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The Algebraic Surfaces of the Enneper Family of Maximal Surfaces in Three Dimensional Minkowski Space

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Abstract: We consider the Enneper family of real maximal surfaces via Weierstrass data $(1, \zeta^m)$ for $\zeta \in \mathbb{C}$, $m \in \mathbb{Z}_{\geq 1}$. We obtain the irreducible surfaces of the family in the three dimensional Minkowski space $\mathbb{E}^{2,1}$. Moreover, we propose that the family has degree $(2m+1)^2$ (resp., class $2m(2m+1)$) in the cartesian coordinates x, y, z (resp., in the inhomogeneous tangential coordinates a, b, c).

Keywords: Weierstrass representation; Enneper maximal surface; algebraic surface; degree; class

MSC: primary 53A35; secondary 53C42, 65D18

1. Introduction

A minimal surface is a surface of vanishing mean curvature in three dimensional Euclidean space $\mathbb{E}^3$. There are many classical and modern minimal surfaces in the literature. See Darboux [1,2], Dierkes [3], Fomenko and Tuzhilin [4], Gray, Salamon, and Abbena [5], Nitsche [6], Osserman [7], Spivak [8] for some books, Lie [9], Schwarz [10], Small [11,12], and Weierstrass [13,14] for some papers related to minimal surfaces in Euclidean geometry.

Lie [9] studied the algebraic minimal surfaces and gave a table classifying these surfaces. See also Enneper [15], Güler [16], Nitsche [6], and Ribaucour [17] for details.

Weierstrass [13] revealed a representation for minimal surfaces in three dimensional Euclidean space $\mathbb{E}^3$. Almost one hundred years later, Kobayashi [18] gave an analogous Weierstrass-type representation for conformal spacelike surfaces with mean curvature identically 0, called maximal surfaces, in three dimensional Minkowski space $\mathbb{E}^{2,1}$.

In this paper, we consider the Enneper family of maximal surfaces $\mathcal{E}_m$ for positive integers $m \geq 1$ by using Weierstrass data $(1, \zeta^m)$ for $\zeta \in \mathbb{C}$, and then show that these surfaces are algebraic in $\mathbb{E}^{2,1}$. See Güler [16] for a Euclidean case of Enneper's algebraic minimal surfaces family.

In Section 2, we give this family of real maximal surfaces in (r, θ) and (u, v) coordinates by using Weierstrass representation in $\mathbb{E}^{2,1}$. In Section 3, we find irreducible algebraic equations defining surfaces $\mathcal{E}_m(u, v)$ in terms of running coordinates x, y, z, and a, b, c, and also compute degrees and classes of $\mathcal{E}_m(u, v)$. Finally, we summarize all findings in tables in the last section, then give some open problems.

2. Family of Enneper Maximal Surfaces

Let $\mathbb{E}^{n,1} := \left(\{x = (x_1, \cdots, x_n, x_0)^t | x_i \in \mathbb{R}\}, \langle \cdot, \cdot \rangle\right)$ be the $(n+1)$-dimensional Lorentz–Minkowski (for short, Minkowski) space with Lorentzian metric $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n - x_0 y_0$.

A vector $x \in \mathbb{E}^{n,1}$ is called space-like if $\langle x, x \rangle > 0$, time-like if $\langle x, x \rangle < 0$, and light-like if $x \neq 0$ and $\langle x, x \rangle = 0$. A surface in $\mathbb{E}^{n,1}$ is called space-like (resp. time-like, light-like) if the induced metric on the tangent planes is a Riemannian (resp. Lorentzian, degenerate) metric.

Now, let $\mathbb{E}^{2,1}$ be three dimensional Minkowski space with Lorentzian metric $\langle . , . \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$. We identify $\overrightarrow{x}$ and $\overrightarrow{x^t}$ without further comment.

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Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A *maximal curve* is an analytic function $\vartheta : \mathcal{U} \to \mathbb{C}^n$ such that $\langle \vartheta'(\zeta), \vartheta'(\zeta) \rangle = 0$, where $\zeta \in \mathcal{U}$, and $\vartheta' := \frac{\partial \vartheta}{\partial \zeta}$. In addition, if $\left\langle \vartheta', \overline{\vartheta'} \right\rangle = |\vartheta'|^2 \neq 0$, then ϑ is a regular maximal curve. We then have maximal surfaces in the associated family of a maximal curve, given by the following Weierstrass representation theorem for ZMC (zero mean curvature) surfaces, or maximal surfaces.

Kobayashi [18] found a Weierstrass type representation for space-like conformal maximal surfaces in $\mathbb{E}^{2,1}$:

Theorem 1. *Let $g(\omega)$ be a meromorphic function and let $f(\omega)$ be a holomorphic function, fg^2 is analytic, defined on a simply connected open subset $U \subset C$ such that $f(\omega)$ does not vanish on U except at the poles of $g(\omega)$. Then,*

$$\mathbf{x}(u,v) = Re \int^{\zeta} \left(f\left(1+g^2\right), if\left(1-g^2\right), -2fg \right) d\omega, \quad (\zeta = u + iv) \tag{1}$$

is a space-like conformal immersion with mean curvature identically 0 *(i.e., space-like conformal maximal surface). Conversely, any spacelike conformal maximal surface can be described in this manner.*

Next, we give some facts about Weierstrass data, and a maximal curve to construct some maximal surfaces.

Definition 1. *A pair of a meromorphic function g and a holomorphic function f, (f,g) is called Weierstrass data for a maximal surface.*

Lemma 1. *The curve of Enneper of order m:*

$$\varepsilon_m(\zeta) = \left(\zeta + \frac{\zeta^{2m+1}}{2m+1},\ i\left(\zeta - \frac{\zeta^{2m+1}}{2m+1}\right),\ -\frac{2\zeta^{m+1}}{m+1} \right) \tag{2}$$

is a maximal curve, $\zeta \in \mathbb{C} - \{0\}$, $i = \sqrt{-1}$, $m \neq -1, -1/2$.

Therefore, we have $\langle \varepsilon_m, \varepsilon_m \rangle = 0$ by using (2). Hence, in $\mathbb{E}^{2,1}$, the Enneper maximal surface is given by

$$\mathcal{E}_m(u,v) = Re \int \varepsilon_m(\zeta) d\zeta, \tag{3}$$

where $\zeta = u + iv$. $Im \int \varepsilon_m(\zeta) d\zeta$ gives the adjoint minimal surface $\mathcal{E}^*_m(u,v)$ of the surface $\mathcal{E}_m(u,v)$ in (3). Then, we get the following:

Corollary 1. *The Weierstrass data $(1, \zeta^m)$ of (3) is a representation of the Enneper maximal surface, where integer $m \geq 1$.*

Considering the findings above with $\zeta = re^{i\theta}$, we get the following Enneper family of maximal surfaces:

$$\mathcal{E}_m(r,\theta) = \begin{pmatrix} r\cos(\theta) + \frac{1}{2m+1} r^{2m+1} \cos[(2m+1)\theta] \\ -r\sin(\theta) + \frac{1}{2m+1} r^{2m+1} \sin[(2m+1)\theta] \\ -\frac{2}{m+1} r^{m+1} \cos[(m+1)\theta] \end{pmatrix} \tag{4}$$

where $m \neq -1, -1/2$. See Figure 1 for Enneper maximal surfaces $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ in (r,θ) coordinates.

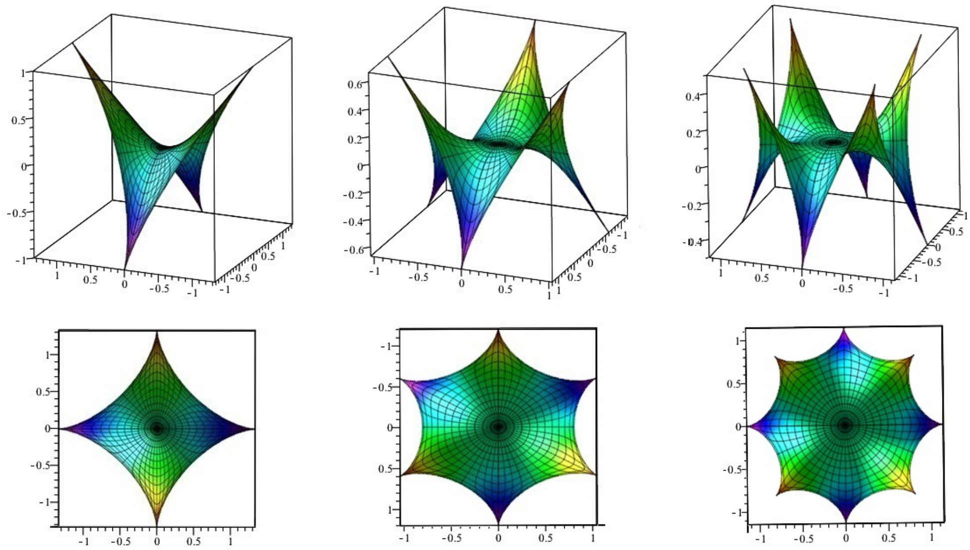

Figure 1. Enneper maximal surfaces, and its top views (**Left**): $\mathcal{E}_1(r,\theta)$, (**Middle**): $\mathcal{E}_2(r,\theta)$, (**Right**): $\mathcal{E}_3(r,\theta)$.

Hence, using the binomial formula, we obtain more clear representation of $\mathcal{E}_m(u,v)$ in (3):

$$\begin{aligned} x(u,v) &= Re\left\{u+iv+\tfrac{1}{2m+1}\sum_{k=0}^{2m+1}\binom{2m+1}{k}u^{2m+1-k}(iv)^k\right\}, \\ y(u,v) &= Re\left\{iu-v+\tfrac{i}{2m+1}\sum_{k=0}^{2m+1}\binom{2m+1}{k}u^{2m+1-k}(iv)^k\right\}, \\ z(u,v) &= Re\left\{-\tfrac{2}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}u^{m+1-k}(iv)^k\right\}. \end{aligned} \tag{5}$$

We study surface $\mathcal{E}_m(u,v)$ in (u,v) coordinates for $m=1,2,\dots,5$, taking $\zeta=u+iv$ at Cartesian coordinates x,y,z, and also in inhomogeneous tangential coordinates a,b,c, by using Weierstrass representation equation.

Next, we give a theorem about maximality of surface $\mathcal{E}_1(u,v)$ (see Figure 1, Left):

Theorem 2. *The surface*

$$\mathcal{E}_1(u,v)=\begin{pmatrix} \frac{1}{3}u^3-uv^2+u \\ -\frac{1}{3}v^3+u^2v-v \\ -u^2+v^2 \end{pmatrix}=\begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v)\end{pmatrix} \tag{6}$$

which has Weierstrass data $(1,\zeta)$*, is an Enneper maximal surface in* $\mathbb{E}^{2,1}$.

Proof. The coefficients of the first fundamental form of the surface $\mathcal{E}_1(u,v)$ ($\mathcal{E}_1$, for short) are given by

$$\begin{aligned} E &= (\lambda-1)^2=G, \\ F &= 0, \end{aligned}$$

where $\lambda=u^2+v^2$. That is, conformality holds. Then, the Gauss map $e_1(u,v)$ of $\mathcal{E}_1$ is as follows

$$e_1=\left(-\frac{2u}{\lambda-1},-\frac{2v}{\lambda-1},\frac{\lambda^2+1}{\lambda-1}\right), \tag{7}$$

where $\lambda \neq 1$. The coefficients of the second fundamental form of $\mathcal{E}_1$ are given by

$$\begin{aligned} L &= -\frac{2(3\lambda+1)}{\lambda-1} = -N, \\ M &= 0. \end{aligned}$$

Then, we obtain the following mean curvature and the Gaussian curvature of the surface $\mathcal{E}_1$:

$$\begin{aligned} H &= 0, \\ K &= \frac{4(3\lambda+1)^2}{(\lambda-1)^6}, \end{aligned}$$

respectively. Here, $H = \langle\sigma,\sigma\rangle \frac{EN+GL-2FM}{2(EG-F^2)}$, $K = \langle\sigma,\sigma\rangle \frac{LN-M}{EG-F^2}$, where $\langle\sigma,\sigma\rangle = -1$. Hence, the Enneper surface is maximal surface with positive Gaussian curvature. □

Therefore, we obtain the following parametric equations of the higher order maximal Enneper surfaces $\mathcal{E}_m(u,v) = (x(u,v), y(u,v), z(u,v))$ (see Figure 2 Middle for $\mathcal{E}_2$, and Figure 2 Right for $\mathcal{E}_3$.):

$$\mathcal{E}_2(u,v) = \begin{pmatrix} \frac{1}{5}u^5 - 2u^3v^2 + uv^4 + u \\ \frac{1}{5}v^5 - 2u^2v^3 + u^4v - v \\ -\frac{2}{3}u^3 + 2uv^2 \end{pmatrix}, \tag{8}$$

$$\mathcal{E}_3(u,v) = \begin{pmatrix} \frac{1}{7}u^7 - 3u^5v^2 + 5u^3v^4 - uv^6 + u \\ -\frac{1}{7}v^7 + 3u^2v^5 - 5u^4v^3 + u^6v - v \\ -\frac{1}{2}u^4 + 3u^2v^2 - \frac{1}{2}v^4 \end{pmatrix}, \tag{9}$$

$$\mathcal{E}_4(u,v) = \begin{pmatrix} \frac{1}{9}u^9 - 4u^7v^2 + 14u^5v^4 - \frac{23}{3}u^3v^6 + uv^8 + u \\ \frac{1}{9}v^9 - 4u^2v^7 + 14u^4v^5 - \frac{23}{3}u^6v^3 + u^8v - v \\ -\frac{2}{5}u^5 + 4u^3v^2 - 2uv^4 \end{pmatrix}, \tag{10}$$

$$\mathcal{E}_5(u,v) = \begin{pmatrix} \frac{1}{11}u^{11} - 5u^9v^2 + 30u^7v^4 - 42u^5v^6 + 15u^3v^8 + uv^{10} + u \\ -\frac{1}{11}v^{11} + 5u^2v^9 - 30u^4v^7 + 42u^6v^5 - 15u^8v^3 + u^{10}v - v \\ -\frac{1}{3}u^6 + 5u^4v^2 - 5u^2v^4 + \frac{1}{3}v^6 \end{pmatrix}. \tag{11}$$

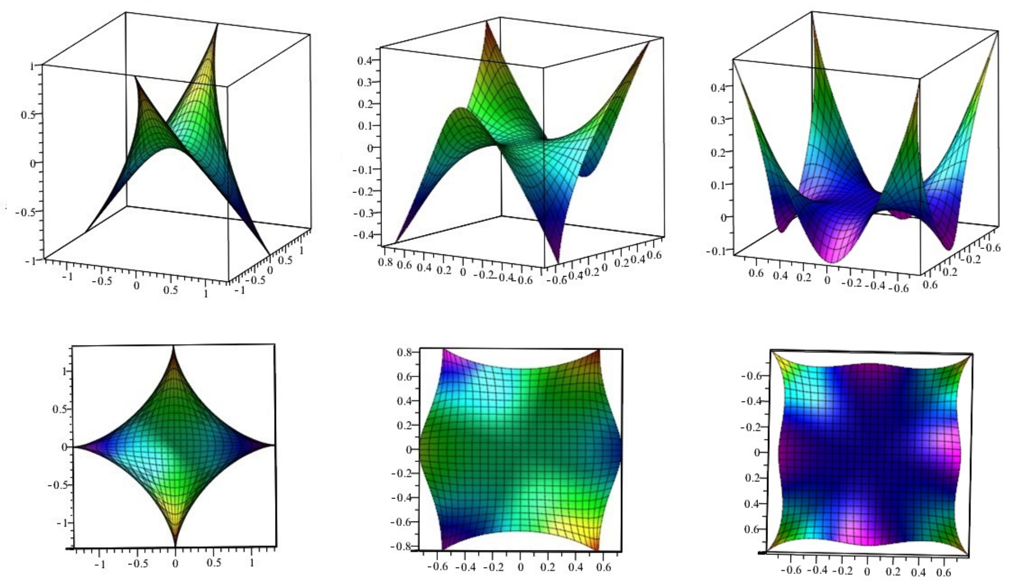

Figure 2. Enneper maximal surfaces (**Left**): $\mathcal{E}_1(u,v)$, (**Middle**): $\mathcal{E}_2(u,v)$, (**Right**): $\mathcal{E}_3(u,v)$.

3. Degree and Class of Enneper Maximal Surfaces

In this section, using some elimination techniques, we derive the irreducible algebraic surface equation, degree and class of Enneper maximal surfaces family $\mathcal{E}_m(u,v)$ for integers $1 \leq m \leq 5$ in three dimensional Minkowski space $\mathbb{E}^{2,1}$.

Let us see some basic notions of the surfaces.

Definition 2. *The set of roots of a polynomial $Q(x,y,z) = 0$ gives an algebraic surface equation. An algebraic surface $\mathbf{s}$ is said to be of degree $\mathbf{d}$ when $\mathbf{d} = \deg(\mathbf{s})$.*

Definition 3. *At a point (u,v) on a surface $\mathbf{s}(u,v) = (x(u,v), y(u,v), z(u,v))$, the tangent plane is given by*

$$Xx + Yy - Zz + P = 0, \tag{12}$$

where $e = (X(u,v), Y(u,v), Z(u,v))$ is the Gauss map, and $P = P(u,v)$. Then, in inhomogeneous tangential coordinates a, b, c, we have the following surface:

$$\widehat{\mathbf{s}}(u,v) = (a,b,c) = (X/P,\ Y/P,\ Z/P). \tag{13}$$

Therefore, we can obtain an algebraic equation $\hat{Q}(a,b,c) = 0$ of $\widehat{\mathbf{s}}(u,v)$ in inhomogeneous tangential coordinates.

Definition 4. *The maximum degree of the algebraic equation $\hat{Q}(a,b,c) = 0$ of $\widehat{\mathbf{s}}(u,v)$ in inhomogeneous tangential coordinates gives the class of $\widehat{\mathbf{s}}(u,v)$.*

See [6], for details of a Euclidean case. Hence, we obtain the following findings for degrees and classes of Enneper maximal surfaces that we use:

3.1. Degree

We compute the irreducible algebraic surface equation $Q_1(x,y,z) = 0$ (see Figure 3, Left) of Enneper's maximal surface $\mathcal{E}_1(u,v)$ in (6) by using some elimination techniques. We find the following algebraic equation:

$$\begin{aligned} Q_1(x,y,z) \quad = \quad & 64z^9 + 432x^2z^6 - 432y^2z^6 - 1215x^4z^3 - 6318x^2y^2z^3 + 3888x^2z^5 \\ & -1215y^4z^3 + 3888y^2z^5 - 1152z^7 + 729x^6 - 2187x^4y^2 - 4374x^4z^2 \\ & +2187x^2y^4 + 6480x^2z^4 - 729y^6 + 4374y^4z^2 - 6480y^2z^4 + 729x^4z \\ & -1458x^2y^2z - 3888x^2z^3 + 729y^4z - 3888y^2z^3 + 5184z^5. \end{aligned} \tag{14}$$

Then, its degree number is 9.

Next, we continue our computations to find $Q_m(x,y,z) = 0$ for integers $m = 2,3$. We compute the following irreducible algebraic surface equations $Q_2(x,y,z) = 0$ (see Figure 3, Middle) and $Q_3(x,y,z) = 0$ (see Figure 3, Right) of the surfaces $\mathcal{E}_2(u,v)$ and $\mathcal{E}_3(u,v)$, respectively,

$$\begin{aligned} Q_2(x,y,z) \quad = \quad & 847\,288\,609\,443z^{25} - 4358\,480\,501\,250x^3z^{20} \\ & +13\,075\,441\,503\,750xy^2z^{20} - 131\,157\,978\,046\,875x^6z^{15} \\ & -474\,186\,536\,015\,625x^4y^2z^{15} + 107 \text{ other lower degree terms,} \end{aligned}$$

$$Q_3(x,y,z) = 2475\,880\,078\,570\,760\,549\,798\,248\,448z^{49}$$
$$+5079\,604\,062\,565\,768\,134\,821\,675\,008x^4z^{42}$$
$$-30\,477\,624\,375\,394\,608\,808\,930\,050\,048x^2y^2z^{42}$$
$$+5079\,604\,062\,565\,768\,134\,821\,675\,008y^4z^{42}$$
$$-633\,850\,350\,654\,216\,217\,766\,624\,493\,568x^8z^{35}$$
$$+446 \text{ other lower degree terms.}$$

Therefore, $Q_m(x,y,z) = 0$ are the algebraic maximal surfaces of the surfaces $\mathcal{E}_m(u,v)$, where $m = 2,3$, and they have degree numbers 25 and 49, respectively.

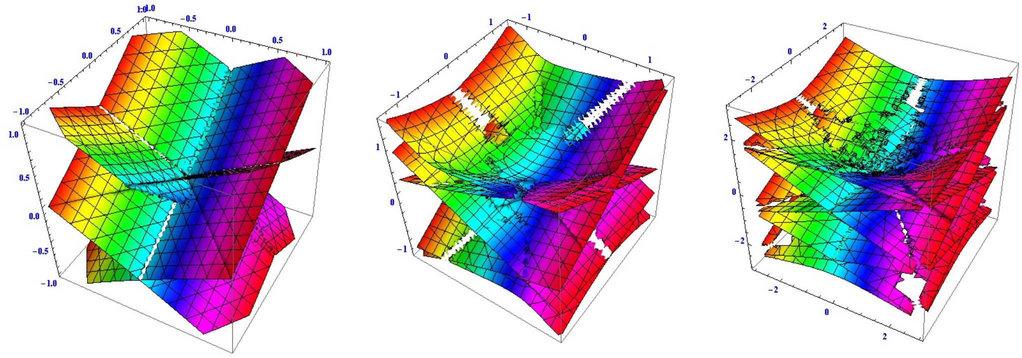

Figure 3. Enneper algebraic maximal surfaces (**Left**): $Q_1(x,y,z) = 0$, (**Middle**): $Q_2(x,y,z) = 0$, (**Right**): $Q_3(x,y,z) = 0$.

3.2. Class

Now, we introduce the class of the surfaces $\mathcal{E}_m(u,v)$ for integers $1 \le m \le 4$. The case $m = 5$, marked with "*" presented in tables of Section 4. Computing the irreducible algebraic surface equations $\hat{Q}_m(a,b,c) = 0$, we obtain the Gauss maps $e_m(u,v)$ (see Figure 4 for e_1, e_2, e_3) for integers $1 \le m \le 5$ of the surfaces $\mathcal{E}_m(u,v)$, and we also generalize them as follows:

$$e_1 = \left(-2\frac{u}{\lambda-1}, -2\frac{v}{\lambda-1}, \frac{\lambda+1}{\lambda-1}\right),$$

$$e_2 = \left(-2\frac{u^2-v^2}{\lambda^2-1}, -2\frac{2uv}{\lambda^2-1}, \frac{\lambda^2+1}{\lambda^2-1}\right), \tag{15}$$

$$e_3 = \left(-2\frac{u^3-3uv^2}{\lambda^3-1}, -2\frac{3u^2v-v^3}{\lambda^3-1}, \frac{\lambda^3+1}{\lambda^3-1}\right), \tag{16}$$

$$e_4 = \left(-2\frac{u^4-6u^2v^2+v^4}{\lambda^4-1}, -2\frac{4u^3v-4uv^3}{\lambda^4-1}, \frac{\lambda^4+1}{\lambda^4-1}\right), \tag{17}$$

$$e_5 = \left(-2\frac{u^5-10u^3v^2+5uv^4}{\lambda^5-1}, -2\frac{5u^4v-10u^2v^3+v^5}{\lambda^5-1}, \frac{\lambda^5+1}{\lambda^5-1}\right), \tag{18}$$

$$\vdots$$

$$e_m = \left(-2\frac{Re(\zeta^m)}{|\zeta|^m-1}, -2\frac{Im(\zeta^m)}{|\zeta|^m-1}, \frac{|\zeta|^m+1}{|\zeta|^m-1}\right), \ (\zeta = u+iv, \ |\zeta| = \lambda). \tag{19}$$

Using (6), (7), (12) and (13), with $P_1(u,v) = \frac{(\lambda-3)(-u^2+v^2)}{3(\lambda-1)}$, we get the following surface $\widehat{\mathcal{E}}_1(u,v)$ (see Figure 5, Left) in inhomogeneous tangential coordinates:

$$a = \frac{6u}{(-u^2+v^2)(\lambda-3)}, \ b = \frac{6v}{(-u^2+v^2)(\lambda-3)}, \ c = -\frac{3(\lambda+1)}{(-u^2+v^2)(\lambda-3)}.$$

where $\lambda = u^2 + v^2$, $\lambda \neq 3$, $u, v \neq 0$. Therefore, we compute Enneper's irreducible algebraic maximal surface equation $\hat{Q}_1(a,b,c) = 0$ (see Figure 6, Left) of the surface $\widehat{\mathcal{E}_1}(u,v)$:

$$\hat{Q}_1(a,b,c) = 4a^6 - 4a^4b^2 - 3a^4c^2 - 4a^2b^4 + 6a^2b^2c^2 + 4b^6 - 3b^4c^2 \\ -18a^4c + 12a^2c^3 + 18b^4c - 12b^2c^3 + 9a^4 + 18a^2b^2 + 9b^4.$$

So, Enneper's maximal surface $\mathcal{E}_1(u,v)$ in (6) has class number 6.

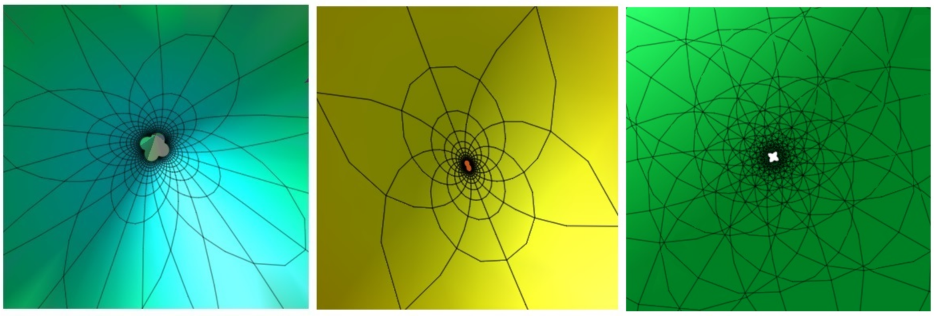

Figure 4. Top views of the Gauss maps of the surfaces $\mathcal{E}_{m=1,2,3}(u,v)$ (**Left**): $e_1(u,v)$, (**Middle**): $e_2(u,v)$, (**Right**): $e_3(u,v)$.

Next, we continue our computations to find $\hat{Q}_m$ for integers $2,3,4$. To find the class of surface $\mathcal{E}_2(u,v)$ (see Figure 5, Middle), we use (9), (12), (13) and (16). Calculating $P_2(u,v) = -\frac{4(u^3-3uv^2)(\lambda^2-5)}{15(\lambda^2-1)}$, we get the following surface $\widehat{\mathcal{E}_2}$ inhomogeneous tangential coordinates:

$$a = \frac{15(u^2-v^2)}{2(u^3-3uv^2)(\lambda^2-5)}, \; b = \frac{15uv}{(u^3-3uv^2)(\lambda^2-5)}, \; c = -\frac{15(\lambda^2+1)}{4(u^3-3uv^2)(\lambda^2-5)},$$

where $\lambda = u^2 + v^2$, $\lambda^2 \neq 5$, $u, v \neq 0$. In the inhomogeneous tangential coordinates a, b, c, we find the following irreducible algebraic surface equation $\hat{Q}_2(a,b,c) = 0$ (see Figure 6, Middle) of the surface $\widehat{\mathcal{E}_2}(u,v)$:

$$\hat{Q}_2(a,b,c) = 2176\,782\,336a^{16}b^4 + 5804\,752\,896a^{14}b^6 - 4837\,294\,080a^{14}b^4c^2 \\ +2902\,376\,448a^{12}b^8 - 8062\,156\,800a^{12}b^6c^2 + 120 \text{ other lower degree terms.}$$

Hence, $\hat{Q}_2(a,b,c) = 0$ is the algebraic surface of the surface $\widehat{\mathcal{E}_2}(u,v)$, and Enneper's maximal surface $\mathcal{E}_2(u,v)$ in (8) has class number 20.

Using similar ways, we compute the irreducible algebraic surface equation $\hat{Q}_3(a,b,c) = 0$ (see Figure 6, Right) of surface $\widehat{\mathcal{E}_3}(u,v)$ (see Figure 5, Right) as follows:

$$\hat{Q}_3(a,b,c) = 26\,623\,333\,280\,885\,243\,904a^{42} - 718\,829\,998\,583\,901\,585\,408a^{40}b^2 \\ -104\,829\,374\,793\,485\,647\,872a^{40}c^2 + 6868\,819\,986\,468\,392\,927\,232a^{38}b^4 \\ +2935\,222\,494\,217\,598\,140\,416a^{38}b^2c^2 + 774 \text{ other lower degree terms.}$$

$\hat{Q}_3(a,b,c) = 0$ is the algebraic surface of the surface $\widehat{\mathcal{E}}_3(u,v)$, and Enneper's maximal surface $\mathcal{E}_3(u,v)$ in (9) has class number 42. We also compute the following irreducible algebraic surface equation $\hat{Q}_4(a,b,c) = 0$ of the surface $\widehat{\mathcal{E}}_4(u,v)$:

$$\begin{aligned} \hat{Q}_4(a,b,c) &= 4294967296000000000000000000000000000000000000000000000a^{64}b^{8} \\ &\quad -24739011624960000000000000000000000000000000000000000000a^{62}b^{8}c^{2} \\ &\quad -96207267430400000000000000000000000000000000000000000000a^{60}b^{12} \\ &\quad +24739011624960000000000000000000000000000000000000000000a^{60}b^{10}c^{2} \\ &\quad +66795331387392000000000000000000000000000000000000000000a^{60}b^{8}c^{4} \\ &\quad +2604 \text{ other lower degree terms.} \end{aligned}$$

$\hat{Q}_4(a,b,c) = 0$ is an algebraic surface of $\widehat{\mathcal{E}}_4(u,v)$, and Enneper's maximal surface $\mathcal{E}_4(u,v)$ in (10) has class number 72.

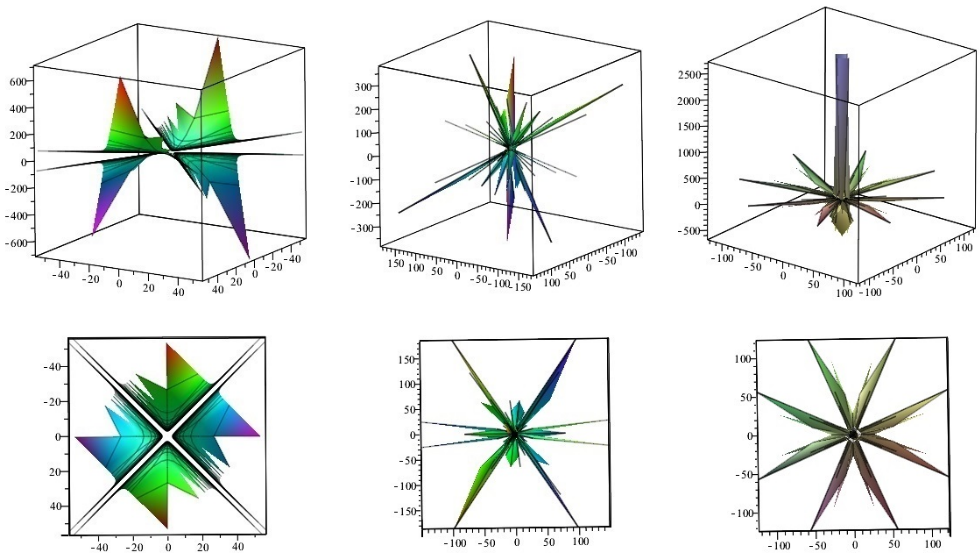

Figure 5. Enneper maximal surfaces in inhomogeneous tangential coordinates (**Left**): $\widehat{\mathcal{E}}_1(u,v)$, (**Middle**): $\widehat{\mathcal{E}}_2(u,v)$, (**Right**): $\widehat{\mathcal{E}}_3(u,v)$.

We obtain the following functions $P_i(u,v)$, where $1 \leq i \leq 6$,

$$P_1(u,v) = -\frac{(u^2-v^2)(\lambda-3)}{3(\lambda-1)}, \qquad P_2(u,v) = -\frac{4(u^3-3uv^2)(\lambda^2-5)}{15(\lambda^2-1)},$$

$$P_3(u,v) = -\frac{3(u^4-6u^2v^2+v^4)(\lambda^3-7)}{14(\lambda^3-1)}, \qquad P_4(u,v) = -\frac{8(u^5-10u^3v^2+5uv^4)(\lambda^4-9)}{45(\lambda^4-1)},$$

$$P_5(u,v) = -\frac{5(u^6-15u^4v^2+15u^2v^4-v^6)(\lambda^5-11)}{33(\lambda^5-1)}, \qquad P_6(u,v) = -\frac{12(u^7-21u^5v^2+35u^3v^4-7uv^6)(\lambda^6-13)}{91(\lambda^6-1)}.$$

We generalize the above functions, and give the following results:

Corollary 2. *The functions $P_{m\geq 1}$ for integers m, are given by*

$$\begin{aligned} P_{2k-1} &= -\frac{2(2k-1)\left[\lambda^{2k-1}-(2k+1)\right]}{(k+1)(2k+1)\left(\lambda^{2k-1}-1\right)} Re\left(\zeta^{2k}\right), \\ P_{2k} &= -\frac{4k\left[\lambda^{2k}-(2k+1)\right]}{(2k+1)(4k+1)\left(\lambda^{2k}-1\right)} Re\left(\zeta^{2k+1}\right), \end{aligned}$$

where integers $k \geq 1$, $\zeta = u + iv$ *and* $|\zeta| = \lambda$.

So far, we find surfaces $\widehat{\mathcal{E}}_1$ and $\widehat{\mathcal{E}}_2$. By using $\mathcal{E}_3 - \mathcal{E}_5$, $e_3 - e_5$, and also (12), (13), we obtain the following surfaces: $\widehat{\mathcal{E}}_m(u,v) = (a,b,c)$:

$$\begin{aligned}
\widehat{\mathcal{E}}_1 &= -\frac{3}{(u^2-v^2)(\lambda-3)}\begin{pmatrix} -2u \\ -2v \\ \lambda+1 \end{pmatrix}, \\
\widehat{\mathcal{E}}_2 &= -\frac{15}{4(u^3-3uv^2)(\lambda^2-5)}\begin{pmatrix} -2(u^2-v^2) \\ -4uv \\ \lambda^2+1 \end{pmatrix}, \\
\widehat{\mathcal{E}}_3 &= -\frac{14}{3(u^4-6u^2v^2+v^4)(\lambda^3-7)}\begin{pmatrix} -2(u^3-3uv^2) \\ -2(u^2v-v^3) \\ \lambda^3+1 \end{pmatrix}, \\
\widehat{\mathcal{E}}_4 &= -\frac{45}{8(u^5-10u^3v^2+5uv^4)(\lambda^4-9)}\begin{pmatrix} -2(u^4-6u^2v^2+v^4) \\ -2(4u^3v-4uv^3) \\ \lambda^4+1 \end{pmatrix}, \\
\widehat{\mathcal{E}}_5 &= -\frac{33}{5(u^6-15u^4v^2+15u^2v^4-v^6)(\lambda^5-11)}\begin{pmatrix} -2(u^5-10u^3v^2+5uv^4) \\ -2(5u^4v-10u^2v^3+v^5) \\ \lambda^5+1 \end{pmatrix}.
\end{aligned}$$

We also generalize the above functions, and find the following results:

Corollary 3. *The surfaces* $\widehat{\mathfrak{S}}_{m\geq 1}(u,v)$ *for integers* m, *are given by*

$$\begin{aligned}
\widehat{\mathcal{E}}_{2k-1}(u,v) &= -\frac{k(4k-1)}{(2k-1)\left[\lambda^{2k-1}-(2k+1)\right]Re\left(\zeta^{2k}\right)}\begin{pmatrix} -2Re\left(\zeta^{2k-1}\right) \\ -2Im\left(\zeta^{2k-1}\right) \\ |\zeta|^{2k-1}+1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \\
\widehat{\mathcal{E}}_{2k}(u,v) &= -\frac{(2k+1)(4k+1)}{4k\left[\lambda^{2k}-(4k+1)\right]Re\left(\zeta^{2k+1}\right)}\begin{pmatrix} -2Re\left(\zeta^{2k}\right) \\ -2Im\left(\zeta^{2k}\right) \\ |\zeta|^{2k}+1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},
\end{aligned}$$

where integers $k \geq 1$, $\zeta = u + iv$ *and* $|\zeta| = \lambda$.

Corollary 4. *In* $\mathbb{E}^{2,1}$, *the relations between the Enneper maximal surface* $\widehat{\mathcal{E}}_{m\geq 1}(u,v)$ *in the inhomogeneous tangential coordinates and the Gauss map* $e_{m\geq 1}(u,v)$ *of the Enneper maximal surface* $\mathcal{E}_{m\geq 1}(u,v)$ *in the cartesian coordinates are given by*

$$\begin{aligned}
\widehat{\mathcal{E}}_{2k-1}(u,v) &= -\frac{k(4k-1)\left(\lambda^{2k-1}-1\right)}{(2k-1)\left[\lambda^{2k-1}-(2k+1)\right]Re\left(\zeta^{2k}\right)}\, e_{2k-1}(u,v), \\
\widehat{\mathcal{E}}_{2k}(u,v) &= -\frac{(2k+1)(4k+1)\left(\lambda^{2k}-1\right)}{4k\left[\lambda^{2k}-(2k+1)\right]Re\left(\zeta^{2k+1}\right)}\, e_{2k}(u,v),
\end{aligned}$$

where integers $k \geq 1$, $\zeta = u + iv$ *and* $|\zeta| = \lambda$.

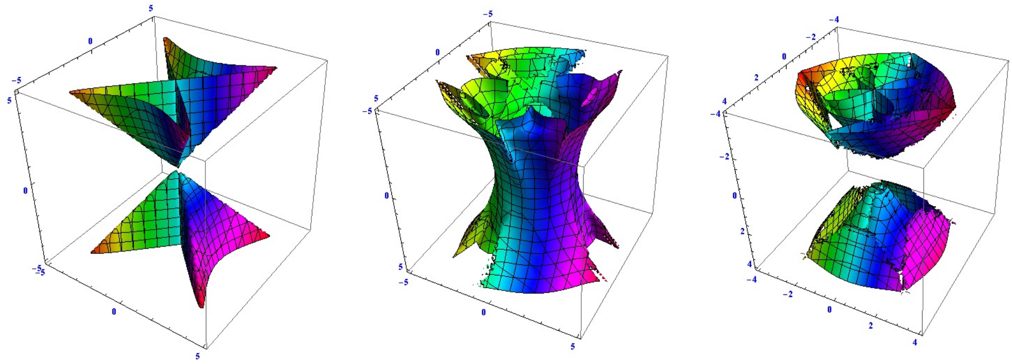

Figure 6. Enneper's algebraic maximal surfaces in inhomogeneous tangential coordinates (**Left**): $\hat{Q}_1(a, b, c) = 0$, (**Middle**): $\hat{Q}_2(a, b, c) = 0$, (**Right**): $\hat{Q}_3(a, b, c) = 0$.

4. Conclusions

To reveal the irreducible algebraic surface equations of the Enneper maximal surfaces $\mathcal{E}_m(u, v)$ in $\mathbb{E}^{2,1}$, we have tried a series of standard techniques in elimination theory: only Sylvester by hand for $Q_1(x, y, z) = 0$, and then projective (Macaulay) and sparse multivariate resultants implemented in the Maple software [19] package multires for $Q_m(x, y, z) = 0$ and $\hat{Q}_m(a, b, c) = 0$.

Maple's native implicitization command Implicitize, and implicitization based on Maple's native implementation of the Groebner Basis. For the latter, we implemented in Maple the method in [20] (Chapter 3, p. 128). Under reasonable time, we only succeed for $m = 1, 2$ in all above methods.

For $m = 3$, the successful method we have tried was to compute the equation defining the elimination ideal using the Groebner Basis package FGb of Faugère in [21].

The time required to output the irreducible algebraic surface equations $Q_m(x, y, z) = 0$ (resp. $\hat{Q}_m(a, b, c) = 0$) for integers $1 \leq m \leq 3$ and polynomials defining the elimination ideal was under reasonable seconds determined by the following Table 1 (resp. Table 2).

For the degree (resp. class) of the irreducible algebraic surface equation $Q_4(x, y, z) = 0$ (resp. $\hat{Q}_5(a, b, c) = 0$) of the surface $\mathcal{E}_4(u, v)$ (resp. $\widehat{\mathcal{E}}_5(u, v)$), marked with "$*$" in Table 1 (resp. Table 2), was rejected (i.e., "out of memory") by Maple 17 on a laptop Pentium Core i5-4310M 2.00 GHz, 4 GB RAM, with the time given in CPU seconds.

Hence, we propose the following:

Proposition 1. *For integers $m \geq 1$, degree number of the irreducible algebraic surfaces $Q_m(x, y, z) = 0$ in the Cartesian coordinates is of $(2m + 1)^2$, and class number of irreducible algebraic surfaces $\hat{Q}_m(a, b, c) = 0$ in inhomogeneous tangential coordinates is of $2m(2m + 1)$ of the $(1, \zeta^m)$-type real Enneper maximal surfaces $\mathcal{E}_m(u, v)$.*

Open Problems

Here, we give some problems that we could not find the answers in this paper:

Problem 1. *Find the irreducible Enneper algebraic maximal surface eq. $Q_{m\geq 4}(x, y, z) = 0$ in the cartesian coordinates by using the parametric equation of the Enneper maximal surface $\mathcal{E}_{m\geq 4}(u, v)$.*

Problem 2. *Find the irreducible Enneper algebraic maximal surface eq. $\hat{Q}_{m\geq 5}(a, b, c) = 0$ in the inhomogeneous tangential coordinates by using the parametric equation of the Enneper maximal surface $\widehat{\mathcal{E}}_{m\geq 5}(u, v)$.*

Finally, we give all findings in Tables 1 and 2.

Table 1. Results for the Enneper algebraic maximal surfaces $Q_m(x,y,z) = 0$.

Algebraic Surface	Degree of Surface	Number of Terms	Gröbner Time (s)	FGb Time (s)
Q_1	9	23	0.266	0.041
Q_2	25	112	321.953	0.835
Q_3	49	451	*	266.854
Q_4	81	*	*	*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Q_m	$(2m+1)^2$	*	*	*

Table 2. Results for the Enneper algebraic maximal surfaces $\hat{Q}_m(a,b,c) = 0$.

Algebraic Surface	Class of Surface	Number of Terms	Gröbner Time (s)	FGb Time (s)
$\hat{Q}_1$	6	14	0.94	0.030
$\hat{Q}_2$	20	125	61.152	0.114
$\hat{Q}_3$	42	779	*	125.904
$\hat{Q}_4$	72	2609	*	1306.718
$\hat{Q}_5$	110	*	*	*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\hat{Q}_m$	$2m(2m+1)$	*	*	*

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References

1. Darboux, G. *Leçons Sur la Théorie Générale des Surfaces, I, II, (French) [Lessons on the General Theory of Surfaces, I, II]*; Reprint of the Second (1914) Edition (I) and the Second (1915) Edition (II), Les Grands Classiques Gauthier-Villars [Gauthier-Villars Great Classics], Cours de Géométrie de la Faculté des Sciences [Course on Geometry of the Faculty of Science]; Éditions Jacques Gabay: Sceaux, France, 1993.
2. Darboux, G. *Leçons Sur la Théorie Générale des Surfaces, III, IV, (French) [Lessons on the General Theory of Surfaces, III, IV]*; Reprint of the 1894 Original (III) and the 1896 Original (IV), Les Grands Classiques Gauthier-Villars [Gauthier-Villars Great Classics], Cours de Géométrie de la Faculté des Sciences [Course on Geometry of the Faculty of Science]; Éditions Jacques Gabay: Sceaux, France, 1993.
3. Dierkes, U.; Hildebrandt, S. *Sauvigny F. Minimal Surfaces*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2010.
4. Fomenko, A.T.; Tuzhilin, A.A. *Elements of the Geometry and Topology of Minimal Surfaces in Three-Dimensional Space, Translated from the Russian by E.J.F. Primrose*; Translations of Mathematical Monographs, 93; American Math. Soc.: Providence, RI, USA, 1991.
5. Gray, A.; Salamon, S.; Abbena, E. *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 3rd ed.; Chapman & Hall/CRC Press: Boca Raton, FL, USA, 2006.
6. Nitsche, J.C.C. *Lectures on Minimal Surfaces. Volume 1. Introduction, Fundamentals, Geometry and Basic Boundary Value Problems*; Cambridge University Press: Cambridge, MA, USA, 1989.
7. Osserman, R. *A Survey of Minimal Surfaces*; Van Nostrand Reinhold Co.: New York, NY, USA, 1969.
8. Spivak, M. A *Comprehensive Introduction to Differential Geometry*, 3rd ed.; Publish or Perish, Inc.: Houston, TX, USA, 1999; Volume IV.
9. Lie, S. Beiträge zur Theorie der Minimalflächen. *Math. Ann.* **1878**, *14*, 331–416. [CrossRef]
10. Schwarz, H.A. Miscellen aus dem Gebiete der Minimalflachen. *J. für die reine und Angew. Math. (Crelle's J.)* **1875**, *80*, 280–300.
11. Small, A.J. Minimal surfaces in $\mathbb{R}^3$ and algebraic curves. *Differ. Geom. Appl.* **1992**, *2*, 369–384. [CrossRef]

12. Small, A.J. Linear structures on the collections of minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$. *Ann. Glob. Anal. Geom.* **1994**, *12*, 97–101. [CrossRef]
13. Weierstrass, K. *Untersuchungen über die Flächen, deren Mittlere Krümmung überall Gleich Null ist*; Akademie der Wissenschaften zu Berlin: Berlin, Germany, 1866; pp. 612–625.
14. Weierstrass, K. *Über die analytische Darstellbarkeit Sogenannter Willkürlicher Functionen einer reellen Veränderlichen*; Akademie der Wissenschaften zu Berlin: Berlin, Germany, 1885; pp. 633–639, 789–805.
15. Enneper, A. Untersuchungen über einige Punkte aus der allgemeinen Theorie der Flächen. *Math. Ann.* **1870**, *2*, 58–623. [CrossRef]
16. Güler, E. Family of Enneper minimal surfaces. *Mathematics* **2018**, *6*, 281. [CrossRef]
17. Ribaucour, A. Etude des elassoides ou surfaces a courbure moyenne nulle. *Mem. Cour. et Mem. Sav. Etr. Acad. R. Sci. Belg. Bruxelles* **1882**, *44*, 236.
18. Kobayashi, O. Maximal surfaces in the 3-dimensional Minkowski space L^3. *Tokyo J. Math.* **1983**, *6*, 297–309. [CrossRef]
19. *Maple Software, (Version 17)*; Waterloo Maple Inc.: Waterloo, ON, Canada. Available online: https://www.maplesoft.com/ (accessed on 10 October 2021).
20. Cox, D.; Little, J.; O'Shea, D. *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 3rd ed.; Undergraduate Texts in Mathematics; Springer: New York, NY, USA, 2007.
21. Faugère, J.C. FGb: A library for computing Gröbner bases. In Proceedings of the Third International Congress Conference on Mathematical Software (ICMS'10), Kobe, Japan, 13–17 September 2010; Springer: Berlin/Heidelberg, Germany, 2010; pp. 84–87.