-s}[z] \,=\, V \qquad \text{on } (0,1), \tag{75}$$

and solve it; by Proposition 3, the unique solution is given by $z = D_+^{1-s}[V] = D\,I_+^{s}[V]$, and fulfils $V = I_+^{1-s}[z]$. Moreover $(d/dx)_+^{s}[z] = (d/dx)I_+^{1-s}[z] = (d/dx)V = 0$ a.e. on

$(0,1)$, whereas $D^s_+[z] = D\, I^{1-s}_+[z] = DV$, which is a nontrivial bounded measure. Explicitly $z = D^{1-s}_+[V]$ fulfills $z \in U^{s,1}_+ \backslash W^{s,1}_+$. So far, we have proved the first embedding chain in (74) for $s \in (1-\alpha, 1) = (1 - \ln 2/\ln 3,\ 1)$.

In the sequel, we show that, given any $\sigma \in (0,1)$, we can adapt the Cantor–Vitali function in such a way that it is s-Hölder continuous for any $s \in (0,\sigma]$; hence, we recover the strict embedding for any s in $(1-\sigma, 1)$, and hence, for any s in $(0,1)$, due to the generic choice of σ.

Indeed, given $\tau \in (0,1)$, we can replace the construction of Cantor $1/3$ - middle set $\mathcal{C}$ (say, a set whose Hausdorff dimension is $\ln 2/\ln 3$, which leads to the $\alpha = \alpha(1/3) = \ln 2/\ln 3$ Hölder continuous Cantor–Vitali function $V_{1/3} := V$) with the Cantor-like τ - middle set $\mathcal{C}_\tau$, with Hausdorff dimension $\dim(\mathcal{C}_\tau) = \ln 2/\big(\ln 2 - \ln(1-\tau)\big)$, which leads to the $\alpha(\tau)$ Hölder continuous Cantor–Vitali generalized function $V = V_\tau$, where

$$\alpha(\tau) = \dim(\mathcal{C}_\tau) = \ln 2/\big(\ln 2 - \ln(1-\tau)\big)\,.$$

Notice that $\alpha(\tau) \to 1_-$ as $\tau \to 0_+$ and $\alpha(\tau) \to 0_+$ as $\tau \to 1_-$, so that $\alpha(\tau)$ spans the interval $(0,1)$ as τ runs over $(0,1)$. Moreover, $V_\tau \in (BV \cap C^0) \backslash W^{1,1}$ for $\tau \in (0,1)$.

Again by Proposition 3, we get that V_τ is representable, say there exists (unique) $z_\tau \in L^1(0,1)$ s.t. $z_\tau = D^{1-s}_+[V_\tau]$ s.t. $V_\tau = I^{1-s}_+[z_\tau]$ for $s \in (1-\alpha(\tau), 1)$, and we claim that $V_\tau(0) = 0$, $I^s_+[V_\tau](0) = 0$, $z_\tau \in U^{s,1}_+ \backslash W^{s,1}_+$ for $s \in (1-\alpha(\tau), 1)$: indeed these claims about the generalized Cantor–Vitali function V_τ can be proved by the same procedure dealing with the definition of $V = V_{1/3}$, as it is sketched below.

The function V_τ is of bounded variations since is monotone, as it is the uniform limit of a sequence of monotone nondecreasing functions.

Continuity of V_τ follows from uniform convergence of standard iterative approximations by piecewise linear functions. The absolutely continuous part of the distributional derivative $D\, V_\tau$ is identically 0 since V_τ is locally constant on an open set of Lebesgue measure 1: indeed, it is a union of open intervals, which is iteratively obtained by approximation with finite unions A_n whose measure ℓ_n fulfills the recursive scheme: $\ell_1 = \tau$, $\ell_{n+1} = \ell_n + 2^n \tau \frac{1-\ell_n}{2^n}$, so that $\ell_n = 1 - (1-\tau)^n \ \to\ 1$ as $n \to \infty$.

The worst case for differential quotients of n-th approximations of V_τ is provided by $(1/2)^n/\big((1-\ell_n)/2^n\big) = 1/(1-\tau)^n$, so that $\alpha(\tau)$ is the biggest real α s.t.

$$(1/2)^n/\big((1-\ell_n)/2^n\big)^\alpha = 2^{n(\alpha-1)}/(1-\tau)^{n\alpha}$$

is uniformly bounded for $n \in \mathbb{N}$, say

$$0 < \alpha \ \le\ \alpha(\tau) = \ln 2/\big(\ln 2 - \ln(1-\tau)\big)\,.$$

So $I^s_+[V_\tau]$ belongs to $\mathcal{C}^1(a,b)$ for every $s \in (1-\alpha(\tau), 1)$ by Theorem 3.1 of [9] and taking into account that $V_\tau(0) = 0$. Therefore, $I^{1-s}_+[V_\tau] \in W^{1,1}$ that is $V_\tau \in W^{s,1}_+(0,1)$ for $s \in (0, \alpha(\tau))$.

Moreover, $I^{1-s}_+[V_\tau](0) = 0$. Indeed, by continuity of V_τ in $[0,1]$, we obtain

$$\Gamma(s)\, I^s_+[V_\tau](0) = \lim_{x\to 0} \int_0^x \frac{V_\tau(t)}{(x-t)^{1-s}}dt = \lim_{x\to 0}\left(\int_0^x \frac{V_\tau(t) - V_\tau(x)}{(x-t)^{1-s}}dt + \frac{x^s V_\tau(x)}{s}\right).$$

Since $V_\tau \in \mathcal{C}^{0,\alpha(\tau)}(a,b)$, we get $|V_\tau(t) - V_\tau(x)| \le C|x-t|^{\alpha(\tau)}$. Thus

$$\left|\int_0^x \frac{V_\tau(t)}{(x-t)^{1-s}}dt\right| \le C\int_0^x (x-t)^{\alpha(\tau)+s-1}dt + \frac{x^s V_\tau(x)}{s} = \frac{Cx^{s+\alpha(\tau)}}{s+\alpha(\tau)} + \frac{x^s V_\tau(x)}{s}\,.$$

Summarizing, if V_τ is the generalized Cantor–Vitali function and $\mathcal{U}_\tau(x) = V_\tau(1-x)$,

$$z_\tau \ :=\ D^{1-s}_+[V_\tau] \ \in\ U^{s,1}_+ \backslash W^{s,1}_+ \qquad s \in (1-\alpha(\tau), 1)\,, \tag{76}$$

since $D^s_+[z_\tau] = D\,\mathcal{V}_\tau$ is a nontrivial Cantor measure with no atomic part, whereas $(d/dx)^s_+[z_\tau] = 0$ a.e.; moreover,

$$u_\tau := D^{1-s}_-[\mathcal{U}_\tau] \in U^{s,1}_- \setminus W^{s,1}_- \qquad s \in (1-\alpha(\tau), 1)\,, \tag{77}$$

where, to achieve (77), we exploit Proposition 2 to solve the backward Abel integral equation in the distributional framework $I^{1-s}[u_\tau] = \mathcal{U}_\tau$; indeed, $\mathcal{U}_\tau \in BV(0,1)$, $I^s_-[\mathcal{U}_\tau](1) = 0$, then the unique solution $v \in L^1(0,1)$ of $I^{1-s}_-[v] = \mathcal{U}_\tau$ is $v = u_\tau = D^{1-s}_-[\mathcal{U}_\tau]$, which fulfills $I^{1-s}_-[u_\tau] = \mathcal{U}_\tau$. Hence, by evaluating the distributional derivative D, we get $D^s_-[u_\tau] = -D\,\mathcal{U}_\tau$ which is a nontrivial Cantor measure with no atomic part, whereas $(d/dx)^s_-[u_\tau] = 0$ a.e. $\square$

We list some properties concerning the comparison of bilateral Riemann–Liouville fractional Sobolev spaces $W^{s,1}$ with classical spaces: Gagliardo fractional Sobolev spaces $W^{s,1}_G$, functions of bounded variation $BV(0,1)$ and $SBV(0,1)$, De Giorgi's space of special bounded variation functions, whose derivatives have no Cantor part ([21,35] for example).

Theorem 7. *Let be $s, r \in (0,1)$ such that $r > s$. Then*

$$W^{r,1}_G(0,1) \cap I^s_+(L^1(a,b)) \cap I^s_-(L^1(a,b)) \subset W^{s,1}(0,1)$$

with continuous injection, say $\forall u \in W^{r,1}_G(0,1) \cap I^s_+(L^1(a,b)) \cap I^s_-(L^1(a,b))$

$$\|u\|_{W^{s,1}} := \|u\|_{L^1(a,b)} + \|I^{1-s}_+ u\|_{W^{1,1}(0,1)} + \|I^{1-s}_- u\|_{W^{1,1}(0,1)} \le C\|u\|_{W^{r,1}_G(0,1)}\,,$$

Proof. Straightforward consequence of Theorem 3.2 of [25] and Definition 9. $\square$

In [25], we have compared $W^{s,1}_+(0,1)$ and $W^{s,1}_-(0,1)$ with $SBV(0,1)$, and proved

$$SBV(0,1) \subset \bigcap_{s\in(0,1)} W^{s,1}(0,1).$$

This inclusion was refined by a recent result (Theorem 3.4 in [2]) showing

$$BV(a,b) \subset \bigcap_{s\in(0,1)} W^{s,1}(a,b). \tag{78}$$

On the other hand, for every $s \in (0,1)$, $W^{s,1}(a,b)$ is contained neither in $W^{1,1}_G(a,b)$ nor in $BV(a,b)$, due to remarkable examples of Weierstrass-type functions. Indeed a Weierstrass function w can be defined ([36]) so that w belongs to $W^{s,1}$, but w does not belong to $BV(0,1)$ since it is nowhere differentiable. Fix $q > 1$ and set

$$w(x) = \sum_{n=0}^{\infty} q^{-n}\big(\exp(iq^n x) - \exp(iq^n a)\big)\,. \tag{79}$$

Notice that the the constant subtraction entails $w(a) = 0$, thus preventing a singularity of $D^s_{a+}[w](x)$ at $x = a$.

Theorem 8. *Let $s, r \in (0,1)$ be such that $r > s$. Then*

$$W^{r,1}_G(0,1) \cap I^s_+(L^1(a,b)) \cap I^s_-(L^1(a,b)) \subset W^{s,1}(0,1)$$

with continuous injection. Precisely,

$$\|u\|_{W^{s,1}} := \|u\|_{L^1(a,b)} + \|I^{1-s}_+ u\|_{W^{1,1}(0,1)} + \|I^{1-s}_- u\|_{W^{1,1}(0,1)} \le C\|u\|_{W^{r,1}_G(0,1)}\,,$$

$\forall u \in W^{r,1}_G(0,1) \cap I^s_+(L^1(a,b)) \cap I^s_-(L^1(a,b))$.

We emphasize that in the case of an unbounded interval (a,b), there is no chance for a compactness statement analogous to Theorem 3 in $W^{s,1}$(a,b), since the Rellich theorem cannot be applied.

On the other hand, the property $u \in W^{s,1}(\mathbb{R})$ entails a stronger qualitative condition on u than in the case of $u \in W^{s,1}(a,b)$ with a boundedness of (a,b), as clarified by the next remark.

Remark 10. *If $u \in W^{s,1}(\mathbb{R})$, $0 < s < 1$, then $\int_{\mathbb{R}} u(t)\,dt = 0$ and $|\xi|^{s-1}\widehat{u}(\xi)$ is bounded in a neighborhood of $\xi = 0$. Property $\int_a^b u(t)\,dt = 0$ may fail for $u \in W^{s,1}(a,b)$ if $(a,b) \neq \mathbb{R}$.*

Indeed, $(a,b) = \mathbb{R}$ entails $\widehat{u} \in C^0 \cap L^\infty(\mathbb{R})$, hence $u \in L^1(\mathbb{R})$, $u \in W^{s,1}(\mathbb{R}) \subset W_e^{s,1}(\mathbb{R})$, hence

$$I_e^{1-s}[u] \;=\; u * \frac{1}{2\,\Gamma(1-s)\,|t|^s} \;\in\; W_G^{1,1}(\mathbb{R}) \;\subset\; L^1(\mathbb{R}),$$

then, exploiting the Fourier transform $\mathcal{F}$, $|\xi|^{s-1}\widehat{u}(\xi) \in C^0 \cap L^\infty(\mathbb{R})$, hence $|\xi|^{s-1}\widehat{u}(\xi)$ is bounded and $\int_{\mathbb{R}} u(t)dt = \widehat{u}(0) = 0$.

If $(a,b) \neq \mathbb{R}$, then, referring to (8) and (10), neither $I_e^{1-s}[u]$ nor $I_o^{1-s}[u]$ belong to $L^1(\mathbb{R})$ (or even to $L^p(\mathbb{R})$ with $1 < p \leq 2$, when $s > 1/2$), moreover the summability of I_o^{1-s} may fail at infinity due to a decay of order $|x|^{-s}$, therefore $\widehat{u}$ may be unbounded around $\xi = 0$.

Remark 11. *Notwithstanding Remark 10 (excluding nontrivial constant functions from the space $W^{s,1}(\mathbb{R})$), if we restrict to bounded intervals, a constant function $u \equiv K$ belongs to $W^{s,1}(a,b)$, for every bounded interval (a,b) and every value of K. Indeed,*

$$I_+^{1-s}[K](x) = \frac{K}{\Gamma(2-s)}(x-a)^{1-s} \in L^1(a,b),$$

$$D_+^s[K](x) = D_x\Big[I_+^{1-s}[K]\Big](x) = \frac{d}{dx}I_+^{1-s}[K](x) \;=\; \frac{K}{\Gamma(1-s)}\,(x-a)^{-s} \in L^1(a,b).$$

4. Bilateral Fractional Bounded Variation Space

Possible naïve definitions could be provided, for $s \in (0,1)$, by

$$A_+^s \;=\; \left\{u \in L^1(a,b) \mid \left(\frac{d}{dx}\right)_+^s u \text{ is a bounded measure}\right\},$$

$$A_-^s \;=\; \left\{u \in L^1(a,b) \mid \left(\frac{d}{dx}\right)_-^s u \text{ is a bounded measure}\right\},$$

$$A^s \;=\; A_+^s \cap A_-^s,$$

which refer to L^1-functions whose classical pointwise-defined Riemann–Liouville fractional derivative of prescribed order $s \in (0,1)$ is a bounded measure.

Actually, if the Riemann–Liouville fractional derivative $D^s[u](x)$ of u exists for a.e. x for some $s \in (0,1)$, then $I_{a+}^s[u]$ is differentiable almost everywhere, referring to the same s; nevertheless, we have no information on the distributional derivative of the fractional integral $I_{a+}^s[u]$.

These differential properties are not completely described by the point-wise derivative, though it exists almost everywhere. This shows that the previous definitions of A^s, A_+^s and A_-^s are not suitable to obtain an integration by the parts formula. Therefore, to develop a satisfactory theory of fractional bounded variation spaces, as we did for fractional Sobolev spaces in [25], we introduce a more suitable function space: the *bilateral fractional bounded variation space BV^s*, as defined in the sequel.

Remark 12. *We recall that, as long as these classical fractional derivatives are evaluated on absolutely continuous functions, as it was done in all previous section, using the operators of the classical Definition 2 provides the same results as the distributional Definition 3: for this reason,* *we keep the usual classical notations ($_{RL}D^s_{a+}$, $_{RL}D^s_{b-}$ and the corresponding short forms D^s_+, D^s_-). However, in the present section, we evaluate fractional derivatives on functions of bounded variations, a setting where the two definitions provide different evaluations.*

Next, inspired by [2], where the nonsymmetric spaces are studied also in the case of higher order derivatives, we introduce the bilateral Riemann–Liouville bounded variation space, with the aim to achieve a symmetric framework.

Definition 10. *The (bilateral) Riemann–Liouville fractional bounded variation spaces. For every $s \in (0,1)$, we set*

$$BV^s \; = BV^s_+ \cap BV^s_- \tag{80}$$

where, referring to Definition 3,

$$BV^s_+ = \{u \in L^1(a,b) \mid I^{1-s}_+[u] \in BV(a,b)\} \;=\; \{u \in L^1(a,b) \mid D^s_+[u] \in \mathcal{M}(a,b)\}\,,$$

$$BV^s_- = \{u \in L^1(a,b) \mid I^{1-s}_-[u] \in BV(a,b)\} \;=\; \{u \in L^1(a,b) \mid D^s_-[u] \in \mathcal{M}(a,b)\}\,.$$

Theorem 9. *Assume that the interval (a,b) is bounded and the parameter s fulfills $0<s<1$. Then, the space $BV^s(a,b)$ is a normed space endowed with the norm*

$$\|u\|_{BV^s} := \|u\|_{L^1(a,b)} + \|\, D^s_{a+}[u]\, \|_{\mathcal{M}(a,b)} + \|\, D^s_{b-}[u]\, \|_{\mathcal{M}(a,b)}\,. \tag{81}$$

Contribution $\|D^s_+[u]\|_{\mathcal{M}(a,b)} + \|D^s_-[u]\|_{\mathcal{M}(a,b)}$ in the norm (81) can be replaced by $\|D^s_e[u]\|_{\mathcal{M}(a,b)} + \|D^s_o[u]\|_{\mathcal{M}(a,b)}$.

Moreover, $BV^s(a,b)$ is a Banach space and for every $q \in [1, 1/(1-s))$, there is $C = C(s,q,a,b)$, such that

$$\|u\|_{L^q(a,b)} \;\leq\; C(s,q,a,b)\, \|u\|_{BV^s(a,b)}, \tag{82}$$

Every $u \in BV^s(a,b)$ can be represented by both

$$u(x) \;=\; I^s_{a+}\left[D^s_{a+}[u]\right](x) \;+\; \frac{I^{1-s}_{a+}[u](a_+)}{\Gamma(s)}\,(x-a)^{s-1} \qquad a.e\; x \in (a,b), \tag{83}$$

and

$$u(x) \;=\; I^s_{b-}\left[D^s_{b-}[u]\right](x) \;+\; \frac{I^{1-s}_{b-}[u](b_-)}{\Gamma(s)}\,(b-x)^{s-1} \qquad a.e\; x \in (a,b). \tag{84}$$

Proof. We emphasize that here $I^{1-s}_{a+}[u](a_+)$ and $I^{1-s}_{b-}[u](b_-)$ replace, respectively, $I^{1-s}_{a+}[u](a)$ and $I^{1-s}_{b-}[u](b)$ which were in representations (52) and (53) of $W^{s,1}$ functions, since in the present BV setting, there are not pointwise defined values, though there are well-defined finite right and left limits at every point in (a,b).

The map $u \mapsto \|u\|_{BV^s}$ is a norm on $W^{s,1}(a,b)$, indeed,

$\|u\|_{L^1(a,b)} + \|D^s_{a+}[u]\|_{\mathcal{M}(a,b)}$ is equivalent to the norm $\|I^{1-s}_{a+}[u]\|_{BV^{s,1}(a,b)}$, since $I^{1-s}_{a+}[u]$ belongs to BV, $D^s_{a+}[u] = DI^{1-s}_{a+}[u]$ and $\|I^{1-s}_{a+}[u]\|_{L^1(a,b)} \leq C\|u\|_{L^1(a,b)}$; analogously $\|u\|_{L^1(a,b)}$ $+\|D^s_{b-}[u]\|_{\mathcal{M}(a,b)}$ is a norm for $I^{1-s}_{b-}[u]$, due to $I^{1-s}_{b-}[u] \in BV$, $D^s_{b-}[u] = DI^{1-s}_{b-}[u]$ and $\|I^{1-s}_{a+}[u]\|_{L^1(a,b)} \leq C\|u\|_{L^1(a,b)}$. Therefore, terms $\|I^{1-s}_{\pm}[u]\|_{BV^s(a,b)}$ can be replaced, respectively, by $\|D\left[I^{1-s}_{\pm}[u]\right]\|_{\mathcal{M}(a,b)}$ in the natural norm

$$\|u\|_{BV^s} := \|u\|_{L^1(a,b)} + \|I^{1-s}_+[u]\|_{BV^s(a,b)} + \|I^{1-s}_-[u]\|_{BV^s(a,b)}\,.$$

The other claims follow by the same proof of Theorem 1 for the fractional Sobolev setting, where actually only the Proposition 2 and Corollary 1 about Abel forward and backward integral equation must be suitably tuned as stated in Remark 17. □

Example 6. *The constant functions and* $v(x) = x(1-x)$ *belong to the space* $BV^s(0,1)$. *In general, the space* $C_0^\infty(a,b)$ *of test function on* (a,b) *is contained in* $BV^s(a,b)$.

Example 7. *Heaviside function* H *belongs to* $BV^s(-1,1)$, *thanks to Example 3.*

Example 8. *Function* $H(x)|x|^{s-1}$ *belongs to* $BV_+^s(-1,1)\backslash W_+^{s,1}(-1,1)$ *if* $0 < s < 1$, *since* $I_+^{1-s}[H(x)\,|x|^{s-1}] = \Gamma(s)H(x) \in BV_+^s(-1,1)$ *due to Example 1.*

Due to the unboundedness of $I_-^{1-s}[\,H(x)\,|x|^{s-1}\,](x)$ *in a right neighborhood of* $x=0$ *(due to Example 5), we obtain that* $H(x)\,|x|^{s-1}$ *does not belong to* $BV_-^s(-1,1)$.

In general, for $0<s<1$, $H(x)\,|x|^{-\alpha}$ *belongs to* $BV_+^s(-1,1)\backslash BV_-^s(-1,1)$ *if* $0<\alpha<1-s$.

Theorem 10. ***(integration by parts in*** $\boldsymbol{BV^s(a,b)}$***)***

Next, identities hold true for $0<s<1$, $-\infty<a<b<+\infty$*:*

$$\begin{cases} \displaystyle\int_a^b u(x)\, d\,D_+^s[v](x) \;=\; -\int_a^b D_x u(x)\; I_+^{1-s}[v](x)\,dx \;+\; u(b)\,I_+^{1-s}[v](b) \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, v \in BV_+^s(a,b),\ \forall\, u \in W_G^{1,1,}(a,b)\,, \end{cases} \tag{85}$$

$$\begin{cases} \displaystyle\int_a^b u(x)\, d\,D_-^s[v](x) \;=\; +\int_a^b D_x u(x)\; I_-^{1-s}[v](x)\,dx \;+\; u(a)\,I_-^{1-s}[v](a) \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, v \in BV_-^s(a,b),\ \forall\, u \in W_G^{1,1,}(a,b)\,, \end{cases} \tag{86}$$

$$\begin{cases} \displaystyle\int_a^b u(x)\, d\,D_e^s[v](x) \;= \\ \displaystyle\quad -\int_a^b D_x u(x)\; I_e^{1-s}[v](x)\,dx + \frac{1}{2}\Big(u(b)\,I_+^{1-s}[v](b) - u(a)\,I_-^{1-s}[v](a)\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, v \in BV^s(a,b),\ \forall\, u \in W_G^{1,1,}(a,b)\,, \end{cases} \tag{87}$$

$$\begin{cases} \displaystyle\int_a^b u(x)\, d\,D_o^s[v](x) \;= \\ \displaystyle\quad -\int_a^b D_x u(x)\; I_0^{1-s}[v](x)\,dx + \frac{1}{2}\Big(u(b)\,I_+^{1-s}[v](b) + u(a)\,I_-^{1-s}[v](a)\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, v \in BV^s(a,b),\ \forall\, u \in W_G^{1,1,}(a,b)\,, \end{cases} \tag{88}$$

Proof. Exactly the same proof of Theorem 2, but the facts that, here, the distributional derivatives D_x in BV replaces the almost everywhere pointwise derivative d/dx in $W_G^{1,1}$ and the integrals at the left-hand side are evaluated with respect to the measures $D_+^s[v]$, $D_-^s[v]$, $D_e^s[v]$ and $D_o^s[v]$, in place of Lebesgue measure. □

Theorem 11. **[*Compactness in*** $\boldsymbol{BV^s(a,b)}$**]**

Assume that $0<s<1$, *the interval* (a,b) *is bounded and*

$$\|u_n\|_{BV^s(a,b)} \;\leq\; C\,. \tag{89}$$

Then, there exist $u \in L^1(a,b)$ *and a subsequence such that, without relabeling,*

$$\begin{cases} (i) & u_n \rightharpoonup u & \textit{weakly in } L^q(a,b)) \quad \forall q \in [1, 1/(1-s))\,, \\ (ii) & I_+^{1-s}[u_n] \to I_+^{1-s}[u] & \textit{strongly in } L^p(a,b)\,,\ \forall p < +\infty\,, \\ (iii) & I_-^{1-s}[u_n] \to I_-^{1-s}[u] & \textit{strongly in } L^p(a,b)\,,\ \forall p < +\infty\,. \end{cases}$$

$$I_+^{1-s}[u_n] \rightharpoonup I_+^{1-s}[u]\,, \qquad I_-^{1-s}[u_n] \rightharpoonup I_-^{1-s}[u] \qquad \textit{weakly in } BV(a,b). \tag{90}$$

Proof. The proof can be achieved by exactly the same argument used in the proof of compactness in $W^{s,1}(a,b)$ (Theorem 3). □

Remark 13. *We emphasize that*

$$BV(a,b) \underset{\neq}{\subset} BV^s(a,b) \qquad \forall s \in (0,1), \tag{91}$$

since $BV \underset{\neq}{\subset} W^{s,1}$ *and* $W^{s,1} \subset BV^s$.

Moreover,

$$BV(a,b) \underset{\neq}{\subset} \bigcap_{\sigma \in (0,1)} W^{\sigma,1}(a,b) \underset{\neq}{\subset} W^{s,1}(a,b) \underset{\neq}{\subset} BV^s_+(a,b) \qquad \forall s \in (0,1). \tag{92}$$

Indeed, the first embedding follows by (78) *and is strict due to* (79)*; the second embedding is obviously strict, about the third embedding notice that* $H(x)|x|^{s-1} \in BV^s_+(-1,1) \backslash W^{s,1}_+(-1,1)$ *(see Example 8).*

In addition, we can rewrite (74) *as follows*

$$W^{s,1}(a,b) \underset{\neq}{\subset} BV^s_+(a,b)\,, \qquad W^{s,1}(a,b) \underset{\neq}{\subset} BV^s_-(a,b)\,, \qquad \forall s \in (0,1). \tag{93}$$

since, referring to notations (76) *and* (77) *in the proof of Theorem 6,*

$$\exists\, z_\tau \in BV^s_+(-1,1) \backslash W^{s,1}(-1,1)\,, \qquad s \in \Big(1 - \ln 2 / \big(\ln 2 - \ln(1-\tau)\big)\,,\, 1\Big)\,, \tag{94}$$

$$\exists\, u_\tau \in BV^s_-(-1,1) \backslash W^{s,1}(-1,1)\,, \qquad s \in \Big(1 - \ln 2 / \big(\ln 2 - \ln(1-\tau)\big)\,,\, 1\Big) \tag{95}$$

5. Abel Equation in $\mathcal{D}'(\mathbb{R})$ and Some Useful Relationships

Here, for the reader's convenience, first, we recall some basic algebra of fractional differential calculus, then we extend to the distributional setting some classical results about Abel integral equations: these suitably tuned claims are exploited in Sections 3 and 4 to prove the main properties of $W^{\alpha,1}(a,b)$ and $BV^\alpha(a,b)$, with $\alpha \in (0,1)$: Theorems 1, 3, 5, 6, 9 and 11.

All the results stated in this section are independent of the ones of previous sections.

To avoid confusion with the standard notation of variable s in the Laplace transform, here, we label by α, instead of s, the index of fractional integral, fractional derivative and fractional Sobolev space.

All along this section: the *Laplace transformable function* refers to a measurable function v on $\mathbb{R}$ with support contained in $[0,+\infty)$ such that there exists $\lambda \in \mathbb{R}$ for which $e^{-\lambda x}v(x)$ is a Lebesgue-integrable function; the *Laplace transformable distribution* refers to a distribution v on $\mathbb{R}$ with support contained in $[0,+\infty)$ such that there exists $\lambda \in \mathbb{R}$ for which $e^{-\lambda x}v(x)$ is a tempered distribution; and in all cases, $V = \mathcal{L}\{v\}$ denotes the Laplace transform of v.

First we recall some relationships concerning fractional integral of powers of x in $(0,1)$:

$$I^{1-\alpha}_{0+}[x^\beta] = \frac{\Gamma(1+\beta)}{\Gamma(2+\beta-\alpha)}\, x^{1+\beta-\alpha} \qquad \alpha \in (0,1)\,,\ \beta > -1\,, \tag{96}$$

$$I^{\alpha}_{0+}[x^\beta] = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha)}\, x^{\beta+\alpha} \qquad \alpha \in (0,1)\,,\ \beta > -1\,, \tag{97}$$

$$I^{1-\alpha}_{1-}[x^\beta] = \left(\frac{\Gamma(\alpha-\beta-1)}{\Gamma(-\beta)} - \frac{B(x,\alpha-\beta-1,1-\alpha)}{\Gamma(1-\alpha)}\right) x^{1+\beta-\alpha} \qquad \alpha,\beta \in (0,1), \tag{98}$$

where $B = B(x,\mu,\nu)$ is the incomplete Beta function: $B(x,\nu,\mu) = \int_0^x y^{\nu-1}(1-y)^{\mu-1}dy$.

Hence, since both conditions $I^{1-\alpha}_{0+}[x^\beta] \in W^{1,1}_G(0,1)$ and $I^{1-\alpha}_{0+}[x^\beta](0) = 0$ hold true when $\beta > \alpha - 1$, one obtains the fractional derivative of power functions of x in $(0,1)$:

$$\begin{aligned} D^\alpha_{0_+}[x^\beta] &= D_x\, I^{1-\alpha}_{0_+}[x^\beta] = \frac{d}{dx}\, I^{1-\alpha}_{0_+}[x^\beta] \\ &= \frac{\Gamma(1+\beta)}{(1+\beta-\alpha)\Gamma(1+\beta-\alpha)}\,\frac{d}{dx}x^{1+\beta-\alpha} \\ &= \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}\,x^{\beta-\alpha} \qquad \alpha\in(0,1)\,,\ \beta>\alpha-1\,. \end{aligned} \tag{99}$$

Moreover,

$$I^{1-\alpha}_{0+}[x^{\alpha-1}] = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{t^{\alpha-1}dt}{(x-t)^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{dy}{y^{1-\alpha}(1-y)^\alpha} = \frac{B(\alpha,1-\alpha)}{\Gamma(1-\alpha)} = \Gamma(\alpha)$$

entails

$$D^\alpha_{0_+}[x^{\alpha-1}] \equiv 0\,, \qquad \alpha\in(0,1)\,. \tag{100}$$

In the particular case $\alpha = 1/2$ we obtain

$$I^{1/2}_{0+}[x^{-1/2}](x) = \sqrt{\pi} \qquad \text{and} \qquad D^{1/2}_{0+}[x^{-1/2}] \equiv 0\,. \tag{101}$$

Thus $D^{1/2}$ has a nontrivial kernel, as it is the case of the linear operators D^α.

More in general, by (100), we know that

$$D^\alpha_{a_+}[\,H(x-a)(x-a)^{\alpha-1}\,] \equiv 0 \qquad \forall\,\alpha\in(0,1),\ \forall\,a\in\mathbb{R},\ x\in\mathbb{R}. \tag{102}$$

The converse holds too (see Proposition 8).

Proposition 1. *(Semigroup property of fractional integral I^α_{a+})*

For every $0<\alpha<1$, $v\in L^1(\mathbb{R})$, $\operatorname{spt} v \subset [a,+\infty)$ with $a\in\mathbb{R}$, we have

$$I^{1-\alpha}_{a+}\big[I^\alpha_{a+}[v]\big](x) = I^1_{a+}[v] = \int_a^x v(t)\,dt \qquad x\in\mathbb{R}, \tag{103}$$

$$D^{1-\alpha}_{a+}\big[D^\alpha_{a+}[v]\big] = D_x[v] \qquad \text{in } \mathcal{D}'(\mathbb{R})\,. \tag{104}$$

In general, if $-\infty<a<b\le+\infty$, $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$, $v\in L^1(a,b)$, then

$$I^\alpha_{a+}\big[I^\beta_{a+}[v]\big](x) = I^{\alpha+\beta}_{a+}[v](x) \qquad x\in\mathbb{R}, \tag{105}$$

$$D^\alpha_{a+}\big[D^\beta_{a+}[v]\big](x) = D^{\alpha+\beta}_{a+}[v] \qquad \mathcal{D}'(\mathbb{R})\,. \tag{106}$$

if $-\infty\le a<b<+\infty$, $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$, $v\in L^1(a,b)$, then

$$I^\alpha_{b-}\big[I^\beta_{b-}[v]\big](x) = I^{\alpha+\beta}_{b-}[v](x) \qquad x\in\mathbb{R}, \tag{107}$$

$$D^\alpha_{b-}\big[D^\beta_{b-}[v]\big](x) = D^{\alpha+\beta}_{b-}[v] \qquad \mathcal{D}'(\mathbb{R})\,. \tag{108}$$

Proof. Consider the trivial extension of v and the standard extension of related subsequent fractional integrals as defined by

$$I^\alpha_{a+}[v] = v * \frac{1}{\Gamma(\alpha)}\,\frac{H(x)}{|x|^{1-\alpha}} \qquad x\in\mathbb{R}\,.$$

We assume first $a = 0$. By denoting V, the Laplace transform of v and taking into account of $\mathcal{L}\{H(x)\, x^\beta\} = \Gamma(\beta+1)/s^{\beta+1}$ and (97), we obtain, for $\mathrm{Re}\, s > 0$,

$$\begin{aligned}\mathcal{L}\Big\{ I_{0+}^{1-\alpha}\left[I_{0+}^{\alpha}[v]\right](x)\Big\} &= \frac{1}{\Gamma(1-\alpha)}\,\frac{1}{\Gamma(\alpha)}\,\mathcal{L}\Big\{ v * \frac{H(x)}{|x|^\alpha} * \frac{H(x)}{|x|^{1-\alpha}} \Big\} \\ &= \frac{1}{s}\, V(s) = \mathcal{L}\Big\{\int_{-\infty}^{x} v(t)dt\Big\} = \mathcal{L}\Big\{\int_{0}^{x} v(t)dt\Big\},\end{aligned}$$

hence, claim (103) follows by the injectivity of the Laplace transform.

$$\begin{aligned}\mathcal{L}\Big\{ D_{0+}^{1-\alpha}\left[D_{0+}^{\alpha}[v]\right](x)\Big\} &= \frac{1}{\Gamma(\alpha)}\,\frac{1}{\Gamma(1-\alpha)}\,\mathcal{L}\Big\{ D_x\Big(\frac{H(x)}{|x|^{1-\alpha}} * D_x\Big(\frac{H(x)}{|x|^{\alpha}} * v\Big)\Big) \Big\} \\ &= \frac{1}{\Gamma(\alpha)}\,\frac{1}{\Gamma(1-\alpha)}\, s\,\frac{\Gamma(\alpha)}{s^\alpha}\, s\,\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\, V(s) = s\,V(s) = \mathcal{L}\{D_x\, v\}\,,\end{aligned}$$

hence, claim (104) follows by the injectivity of the Laplace transform.

In general, we obtain, for $\mathrm{Re}\, s > 0$,

$$\begin{aligned}\mathcal{L}\Big\{ I_{0+}^{\alpha}\left[I_{0+}^{\beta}[v]\right](x)\Big\} &= \frac{1}{\Gamma(\alpha)}\,\frac{1}{\Gamma(\beta)}\,\mathcal{L}\Big\{ v * \frac{H(x)}{|x|^{1-\alpha}} * \frac{H(x)}{|x|^{1-\beta}} \Big\} \\ &= \frac{1}{s^{\alpha+\beta}}\, V(s) = \frac{1}{\Gamma(\alpha+\beta)}\mathcal{L}\Big\{ v * \frac{H(x)}{|x|^{1-(\alpha+\beta)}}\Big\} = \mathcal{L}\Big\{ I_{0+}^{\alpha+\beta}[v](x)\Big\},\end{aligned}$$

which proves (105). Identity (106) is achieved in the same way. The case of general $a \in \mathbb{R}$ is achieved by translation. Moreover, given $v \in L^1(a,b)$ with $-\infty < a < b \le +\infty$, by considering the trivial extension of v on $\mathbb{R}$, we obtain

$$\begin{aligned} I_{b-}^{\alpha}\left[I_{b-}^{\beta}[v]\right] &\overset{(35)}{=} \Big(I_{a+}^{\alpha}\Big[\Big(I_{b-}^{\beta}[v]\Big)^{\vee}\Big]\Big)^{\vee} = \Big(I_{a+}^{\alpha}\Big[\Big(\Big(I_{a+}^{\beta}[\check{v}]\Big)^{\vee}\Big)^{\vee}\Big]\Big)^{\vee} = \Big(I_{a+}^{\alpha}[I_{a+}^{\beta}[\check{v}]]\Big)^{\vee} = \\ &\overset{(105)}{=} \Big(I_{a+}^{\alpha+\beta}[\check{v}]\Big)^{\vee} \overset{(35)}{=} \Big(\Big(I_{b-}^{\alpha+\beta}[v]\Big)^{\vee}\Big)^{\vee} = I_{b-}^{\alpha+\beta}[v]\,.\end{aligned}$$

hence, (107) is proved. Identity (108) is achieved in the same way. □

Proposition 2. *Assume $\alpha \in (0,1)$, $-\infty < a < b \le +\infty$, $I_{a+}^{1-\alpha}[f](a) = 0$ and f belongs to $W_{+}^{\alpha,1}(a,b) := \Big\{v \in L^1(a,b) \,|\, I_{a+}^{1-\alpha}[v] \in W_G^{1,1}(a,b)\Big\}$.*

*Then the **Abel integral equation***

$$I_{a+}^{\alpha}[u](x) = f(x) \qquad \textit{for a.e. } x \textit{ in the interval } (a,b) \tag{109}$$

admits the solution u given by

$$u(x) = D_{a+}^{\alpha}[f](x) \quad \textit{for a.e. } x \textit{ in the interval } (a,b). \tag{110}$$

which is unique among Laplace-transformable functions evaluated with translated variable $x-a$.

Proof. Whenever necessary, we consider the trivial extension (namely, 0 valued) on $(-\infty, a)$ of every function and if necessary on $(b, +\infty)$, without relabeling the function name. Thus, every related fractional integral set as a function defined over (a,b) has a trivial extension, which coincides on (a,b) with the same fractional integral of the trivial extension, namely, it has support contained on $[a, +\infty)$.

First, we assume $a = 0$. In such a case, f is a Laplace-transformable function: we denote by $F(s) = \mathcal{L}\{f\}(s)$ and $\mathrm{U}(s) = \mathcal{L}\{\mathrm{u}\}(s)$ their Laplace transform evaluated at the variable s. If a Laplace transformable solution u exists, then its Laplace transform U must fulfill the transformed equation. We have

$$I^{\alpha}_{0+}[u](x) \;=\; \frac{1}{\Gamma(\alpha)}\int_0^x \frac{u(t)\,dt}{(x-t)^{1-\alpha}} = \frac{1}{\Gamma(\alpha)}\left(H(x)\,x^{\alpha-1}\right) * u$$

$$I^{\alpha}_{0+}[u](x) = f(x)$$

$$\frac{1}{\Gamma(\alpha)}\,\frac{\Gamma(\alpha)}{s^{\alpha}}\,\mathrm{U}(\mathrm{s}) \;=\; \mathrm{F}(\mathrm{s})$$

$$\mathrm{U}(s) \;=\; s^{\alpha}F(s) \;=\; s\,\frac{F(s)}{s^{1-\alpha}}$$

We evaluate $u = \mathcal{L}^{-1}\mathrm{U}$: reminding that $\mathcal{L}\{D_x w(x)\} = s\,\mathcal{L}\{w\}$, where w is any $\mathcal{L}$-transformable distribution, and here, D_x and d/dx denote respectively the distributional derivative on the open set $\mathbb{R}$ and on $(a,+\infty)$. By taking into account that $I^{1-\alpha}_{+}[f]$ belongs to $W^{1,1}_{G}(0,b) \subset L^{\infty}(0,b) \cap C^{0}[0,b]$, we know that $I^{1-\alpha}_{+}[f](0_+)$ is a well-defined real value. Thus, by formula $s\,G(s) = \mathcal{L}\{D_x g\} = \mathcal{L}\{(d/dx)g\} + g(0_+)$ applied to $g = I^{1-\alpha}_{a+}[f]$ under the assumption $I^{1-\alpha}_{a+}[f](0) = 0$, we obtain

$$\begin{aligned} u(x) \;&=\; \mathcal{L}^{-1}\{U(s)\} \;=\; \mathcal{L}^{-1}\left\{s\,\frac{F(s)}{s^{1-\alpha}}\right\} \\ &=\; D_x\left(f * \frac{1}{\Gamma(1-\alpha)\,x^{\alpha}}\right) \;=\; D_x\left(I^{1-\alpha}_{+}[f](x)\right) \\ &=\; \frac{d}{dx}\left(I^{1-\alpha}_{+}[f](x)\right) \;=\; \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dx}\int_0^x \frac{f(t)\,dt}{(x-t)^{\alpha}} \;=\; D^{\alpha}_{+}[f](x)\,, \end{aligned} \tag{111}$$

where the four last equalities are understood in the sense a.e. on $(-\infty,b)$, coherently with the fact that $u \in L^1(-\infty,b)$ because it coincides with the derivative of the function $I^{1-\alpha}_{+}[f] \in W^{1,1}_{G}(0,b)$ and vanishes on $(-\infty,0)$. Moreover u is unique due to the injectivity of the Laplace transform. Then (110) is proved when $a = 0$.

If $a \neq 0$, we can exploit the solution Formula (111) proved in case $a = 0$: assume $I^{\alpha}_{a+}[u] = f$ on (a,b) and set $u(t) = v(t-a)$; then $t \mapsto v(t) = u(t+a)$ and $t \mapsto f(t+a)$ have support on $[0,+\infty)$ and, hence, are Laplace transformable functions.

$$\begin{aligned} I^{\alpha}_{a+}[u](x) \;&=\; \frac{1}{\Gamma(\alpha)}\int_a^x \frac{u(t)dt}{(x-t)^{1-\alpha}} = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{v(t-a)\,dt}{\big((x-a)-(a+t)\big)^{1-\alpha}} \\ &=\; \frac{1}{\Gamma(\alpha)}\int_0^{x-a} \frac{v(\tau)\,d\tau}{\big((x-a)-\tau\big)^{1-\alpha}} = I^{\alpha}_{0+}[v](x-a) = I^{\alpha}_{0+}[u(t+a)](x-a) \end{aligned}$$

Thus we have the Abel equation $I^{\alpha}_{0+}[u(t+a)](x-a) \;=\; f(x)$, that is

$$I^{\alpha}_{0+}[u(t+a)](x) \;=\; f(x+a)\,.$$

By (111), we get $u(x+a) = v(x) = D^{\alpha}_{0+}[f(x+a)](x+a)$, that is

$$u(x) = D^{\alpha}_{a+}[f](x)\,. \tag{112}$$

□

Remark 14. *At a first glance, both technical assumptions in Proposition 2, namely* $f \in W^{\alpha,1}_{+}(a,b)$ *and* $I^{1-\alpha}_{a+}[f](a) = 0$*, may look strange or unnatural.*

However, they cannot be circumvented: actually, they are both necessary conditions for the existence of a solution $u \in L^1(a,b)$ of Equation (109).

Let us check this claim: if such a solution as $u \in L^1(a,b)$ exists, then $f = I^{\alpha}_{a+}[u]$ belongs to $L^1(a,b)$; moreover, due to the semigroup property of fractional integrals (see Proposition 1),

$$I^{1-\alpha}_{a+}[f](x) = I^{1-\alpha}_{a+}[I^{\alpha}_{+}[u]](x) = I^{1}_{a+}[[u]](x) = \int_a^x u(t)\,dt\,, \tag{113}$$

hence, $I^{1-\alpha}_{a+}[f]$ is the primitive of an $L^1(a,b)$ function; thus $I^{1-\alpha}_{a+}[f]$ belongs to $W^{1,1}_G(a,b)$ and $I^{1-\alpha}_{a+}[f](a) = 0$.

Remark 15. *Condition $I^{1-\alpha}_{+}[f](a) = 0$ may be not easy to check. However, it can be replaced by stronger conditions, which are much easier to check. Indeed, if either there exists a finite value $f(a_+) := \lim\limits_{x \to a_+} f(x)$ or f is bounded in a neighborhood of 0, then $I^{1-\alpha}_{+}[f](a) = 0$.*

Remark 16. *For the unnormalized Abel equation , $\Gamma(\alpha)\, I^{\alpha}_{a+}[u] = f$, namely*

$$\int_a^x \frac{u(t)}{(x-t)^{1-\alpha}}\,dt \;=\; f(x) \qquad \text{for } x \text{ in the interval } (a,b), \tag{114}$$

as a straightforward consequence of Proposition 2 and Euler reflection formula, *$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)\ \forall z \in \mathbb{C}\backslash\mathbb{Z}$, under the assumption $f \in W^{s,1}(a,b)$, we recover the next formula for the unique solution u in $L^1(a,b)$:*

$$u(x) = \frac{1}{\Gamma(\alpha)} D^{\alpha}_{a+}[f](x) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\,\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}} = \frac{\sin(\alpha\pi)}{\pi}\,\frac{d}{dx}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}} \tag{115}$$

still under the requirement that necessary conditions $f \in W^{\alpha,1}_{+}(a,b)$ and $I^{1-\alpha}_{+}[f](a) = 0$ hold true.

Now, we remove the assumption $I^{1-\alpha}_{a+}[v](a) = 0$ and look for solutions in $\mathcal{D}'(\mathbb{R})$.

Proposition 3. *Assume that $\alpha \in (0,1)$, $-\infty < a < b \le +\infty$ and f belongs to the space $BV^{\alpha}_{+}(a,b) := \left\{ v \in L^1(a,b) \,|\, I^{1-\alpha}_{a+}[v] \in BV(a,b) \right\}$.*

Then, the ***Abel integral equation in the distributional framework***

$$I^{\alpha}_{a+}[u] \;=\; f \qquad \text{in } \mathcal{D}'(\mathbb{R}), \tag{116}$$

admits a unique solution u among Laplace transformable distributions evaluated at $x-a$ (variable translation), which is the bounded measure on $\mathbb{R}$ with support contained in $[a,+\infty)$ given by

$$u(x) \;=\; D^{\alpha}_{a+}[f](x) \;+\; I^{1-\alpha}_{a+}[f](a_+)\,\delta(x-a) \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{117}$$

In (116), actually u denotes the trivial extension outside (a,b), and

$$I^{\alpha}_{a+}[u] \;=\; u \,*\, \frac{1}{\Gamma(\alpha)}\,\frac{H(x)}{|x|^{1-\alpha}}$$

represents the distributional convolution whose evaluation, namely f, is identically 0 on $(-\infty,a)$ and possibly non-zero on $[b,+\infty)$.

Proof. Same proof of Proposition 2. Only the step in (111) with $a = 0$ has to be slightly modified: denoting by $\widetilde{\widetilde{D}}_x$ and $\widetilde{D}_x$ the distributional derivative respectively in $\mathcal{D}'(\mathbb{R}\backslash\{0\})$ and $\mathcal{D}'(0,+\infty)$, setting $F(s) = \mathcal{L}\{f\}$, $g(x) = I^{1-\alpha}_{a+}[f](x)$, $G(s) = \mathcal{L}\{g\} = F(s)/s^{1-\alpha}$, $\mathcal{L}\{D_x g\} = sG(s)$ and

$$D_x\, g = \widetilde{\widetilde{D}}_x\, g + g(0_+)\,\delta(x) = \widetilde{D}_x\, g + g(0_+)\,\delta(x)\,,$$

we exploit the fact that $I_+^{1-\alpha}[f](0_+)$ is a finite well-defined value (since $f \in BV_+^\alpha(0,b)$ entails $I_+^{1-\alpha}[f] \in BV(0,b)$), and we replace (111) by

$$\begin{aligned}
u(x) \;&=\; \mathcal{L}^{-1}\{U(s)\} \;=\; \mathcal{L}^{-1}\left\{ s\,\frac{F(s)}{s^{1-\alpha}} \right\} \;=\; \mathcal{L}^{-1}\{\, s\, G(s)\,\} \\
&=\; D_x\, g \;=\; \frac{d}{dx}\, g + g(0_+)\,\delta(x) \\
&=\; \frac{d}{dx}\left(f * \frac{1}{\Gamma(1-\alpha)\, x^\alpha} \right) + I_{a+}^{1-\alpha}[f](0_+)\,\delta(x) \\
&=\; \frac{d}{dx}\left(I_+^{1-\alpha}[f](x) \right) + I_{a+}^{1-\alpha}[f](0_+)\,\delta(x) \\
&=\; D_+^\alpha[f](x) \;+\; I_{a+}^{1-\alpha}[f](0_+)\,\delta(x)\,.
\end{aligned}$$

□

Corollary 1. *Assume $\alpha \in (0,1)$, $-\infty \le a < b < +\infty$, the value $f(b_-) := \lim_{x\to b_-} f(x)$ exists and is finite (possibly substituted by weaker condition $I_{b-}^{1-\alpha}[f](b) = 0$) and f belongs to $W_-^{\alpha,1}(a,b) := \left\{ v \in L^1(a,b) \,|\, I_{b-}^{1-\alpha}[v] \in W_G^{1,1}(a,b) \right\}$.*

Then, the ***backward Abel integral equation***

$$I_{b-}^\alpha[u](x) \;=\; f(x) \qquad \text{for a.e. } x \text{ in the interval } (a,b) \tag{118}$$

admits a solution u, unique among Laplace transformable functions evaluated at $b-x$ (sign change and translation), which is given by

$$u(x) \;=\; D_{b-}^\alpha[f](x) \qquad \text{for a.e. } x \text{ in the interval } (a,b). \tag{119}$$

Proof. Taking into account that $b \in \mathbb{R}$, set $v(t) = \check{u}(t) := u(-t)$, and hence, $u : (a,b) \to \mathbb{R}$, $v : (-b,-a) \to \mathbb{R}$, and choose $x \in (a,b)$. Then

$$\begin{aligned}
f(x) \;&=\; I_{b-}^\alpha[u](x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{u(\tau)\,d\tau}{(\tau - x)^{1-\alpha}} \\
&=\; \frac{1}{\Gamma(\alpha)} \int_{x-b}^0 \frac{u(t+b)\,dt}{\big((b+t) - x\big)^{1-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{x-b}^0 \frac{v(-(b+t))\,dt}{\big(-x + (b+t)\big)^{1-\alpha}} \\
&=\; \frac{-1}{\Gamma(\alpha)} \int_0^{x-b} \frac{v(-(b+t))\,dt}{\big(-x + (b+t)\big)^{1-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{-b}^{-x} \frac{v(y)\,dy}{\big(-x - y)\big)^{1-\alpha}} = I_{(-b)+}^\alpha[v](-x)
\end{aligned}$$

Therefore, we can apply Proposition 2 to an Abel equation on $(-b,-a)$:

$$I_{(-b)+}^\alpha[v](x) \;=\; f(-x)$$

$$I_{(-b)+}^\alpha[\check{u}](x) \;=\; \check{f}(x)$$

$$\check{u}(x) \;=\; D_{(-b)+}^\alpha[\check{f}](x)$$

$$\begin{aligned}
u(-x) \;&=\; \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-b}^x \frac{f(-t)\,dt}{(x-t)^\alpha} = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_b^{-x} \frac{f(\tau)\,d\tau}{(x+\tau)^\alpha} \\
&=\; \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-x}^b \frac{f(\tau)\,d\tau}{(x+\tau)^\alpha} \;=\; \frac{d}{dx}\left(I_{b-}^{1-\alpha}[f](-x) \right) \overset{\boxed{\text{chain-rule}}}{=} \\
&=\; -\frac{d}{dx}\left(I_{b-}^{1-\alpha}[f] \right)(-x) \;=\; D_{b-}^\alpha[f](-x) \qquad x \in (-b,-a)\,.
\end{aligned}$$

$$u(x) \;=\; D^{\alpha}_{b-}[f](x) \qquad x \in (a,b) \quad \square$$

Corollary 2. *Assume that* $\alpha \in (0,1)$, $-\infty < a < b \leq +\infty$ *and* f *belongs to the space* $BV^{\alpha}_{-}(a,b) := \left\{ v \in L^1(a,b) \,|\, I^{1-\alpha}_{b-}[v] \in BV(a,b) \right\}$.

Then the ***backward Abel integral equation in the distributional framework***

$$I^{\alpha}_{b-}[u](x) \;=\; f(x) \qquad \text{in } \mathcal{D}'(\mathbb{R}), \tag{120}$$

admits a unique solution u *among Laplace transformable distributions evaluated at* $b - x$ *(say with sign change and translation), which is the bounded measure with support contained in* $(-\infty, b]$ *given by*

$$u(x) \;=\; D^{\alpha}_{b-}[f](x) \;+\; I^{1-\alpha}_{b-}[f](b_-)\,\delta(x-b) \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{121}$$

In (120), *actually* u *denotes the trivial extension outside* (a,b), *and*

$$I^{\alpha}_{b-}[u] \;=\; u * \frac{1}{\Gamma(\alpha)}\,\frac{H(-x)}{|x|^{1-\alpha}}\,.$$

represents the distributional convolution whose evaluation, namely f, *is identically* 0 *on* $[b, +\infty)$ *and possibly non-zero on* $(-\infty, a)$.

Proof. Same proof of Corollary 1, but exploiting Proposition 3 instead of Proposition 2. Notice that the trivial extension of a function in $L^1(a,b)$ has compact support and can be dealt with as a Laplace transformable distribution evaluated at the variable $(b-x)$. $\square$

Example 9. *We mention some basic examples of solution* u *for Abel integral equation* $I^{\alpha}_{0_+}[u] = f$ *on* $(0,b)$ *with* $0 < b \leq +\infty$ *and, more in general for distributional Abel integral equation* $I^{\alpha}_{a_+}[u] = f$ *and* $I^{\alpha}_{b_-}[u] = f$ *with support condition.*

1. *If* $f = x^{\alpha}$, *then* $u = D^{\alpha}_{a+}[t^{\alpha}](x) = \Gamma(\alpha+1)\,H(x)$, *for* $\alpha \in (0,1)$, *due to Proposition* 2.
2. *If* $f = H(x)$, *then* $u = D^{\alpha}_{a+}[H](x) \;=\; \dfrac{x^{-\alpha}}{\Gamma(1-\alpha)}$, *for* $\alpha \in (0,1)$, *due to Proposition* 2.
3. *If* $f = x^{\beta}$, *then* $u = \dfrac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)}\,x^{\beta-\alpha}$, *for* $\alpha \in (0,1)$, $\beta > \alpha - 1$, *due to Proposition* 2.
   These relationships are deduced by Proposition 2*: in the first and second item, notice that* $f \in L^{\infty}(a,b)$ *entails* $I^{1-\alpha}_{0+}[u](0) = 0$ *(see Remark* 15*), while in third item* $\beta > \alpha - 1$ *entails both* $I^{1-\alpha}_{+}[x^{\beta}](x) = \frac{\Gamma(1+\beta)}{\Gamma(2+\beta-\alpha)}\,x^{1+\beta-\alpha} \in W^{1,1}_{G}(0,1)$ *and* $I^{1-\alpha}_{0+}[x^{\beta}](0) = 0$. *Thus, we get the three claims above by applying the relationships*

$$D^{\alpha}_{0_+}[x^{\beta}] \;=\; \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}\,x^{\beta-\alpha}\,, \qquad (1-\alpha) \neq \alpha \in (0,1),\; \beta > \alpha - 1\,, \tag{122}$$

$$D^{\alpha}_{0_+}[x^{\alpha-1}] \;=\; 0\,, \qquad\qquad \alpha \in (0,1)\,. \tag{123}$$

4. *If* $f(x) = (x-a)^{\alpha-1}$, $x \in (a,b)$, $\alpha \in (0,1)$, *then the solution* u *with support on* $[a, +\infty)$ *to distributional backward Abel equation* $I^{\alpha}_{a+}[u] \;=\; (x-a)^{\alpha-1}H(x-a)$ *is given by* $u(x) = \Gamma(\alpha)\,\delta(x-a)$.
   Indeed $I^{1-\alpha}_{a+}[(t-a)^{\alpha-1}](x) \;=\; \Gamma(\alpha)$, $D^{\alpha}_{a+}[t^{\alpha-1}](x) \;=\; D_x I^{1-\alpha}_{a_+}[t^{\alpha-1}](x) \;\equiv\; 0$ *thus, by Proposition* 3, $u = D^{\alpha}_{a+}[(t-a)^{\alpha-1}](x) + I^{1-\alpha}_{a+}[(t-a)^{1-\alpha}](a_+)\delta(x-a) = \Gamma(\alpha)\delta(x-a)$.
   Then, u *solves the Abel equation since, by representation* (14)*, we obtain*
   $$I^{\alpha}_{a+}[\Gamma(\alpha)\,\delta(x-a)] = \Gamma(\alpha)\,\delta(x-a) * \frac{H(x)\,(x)^{\alpha-1}}{\Gamma(\alpha)} = H(x-a)\,(x-a)^{\alpha-1} \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

5. *If $f(x) = x^2(1-x)$, $x \in (0,1)$, $\alpha \in (0,1)$, then the solution u to backward Abel equation $I^{\alpha}_{1-}[u] = f(x)$ is given by $u(x) = D^{\alpha}_{1-}f(x) = \frac{(1-x)^{1-\alpha}\left(6x^2-4\alpha x+(\alpha-1)\alpha\right)}{\Gamma(4-\alpha)}$, due to Corollary 1 since $I^{1-\alpha}_{b-}[f](b_-) = 0$.*
6. *If $f(x) = (b-x)^{\alpha-1}$, $x \in (a,b)$, $\alpha \in (0,1)$, then the solution u with support on $(-\infty, b]$ to backward distributional Abel equation $I^{\alpha}_{b-}[u] = (b-x)^{\alpha-1}H(b-x)$ is given by $u(x) = \Gamma(\alpha)\,\delta(x-b)$. Indeed*

$$\begin{aligned} I^{1-\alpha}_{b-}[(b-t)^{\alpha-1}](x) &= \frac{1}{\Gamma(1-\alpha)}\int_x^b \frac{1}{(b-t)^{1-\alpha}(t-x)^{\alpha}}\,dt \overset{\boxed{y=(b-t)x/(b-x)}}{=} \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{1}{y^{1-\alpha}(1-y)^{\alpha}}\,dy = \frac{B(\alpha,1-\alpha)}{\Gamma(1-\alpha)} = \Gamma(\alpha)\,, \end{aligned}$$

so $I^{1-\alpha}_{b-}[(b-t)^{\alpha-1}](x) = \Gamma(\alpha)$, $D^{\alpha}_{b-}[t^{\alpha-1}](x) = -D_x I^{1-\alpha}_{b-}[t^{\alpha-1}](x) \equiv 0$ so, by Corollary 2, $u = D^{\alpha}_{b-}[(b-t)^{\alpha-1}](x) + I^{1-\alpha}_{b-}[(b-t)^{\alpha-1}](b_-)\,\delta(b-x) = \Gamma(\alpha)\,\delta(b-x)$.
Then, such u solves the Abel equation since, by representation (14),

$$I^{\alpha}_{-}[\Gamma(\alpha)\,\delta(b-x)] = \Gamma(\alpha)\,\delta(b-x) * \frac{H(-x)\,(x)^{\alpha-1}}{\Gamma(\alpha)} = H(b-x)\,(b-x)^{\alpha-1} \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Lemma 8. *Fix a value $\alpha \in (0,1)$.*

If a Laplace transformable function u fulfils $D^{\alpha}_{0+}[u] \equiv 0$ on the half-line $(0,+\infty)$, then $u(x) = C\,x^{\alpha-1}$, for a suitable constant C.

If a function $u \in L^1(a,b)$, with $-\infty < a < b < +\infty$, fulfils $D^{\alpha}_{a+}[u] \equiv 0$ on (a,b), then $u(x) = C\,(x-a)^{\alpha-1}$, for a suitable constant K.

If a function $u \in L^1(a,b)$, with $-\infty < a < b < +\infty$, fulfils $D^{\alpha}_{b-}[u] \equiv 0$ on (a,b), then $u(x) = C\,(b-x)^{\alpha-1}$, for a suitable constant C.

Proof. The property

$$D_x\, I^{1-\alpha}_{a+}[u] \;=\; D^{\alpha}_{a+}[u] \;\equiv 0$$

entails $I^{1-\alpha}_{a+}[u]$ is constant. Thus, for a suitable constant function K, we have that u fulfills the Abel integral equation: $I^{1-\alpha}_{a+}[u] = K$, moreover $I^{\alpha}_{a+}[K](a) = 0$ since $K \in L^{\infty}(a,b)$, and due to (96) and the boundedness of (a,b)

$$I^{\alpha}_{a+}[K] = K\frac{(x-a)^{\alpha}}{\alpha\,\Gamma(\alpha)} \in W^{1,1}_G(a,b) \qquad \forall\,\alpha \in (0,1).$$

Then, by Proposition 2, the solution u of the Abel equation $I^{1-\alpha}_{a+}[u] = K$ is

$$u(x) \;=\; D^{1-\alpha}_{+}[K] \;=\; D_x\, I^{\alpha}_{a+}[K] = \frac{K}{\Gamma(\alpha)}\,\frac{d}{dx}\,\frac{(x-a)^{\alpha}}{\alpha} \;=\; \frac{K}{\Gamma(\alpha)}\,(x-a)^{\alpha-1}\,.$$

This proves the first and second claim, since an $\mathcal{L}$-transformable function is an L^1 function on every bounded interval. The third one follows in the same way, by applying Corollary 1 to the backward Abel equation $I^{1-\alpha}_{b-}[u] = K$. □

Lemma 8 provides the inverse of (100). Hence, summarizing

$$u \in L^1(a,b) \text{ fulfils } D^{\alpha}_{a+}[u] \equiv 0 \text{ on } (a,b) \qquad \text{iff} \qquad u(x) = C(x-a)^{\alpha-1}\,. \tag{124}$$

Lemma 9. *Assume that the interval (a,b) is bounded, $0 < \alpha < 1$, $v \in L^1(a,b)$, $I^{1-\alpha}_{a+}[v]$ belongs to $W^{1,1}_G(a,b)$ and $I^{1-\alpha}_{a+}[v](a_+) = 0$.*

Then

$$\exists \text{ unique } \mathcal{V} \in L^1(a,b): \quad v = I^{\alpha}_{a+}[\mathcal{V}]\,; \text{ and } \mathcal{V}(x) = D^{\alpha}_{a+}[v] = D_x I^{1-\alpha}_{a+}[v]\,, \tag{125}$$

and $v \in L^q(a,b)$ for every $q \in [0, 1/(1-\alpha))$; moreover, there is $C = C(q)$ such that

$$\|v\|_{L^q(a,b)} \leq C(q)\left(\|v\|_{L^1(a,b)} + \|I^{1-\alpha}_{a+}[v]\|_{W^{1,1}_G(a,b)}\right). \tag{126}$$

The same claims hold true when $I^{\alpha}_{a+}[v]$, $I^{1-\alpha}_{a+}[v]$ and $D^{\alpha}_{a+}[v]$ are replaced, respectively, by $I^{\alpha}_{b-}[v]$, $I^{1-\alpha}_{b-}[v]$ and $D^{\alpha}_{b-}[v]$ in the assumptions and the claims.

Proof. By considering $\mathcal{V}$ as the unknown in the Abel integral equation

$$I^{\alpha}_{a+}[\mathcal{V}] \;=\; v \qquad \text{on } (a,b) \tag{127}$$

we know by Proposition 2 that there is a solution $\mathcal{V} \in L^1(a,b)$ fulfilling the integral equation: such $\mathcal{V}$ is the unique solution in $L^1(a,b)$ and fulfills

$$\mathcal{V}(x) = D^{\alpha}_{a+}[v] = D_x\left(I^{1-\alpha}_{a+}[v]\right). \tag{128}$$

Thus $\mathcal{V}(x) \in L^1(a,b)$. Moreover, by (127), $I^{1-\alpha}_{a+}[v](a) = 0$ and the semigroup property of $s \mapsto I^s_{a+}$ (Proposition 1),

$$I^{1-\alpha}_{a+}[v](x) = I^{1-\alpha}_{a+}[I^{\alpha}_{a+}[\mathcal{V}]] = I^1_{a+}[\mathcal{V}] = \int_a^x \mathcal{V}(t)\,dt\,.$$

Summing up $\mathcal{V} \in L^1(a,b)$, $I^{1-\alpha}_{a+}[\mathcal{V}](a_+) = 0$, $I^{1-\alpha}_{a+}[v](a_+) = 0$, $I^{1-\alpha}_{a+}[v - I^{\alpha}_{a+}[\mathcal{V}]](a_+) \equiv 0$; then $v = I^{\alpha}_{a+}[\mathcal{V}]$, $v \in I^{\alpha}_{a+}\left(L^1(a,b)\right)$ and (125), (126) follow by standard embedding of fractional integrals. □

If we remove the assumption $I^{1-\alpha}_{a+}[v](a_+) = 0$ in Lemma 9, then we must add suitable corrections to both v and $\mathcal{V}$, as stated by the next theorem.

Theorem 12. *Assume that (a,b) bounded, $0 < \alpha < 1$, $v \in L^1(a,b)$, $I^{1-\alpha}_{a+}[v]$ belongs to $BV(a,b)$. Then*

$$\exists\, \mathcal{V} \in \mathcal{M}(\mathbb{R}),\ \mathrm{spt}\,\mathcal{V} \subset [a,+\infty),\ \exists\, \mathfrak{K} \in \mathbb{R}:\ v = I^{\alpha}_{a+}[\mathcal{V}] + \frac{\mathfrak{K}}{\Gamma(\alpha)}(x-a)^{\alpha-1};\ D^{\alpha}_{a+}[v] = \mathcal{V}(x)\,, \tag{129}$$

and $v \in L^q(a,b)$ for every $q \in [1, 1/(1-\alpha))$; moreover, there is $B = B(q,\mathfrak{K},\alpha)$ such that

$$\|v\|_{L^q(a,b)} \leq B(q,\mathfrak{K},\alpha)\left(\|v\|_{L^1(a,b)} + \|I^{1-\alpha}_{a+}[v]\|_{W^{1,1}_G(a,b)}\right). \tag{130}$$

Explicitly, for every given $\alpha \in (0,1)$, we have

$$v = I^{\alpha}_{a+}[D^{\alpha}_{a+}[v]] + \frac{I^{1-\alpha}_{a+}[v](a_+)}{\Gamma(\alpha)}(x-a)^{\alpha-1}, \quad \text{a.e. } x \in (a,b), \forall v \in L^1(a,b) : I^{1-\alpha}_{a+}[v] \in BV(a,b). \tag{131}$$

The same claims hold true when $I^{\alpha}_{a+}[v]$, $I^{1-\alpha}_{a+}[v]$ and $D^{\alpha}_{a+}[v]$ are replaced respectively by $I^{\alpha}_{b-}[v]$, $I^{1-\alpha}_{b-}[v]$ and $D^{\alpha}_{b-}[v]$ in the assumptions and the claims.

Proof. Since $I^{1-\alpha}_{a+}[v]$ belongs to $BV(a,b)$, it has a finite right value $I^{1-\alpha}_{a+}[v](a_+)$ at $x = a$, labeled by $\mathfrak{K}$, say $\mathfrak{K} := I^{1-\alpha}_{a+}[v](a_+)$. By (97), $I^{1-\alpha}_{a+}[(x-a)]^{\alpha-1} \equiv \Gamma(\alpha)$, $0 < \alpha < 1$. We set

$$w(x) = v(x) - \frac{\mathfrak{K}}{\Gamma(\alpha)}(x-a)^{\alpha-1}$$

then $w \in L^1(a,b)$, $I^{1-\alpha}_{a+}[w] \in BV(a,b)$ and $I^{1-\alpha}_{a+}[w](a_+) = 0$. We know by Proposition 3 that there is a solution $\mathcal{W} \in \mathcal{M}(\mathbb{R})$ with $\mathrm{spt}\,\mathcal{W} \subset [a,+\infty)$ fulfilling the integral equation

$$v = I^{\alpha}_{a+}[\mathcal{V}] \qquad \text{in } \mathcal{D}'(\mathbb{R}), \tag{132}$$

such $\mathcal{W}$ is the unique solution with support on $[a,+\infty)$ and fulfills

$$\mathcal{W}(x) \;=\; D^{\alpha}_{a+}[w](x) \tag{133}$$

Thus $\mathcal{W}(x) \in \mathcal{M}$. By (132), $I^{1-\alpha}_{a+}[w](a_+) = 0$ and the semigroup property of $s \mapsto I^s_{a+}$

$$I^{1-\alpha}_{a+}[w](x) = I^{1-\alpha}_{a+}[I^{\alpha}_{a+}[\mathcal{W}]] = I^1_{a+}[\mathcal{W}] = \int_a^x \mathcal{W}(t)\,dt\,.$$

Hence, by setting $\mathcal{V} = \mathcal{W} + I^{1-\alpha}_{a+}[v](a_+)\,\delta(x-a)$ and taking into account (99), we obtain

$$v(x) \;=\; w(x) + \frac{\mathfrak{K}}{\Gamma(\alpha)}(x-a)^{\alpha-1} \;=\; I^{\alpha}_{a+}[\mathcal{V}](x) + \frac{\mathfrak{K}}{\Gamma(\alpha)}(x-a)^{\alpha-1}\,. \tag{134}$$

$$D^{\alpha}_{a+}[v] = D^{\alpha}_{a+}[w] + D^{\alpha}_{a+}\left[\frac{\mathfrak{K}}{\Gamma(\alpha)}(x-a)^{\alpha-1}\right] = D_x I^{1-\alpha}_{a+}[v] \;= \mathcal{V}(x) = D^{\alpha}_{a+}[w]. \tag{135}$$

Thus $\mathcal{V} \in L^1(a,b), I^{1-\alpha}_{a+}[\mathcal{V}](a_+) = 0, I^{1-\alpha}_{a+}[v](a_+) = 0, I^{1-\alpha}_{a+}[v - I^{\alpha}_{a+}[\mathcal{V}]](a_+) \equiv 0$; then $v = I^{\alpha}_{a+}[\mathcal{V}]$, $v \in I^{\alpha}_{a+}\big(L^1(a,b)\big)$. The function $(x-a)^{\alpha-1}$ belongs to $L^q(a,b)$ for every $q \in \big[1, 1/(1-\alpha)\big)$, due to the boundedness of the interval. By standard embedding of fractional integrals, the function $w = I^{\alpha}_{a+}(\mathcal{V})$ belongs to $L^q(a,b)$ for every $q \in \big[1, 1/(1-\alpha)\big)$.

Summarizing, $v \in L^q(a,b)$ for every $q \in \big[1, 1/(1-\alpha)\big)$ and (129) and (130) hold true. □

Remark 17. *We emphasize that in Theorem 12 the fractional integrals and derivatives $I^{1-\alpha}_{a+}$, $I^{1-\alpha}_{b-}$, D^{α}_{a+} and D^{α}_{b-} are understood in the distributional sense provided by Definitions 1 and 3. Referring to Definition 10, with (a,b) bounded, (131) reads as follows*

$$v(x) = I^{\alpha}_{a+}[D^{\alpha}_{a+}[v]](x) + \frac{I^{1-\alpha}_{a+}[v](a_+)}{\Gamma(\alpha)}(x-a)^{\alpha-1} \quad \text{a.e. } x \in (a,b), \forall v \in BV^s_+(a,b), \alpha \in (0,1). \tag{136}$$

Moreover, in a bounded interval (a,b) we have

$$v = D^{\alpha}_{a+}[I^{\alpha}_{a+}[v]] \quad \text{a.e. on } \in (a,b),\ \forall v \in L^1(a,b),\ \forall \alpha \in (0,1), \tag{137}$$

since $D^{\alpha}_{a+}[I^{\alpha}_{a+}[v]] = D_x[I^{1-\alpha}_{a+}[I^{\alpha}_{a+}[v]]] = D_x[[I^1_{a+}[v]] = D_x[\int_a^x v] = v$; whereas

$$v = D^{\alpha}_{a+}[I^{\alpha}_{a+}[v]] + C\,\delta(x-a) \quad \text{on } \mathcal{D}'(\mathbb{R}),\ \forall v \in \mathcal{M}(\mathbb{R}), \operatorname{spt} v \subset [a,b],\ \forall \alpha \in (0,1), \tag{138}$$

where $C = \lim_{x\to a_+} I^1[v](x)$, indeed by Lemma 8 the kernel of D^{α}_{a+} is made by functions of the kind $K(x-a)^{\alpha-1}$, which all belong to $BV^s_+(a,b)$ and fulfill on $\mathbb{R}$

$$C\,\Gamma(\alpha)H(x-a)(x-a)^{\alpha-1} = C\,\Gamma(\alpha)\,\delta(x-a) * \frac{1}{\Gamma(\alpha)}\frac{H(x)}{x^{\alpha-1}} = I^{\alpha}_{a+}[C\,\Gamma(\alpha)\,\delta(x-a)]\,.$$

6. Conclusions

We establish some properties of the bilateral Riemann–Liouville fractional derivative D^s.

We set the notation and study the associated Sobolev spaces of fractional order s, denoted by $W^{s,1}(a,b)$, and the fractional bounded variation spaces of fractional order s, denoted by $BV^s(a,b)$. The basic properties of these spaces are proved: weak compactness properties, and comparison embeddings and strict embeddings with several related spaces, namely,

$$BV(a,b) \underset{\neq}{\subset} \bigcap_{\sigma\in(0,1)} W^{\sigma,1}(a,b) \underset{\neq}{\subset} W^{s,1}(a,b) \underset{\neq}{\subset} BV^s_+(a,b) \qquad \forall s \in (0,1),$$

$$W^{s,1}(a,b) \underset{\neq}{\subset} BV^s_+(a,b)\,, \qquad W^{s,1}(a,b) \underset{\neq}{\subset} BV^s_-(a,b)\,, \qquad \forall s \in (0,1).$$

Spaces $W^{s,1}$ and BV^s are the natural setting for data of Abel integral equations in order to make them well-posed problems in the distributional framework too.

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