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Some Identities on the Twisted q-Analogues of Catalan-Daehee Numbers and Polynomials

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Abstract: In this paper, the author considers twisted q-analogues of Catalan-Daehee numbers and polynomials by using p-adic q-integral on $\mathbb{Z}_p$. We derive some explicit identities for those twisted numbers and polynomials related to various special numbers and polynomials.

Keywords: q-analogue of Catalan-Daehee numbers; q-analogue of Catalan-Daehee polynomials; p-adic q-integral on $\mathbb{Z}_p$; twisted q-analogue of Catalan-Daehee numbers; twisted q-analogue of Catalan-Daehee polynomials

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ we denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normally defined $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-analogue of x is defined by $[x]_q = \dfrac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-integral on $\mathbb{Z}_p$ is defined by [1–3]

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x)d\mu_q(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \end{aligned} \quad (1)$$

From (1), we have

$$q\int_{\mathbb{Z}_p} f(x+1)d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) + (q-1)f(0) + \frac{q-1}{\log q} f'(0), \quad (2)$$

where $f'(0) = \frac{df(x)}{d}\big|_{x=0}$.

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \{w | w^{p^n} = 1\}$ is the cyclic group of order p^n.

For $w \in T_p$, let us take $f(x) = w^x e^{xt}$. Then, by (1), we get

$$\frac{(q-1) + \frac{q-1}{\log q}t}{wqe^t - 1} = \int_{\mathbb{Z}_p} w^x e^{xt} d\mu_q(x) \quad (3)$$

Thus, by (3), we define the twisted q-Bernoulli numbers which are given by the generating function to be

$$\frac{(q-1)+\frac{q-1}{\log q}t}{qwe^t-1}=\sum_{n=0}^{\infty}B_{n,q,w}\frac{t^n}{n!}. \tag{4}$$

From (4), we note that

$$qw(B_{q,w}+1)^n-B_{n,q,w}=\begin{cases} q-1, & \text{if } n=0\\ \dfrac{q-1}{\log q} & \text{if } n=1,\\ 0 & \text{if } n\geq 1,\end{cases}$$

with the usual convention about replacing $B_{q,w}^n$ by $B_{n,q,w}$.

From (2) and (4), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}w^x x^n d\mu_q(x)\frac{t^n}{n!}&=\int_{\mathbb{Z}_p}w^x e^{xt}d\mu_q(x)\\ &=\frac{(q-1)+\frac{q-1}{\log q}t}{wqe^t-1}=\sum_{n=0}^{\infty}B_{n,q,w}\frac{t^n}{n!}.\end{aligned} \tag{5}$$

Thus, by (5), we get

$$\int_{\mathbb{Z}_p}w^x x^n d\mu_q(x)=B_{n,q,w}, \quad (n\geq 0). \tag{6}$$

For $|t|_p<p^{-\frac{1}{p-1}}$, the twisted (λ,q)-Daehee polynomials are defined by generating function to be (cf. [4])

$$\sum_{n=0}^{\infty}D_{n,q,w}(x|\lambda)\frac{t^n}{n!}=\frac{2(q-1)+\lambda\frac{q-1}{\log q}\log(1+t)}{wq^2(1+t)^{\lambda}-1}(1+t)^{\lambda x}. \tag{7}$$

When $x=0$, $D_{n,q,w}(\lambda)=D_{n,q,w}(0|\lambda)$ are called the twisted (λ,q)-Daehee numbers. In particular,

$$D_{0,q,w}(1)=\frac{2(q-1)}{wq^2-1}.$$

The twisted Catalan-Daehee numbers are defined by [5]

$$\frac{\frac{1}{2}\log(1-4t)}{w\sqrt{1-4t}-1}=\sum_{n=0}^{\infty}d_{n,w}t^n. \tag{8}$$

If we take $w=1$ in the twisted Catalan-Daehee numbers, $d_n=d_{n,1}$, are the Catalan-Daehee numbers in [6–8].

We note that

$$\sqrt{1+t}=\sum_{m=0}^{\infty}(-1)^{m-1}\binom{2m}{m}\left(\frac{1}{4}\right)^m\left(\frac{1}{2m-1}\right)t^m. \tag{9}$$

By replacing t by $-4t$ in (9), we get

$$\sqrt{1-4t}=1-2\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{m+1}t^{m+1}=1-2\sum_{m=0}^{\infty}C_m t^{m+1}, \tag{10}$$

where C_m is the Catalan number.

From (8) and (10), Dolgy et al. showed a relation between the Catalan-Daehee numbers and the Catalan numbers in [6];

$$d_n = \begin{cases} 1, & \text{if } n = 0 \\ \dfrac{4^n}{n+1} - \displaystyle\sum_{m=0}^{n-1} \dfrac{4^{n-m-1}}{n-m} C_m, & \text{if } n \geq 1. \end{cases}$$

Catalan-Daehee numbers and polynomials were introduced in [7] and considered the family of linear differential equations arising from the generating function of those numbers in order to derive some explicit identities involving Catalan-Daehee numbers and Catalan numbers. In [8], several properties and identities associated with Catalan-Daehee numbers and polynomials were derived by utilizing umbral calculus techniques. Dolgy et al. gave some new identities for those numbers and polynomials derived from p-adic Volkenborn integrals on $\mathbb{Z}_p$ in [6]. Recently, Ma et al. introduced and studied q-analogues of the Catalan-Daehee numbers and polynomials with the help of p-adic q-integral on $\mathbb{Z}_p$ in [9]. The aim of this paper is to introduce q-analogues of the twisted Catalan-Daehee numbers and polynomials by using p-adic q-integral on $\mathbb{Z}_p$, and derive some explicit identities for those twisted numbers and polynomials related to various special numbers and polynomials.

2. The Twisted Q-Analogues of Catalan-Daehee Numbers

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$ and for $w \in T_p$, we have

$$\int_{\mathbb{Z}_p} w^x (1-4t)^{\frac{x}{2}} d\mu_q(x) = \frac{q-1+\frac{q-1}{\log q}\frac{1}{2}\log(1-4t)}{qw\sqrt{1-4t}-1}. \tag{11}$$

In the view of (11), we define the twisted q-analogue of Catalan-Daehee numbers which are given by the generating function to be

$$\frac{q-1+\frac{q-1}{\log q}\frac{1}{2}\log(1-4t)}{qw\sqrt{1-4t}-1} = \sum_{n=0}^{\infty} d_{n,q,w} t^n. \tag{12}$$

Note that $\lim_{q \to 1} d_{n,q,w} = d_{n,w}$, $(n \geq 0)$, which is the twisted Catalan-Daehee numbers in [5].

From (7) and (12), we have

$$\begin{aligned} \sum_{n=0}^{\infty} d_{n,q,w} t^n &= \frac{1}{2}\left(\frac{2(q-1)+\frac{q-1}{\log q}\log(1-4t)}{w^2q^2(1-4t)-1}\right)\left(qw\sqrt{1-4t}+1\right) \\ &= \frac{1}{2}\left(\sum_{l=0}^{\infty} 4^l D_{l,q,w^2}(1)\frac{(-t)^l}{l!}\right)\left(1+qw-2qw\sum_{m=0}^{\infty} C_m t^{m+1}\right) \\ &= \frac{1+qw}{2}\sum_{n=0}^{\infty}(-4)^n \frac{D_{n,q,w}(1)}{n!} t^n - qw\sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q,w^2}(1) C_m\right) t^n \\ &= \frac{q^2-1}{wq^2-1} + \sum_{n=0}^{\infty}\frac{[2]_{qw}}{2}(-4)^n \frac{D_{n,q,w}(1)}{n!} t^n - qw\sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q,w^2}(1) C_m\right) t^n \\ &= \frac{q^2-1}{wq^2-1} + \sum_{n=1}^{\infty}\left(\frac{[2]_{qw}}{2}\frac{(-4)^n}{n!} D_{n,q,w^2}(1) - qw\sum_{m=0}^{n-1}\frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q,w^2}(1) C_m\right) t^n. \end{aligned} \tag{13}$$

Therefore, by comparing the coefficients on the both sides of (13), we obtain the following theorem.

Theorem 1. *For $n \geq 0$ and $w \in T_p$, we have*

$$d_{n,q,w} = \begin{cases} \dfrac{q^2-1}{wq^2-1}, & \text{if } n=0, \\ \dfrac{1+qw}{2}\dfrac{(-4)^n}{n!}D_{n,q,w^2}(1) - qw\displaystyle\sum_{m=0}^{n-1}\dfrac{(-4)^{n-m-1}}{(n-m-1)!}2^{2n-2m-1}D_{n-m-1,q,w^2}(1)C_m, & \text{if } n \geq 1. \end{cases}$$

Specially, $w = 1$ and $q \to 1$, we have

Corollary 1 (Theorem 1, [6]). *For $n \geq 0$, we have*

$$d_n = \begin{cases} 1, & \text{if } n=0, \\ (-4)^n\dfrac{D_n(1)}{n!} - \displaystyle\sum_{m=0}^{n-1}\dfrac{(-4)^{n-m-1}}{(n-m-1)!}2^{2n-2m-1}D_{n-m-1}(1)C_m, & \text{if } n \geq 1. \end{cases}$$

Now, from (6) and (12), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} d_{n,q,w}t^n &= \frac{q-1+\frac{q-1}{\log q}\frac{1}{2}\log(1-4t)}{qw\sqrt{1-4t}-1} = \int_{\mathbb{Z}_p} w^x(1-4t)^{\frac{x}{2}}d\mu_q(x) \\ &= \sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^m\frac{1}{m!}\big(\log(1-4t)\big)^m\int_{\mathbb{Z}_p} w^x x^m d\mu_q(x) \\ &= \sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^m B_{m,q,w}\sum_{n=m}^{\infty} S_1(n,m)\frac{1}{n!}(-4t)^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} 2^{2n-m}(-1)^n B_{m,q,w}S_1(n,m)\right)\frac{t^n}{n!}, \end{aligned} \tag{14}$$

where $S_1(n,m),\ (n,m \geq 0)$ is the Stirling number of the first kind which is defined by [1–20]

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \geq 0).$$

Here, $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

Therefore, by (14), we obtain the following theorem.

Theorem 2. *For $n \geq 0$ and $w \in T_p$, we have*

$$(-1)^n d_{n,q,w} = \frac{1}{n!}\sum_{m=0}^{n} 2^{2n-m}B_{m,q,w}S_1(n,m).$$

By binomial expansion, we get

$$\int_{\mathbb{Z}_p} w^x(1-4t)^{\frac{x}{2}}d\mu_q(x) = \sum_{n=0}^{\infty}(-4)^n\int_{\mathbb{Z}_p} w^x\binom{\frac{x}{2}}{n}d\mu_q(x)t^n. \tag{15}$$

From (12) and (15), we obtain the following corollary.

Corollary 2. *For $n \geq 0$ and $w \in T_p$, we have*

$$\int_{\mathbb{Z}_p} w^x\binom{\frac{x}{2}}{n}d\mu_q(x) = (-1)^n 2^{-2n}d_{n,q,w} = \frac{1}{n!}\sum_{m=0}^{n}\left(\frac{1}{2}\right)^m B_{m,q,w}S_1(n,m).$$

For the case $w = 1$ and $q \to 1$, we have the following.

Corollary 3 (Theorem 2, [6]). *For $n \geq 0$, we have*

$$(-1)^n d_n = \frac{1}{n!} \sum_{m=0}^{n} 2^{2n-m} B_m S_1(n,m).$$

The twisted q-analogue of λ-Daehee polynomials are given by the p-adic q-integral on $\mathbb{Z}_p$ to be

$$\begin{aligned} \int_{\mathbb{Z}_p} w^y (1+t)^{\lambda y + x} d\mu_q(y) &= \frac{(q-1) + \lambda \frac{q-1}{\log q} \log(1+t)}{qw(1+t)^{\lambda} - 1}(1+t)^x \\ &= \sum_{n=0}^{\infty} \widetilde{D}_{n,q,w}(x|\lambda) \frac{t^n}{n!}. \end{aligned} \tag{16}$$

When $x = 0$, $\widetilde{D}_{n,q,w}(\lambda) = \widetilde{D}_{n,q,w}(0|\lambda)$ $(n \geq 0)$ are called the twisted q-analogue of λ-Daehee numbers. Note that

$$\widetilde{D}_{0,q,w}(\lambda) = \frac{q-1}{wq-1}.$$

From (16), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n 4^n \widetilde{D}_{n,q,w}\left(\frac{1}{2}\right) \frac{t^n}{n!} &= \frac{q - 1 + \frac{1}{2} \frac{q-1}{\log q} \log(1-4t)}{qw(1-4t)^{\frac{1}{2}} - 1} \\ &= \sum_{n=0}^{\infty} d_{n,q,w} t^n. \end{aligned} \tag{17}$$

Thus, by (17), we get

$$d_{n,q,w} = (-1)^n \frac{4^n}{n!} \widetilde{D}_{n,q,w}\left(\frac{1}{2}\right), \quad (n \geq 0).$$

Let us take $t = \frac{1}{4}(1 - e^{2t})$ in (12). Then we have

$$\begin{aligned} \sum_{k=0}^{\infty} d_{k,q,w} \left(\frac{1}{4}\right)^k (1 - e^{2t})^k &= \frac{q - 1 + \frac{q-1}{\log q} t}{qwe^t - 1} = \int_{\mathbb{Z}_p} w^x e^{xt} d\mu_q(x) \\ &= \sum_{n=0}^{\infty} B_{n,q,w} \frac{t^n}{n!}. \end{aligned} \tag{18}$$

On the other hand,

$$\begin{aligned} \sum_{k=0}^{\infty} d_{k,q,w} (-1)^k \left(\frac{1}{4}\right)^k (e^{2t} - 1)^k &= \sum_{k=0}^{\infty} (-1)^k k! d_{k,q,w} \left(\frac{1}{4}\right)^k \frac{1}{k!} (e^{2t} - 1)^k \\ &= \sum_{k=0}^{\infty} (-1)^k k! d_{k,q,w} 2^{-2k} \sum_{n=k}^{\infty} S_2(n,k) 2^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^k k! d_{k,q,w} 2^{n-2k} S_2(n,k) \right) \frac{t^n}{n!}, \end{aligned} \tag{19}$$

where $S_2(n,k)$ $(n, k \geq 0)$ is the Stirling number of the second kind which is defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0).$$

Therefore, by (18) and (19), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$B_{n,q,w} = \sum_{k=0}^{n} (-1)^k S_2(n,k) 2^{n-2k} k! d_{k,q,w}.$$

Now, we observe that

$$\int_{\mathbb{Z}_p} w^y (1-4t)^{\frac{x+y}{2}} d\mu_q(y) = \frac{(q-1) + \frac{q-1}{\log q}\frac{1}{2}\log(1-4t)}{wq\sqrt{1-4t}-1}(1-4t)^{\frac{x}{2}}.$$

We define the twisted Catalan-Daehee polynomials which are given by the generating function to be

$$\frac{q-1+\frac{q-1}{\log q}\frac{1}{2}\log(1-4t)}{qw\sqrt{1-4t}-1}(1-4t)^{\frac{x}{2}} = \sum_{n=0}^{\infty} d_{n,q,w}(x) t^n. \tag{20}$$

When $x = 0$, $d_{n,q,w} = d_{n,q,w}(0)$ $(n \geq 0)$ are the twisted Catalan-Daehee numbers in (12).

Note that

$$\begin{aligned}(1-4t)^{\frac{x}{2}} &= \sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^l \frac{1}{l!}\big(\log(1-4t)\big)^l = \sum_{l=0}^{\infty}\left(\frac{x}{2}\right)^l \sum_{m=l}^{\infty} S_1(m,l)(-4)^m \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m} S_1(m,l)\frac{(-4)^m}{m!}\left(\frac{x}{2}\right)^l\right)t^m.\end{aligned} \tag{21}$$

Thus, by (12), (20) and (21), we get

$$\begin{aligned}\sum_{n=0}^{\infty} d_{n,q,w}(x)t^n &= \frac{q-1+\frac{q-1}{\log q}\frac{1}{2}\log(1-4t)}{qw\sqrt{1-4t}-1}(1-4t)^{\frac{x}{2}} \\ &= \left(\sum_{k=0}^{\infty} d_{k,q,w}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m} S_1(m,l)\frac{(-4)^m}{m!}\left(\frac{x}{2}\right)^l\right)t^m \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m} S_1(m,l)\frac{(-4)^m}{m!}d_{n-m,q,w}\left(\frac{x}{2}\right)^l\right)t^n.\end{aligned} \tag{22}$$

By comparing the coefficients on the both sides (22), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$\begin{aligned}d_{n,q,w}(x) &= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(m,l)(-1)^m \frac{2^{2m-l}}{m!} d_{n-m,q,w} x^l \\ &= \sum_{l=0}^{n}\left(\sum_{m=l}^{n}(-1)^m \frac{2^{2m-l}}{m!} S_1(m,l) d_{n-m,q,w}\right)x^l.\end{aligned}$$

For the case $w = 1$ and $q \to 1$, we have the following.

Corollary 4 (Theorem 5, [6]). *For $n \geq 0$, we have*

$$d_n(x) = \sum_{l=0}^{n}\left(\sum_{m=l}^{n}(-1)^m \frac{2^{2m-l}}{m!} S_1(m,l) d_{n-m}\right)x^l.$$

3. Conclusions

To summarize, we introduced twisted q-analogues of Catalan-Daehee numbers and polynomials and obtained several explicit expressions and identities related to them. We expressed the twisted q-analogues of Catalan-Daehee numbers in terms of the twisted

(λ, q)-Daehee numbers, and of the twisted q-Bernoulli numbers and Stirling numbers of the first kind in Theorems 1 and 2. We also derived an identity involving the twisted q-Bernoulli numbers, twisted q-analogues of Catalan-Daehee numbers and Stirling numbers of the second kind in Theorem 3. In addition, we obtain an explicit expression for the twisted q-analogues of Catalan-Daehee polynomials which involve the twisted q-analogues of Catalan-Daehee numbers and Stirling numbers of the first kind in Theorem 4.

In recent years, many special numbers and polynomials have been studied by employing various methods, including: generating functions, p-adic analysis, combinatorial methods, umbral calculus, differential equations, probability theory and analytic number theory. We are now interested in continuing our research into the application of 'twisted' and 'q-analogue' versions of certain interesting special polynomials and numbers in the fields of physics, science, and engineering as well as mathematics.

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