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A Study of Clairaut Semi-Invariant Riemannian Maps from Cosymplectic Manifolds

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Abstract: In the present note, we characterize Clairaut semi-invariant Riemannian maps from cosymplectic manifolds to Riemannian manifolds. Moreover, we provide a nontrivial example of such a Riemannian map.

Keywords: cosymplectic manifolds; Riemannian map; Clairaut semi-invariant Riemannian map

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1. Introduction

The theory of Riemannian maps between Riemannian manifolds is widely used to compare the geometric structures between two Riemannian manifolds, initiated by Fischer [1]. Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two Riemannian manifolds of dimensions m and n, respectively. Let a Riemannian map $\Pi : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$ be a differentiable map between $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ such that $0 < rank\Pi_* < \min\{m, n\}$, where Π_* represents a differential map of Π. If we denote the *kernel* space of Π_* by $\ker \Pi_*$ and the orthogonal complementary space of $\ker \Pi_*$ by $(\ker \Pi_*)^\perp$ in $T\mathcal{M}_1$, then the $T\mathcal{M}_1$ has the following orthogonal decomposition:

$$T\mathcal{M}_1 = \ker \Pi_* \oplus (\ker \Pi_*)^\perp. \tag{1}$$

We denote the range of Π_* by $range\Pi_*$ and for a point $q \in \mathcal{M}_1$ the orthogonal complementary space of $range\Pi_{*\Pi(q)}$ by $(range\Pi_{*\Pi(q)})^\perp$ in $T_{\Pi(q)}\mathcal{M}_2$. The tangent space $T_{\Pi(q)}\mathcal{M}_2$ has the following orthogonal decomposition:

$$T_{\Pi(q)}\mathcal{M}_2 = (range\Pi_{*\Pi(q)}) \oplus (range\Pi_{*\Pi(q)})^\perp.$$

A differentiable map $\Pi : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$ is called a Riemannian map at $q \in \mathcal{M}_1$ if the horizontal restriction $\Pi^h_{*q} : (\ker \Pi_{*q})^\perp \to (range\Pi_{*\Pi(q)})$ is a linear isometric between the inner product spaces $((\ker \Pi_{*q})^\perp, (g_1)_{(q)}|_{(\ker \Pi_{*q})^\perp})$ and $(range\Pi_{*\Pi_{(q)}}, (g_2)_{(\Pi(q))}|_{(range\Pi_{*q})})$.

Further, the notion of the Riemannian map has been studied from different perspectives, such as invariant and anti-invariant Riemannian maps [2], semi-invariant Riemannian maps [3], slant Riemannian maps [4–6], semi-slant Riemannian maps [7–9], hemi-slant Riemannian maps [10], quasi-hemi-slant Riemannian maps [11] etc.

On the other side, in the theory of the geodesics upon a surface of revolution, the prestigious Clairaut's theorem states that for any geodesic $c(c : I_1 \subset R \to \mathcal{M}_1$ on $\mathcal{M}_1)$ on the revolution surface $\mathcal{M}_1$ the product $rsin\theta$ is constant along c, where $\theta(s)$ is the angle between $c(s)$ and the meridian curve through $c(s)$, $s \in I_1$. This means that it is independent of s. In 1972, Bishop [12] studied Riemannian submersions which are a generalization of Clairaut's theorem. According to him, a submersion $\Pi : \mathcal{M}_1 \to \mathcal{M}_2$ is said to be a Clairaut submersion if there is a function $r : \mathcal{M}_1 \to R^+$ such that for every geodesic making an angle θ with the horizontal subspaces, $rsin\theta$ is constant. This notion has also been studied in Lorentzian spaces, time-like and space-like spaces, by the authors [13–15]. Later, in [16], it was shown that such submersions have their applications in static spacetimes.

Moreover, Clairaut submersions were further generalized in [17]. We recommend the papers [18–32] and the references therein for more details about the further related studies.

In this paper, we are interested in studying the above idea in contact manifolds. Throughout the manuscript, we denote semi-invariant Riemannian maps by SIR maps and Clairaut semi-invariant Riemannian maps by CSIR maps. The article is organized as follows: In Section 2, we gather some basic facts that are needed for this paper. In Section 3, we define a CSIR map from an almost contact metric manifold to a Riemannian manifold and study its geometry. In Section 4, we give a nontrivial example of the CSIR map from cosymplectic manifolds to Riemannian manifolds.

2. Preliminaries

An odd-dimensional smooth manifold $\mathcal{M}_1$ is said to have an almost contact structure [33] if there exist on $\mathcal{M}_1$ a tensor field ϕ of type $(1,1)$, a vector field ξ, and 1-form η such that

$$\phi^2 V_1 = -V_1 + \eta(V_1)\xi,\ \eta \circ \phi = 0,\ \ \phi\xi = 0, \tag{2}$$

$$\eta(\xi) = 1. \tag{3}$$

If there exists a Riemannian metric g_1 on an almost contact manifold $\mathcal{M}_1$ satisfying:

$$\begin{aligned} g_1(\phi V_1, \phi V_2) &= g_1(V_1, V_2) - \eta(V_1)\eta(V_2), \\ g_1(V_1, \phi V_2) &= -g_1(\phi V_1, V_2), \end{aligned} \tag{4}$$

$$g_1(V_1, \xi) = \eta(V_1), \tag{5}$$

where V_1, V_2 are any vector fields on $\mathcal{M}_1$, then $\mathcal{M}_1$ is called an almost contact metric manifold [34] with an almost contact structure (ϕ, ξ, η, g_1) and is represented by $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\mathcal{M}_1 \times R$ is given by

$$J(V_1, \mathcal{F}\frac{d}{dt}) = (\phi V_1 - \mathcal{F}\xi, \eta(V_1)\frac{d}{dt}), \tag{6}$$

where $J^2 = -I$ and $\mathcal{F}$ is a differentiable function on $\mathcal{M}_1 \times R$ that has no torsion, i.e., J is integrable. The condition for normality in terms of ϕ, ξ, and η is given by $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\mathcal{M}_1$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Further, the fundamental 2-form Φ is defined by $\Phi(V_1, V_2) = g_1(V_1, \phi V_2)$.

A manifold $\mathcal{M}_1$ with the structure (ϕ, ξ, η, g_1) is said to be cosymplectic [33] if

$$(\nabla_{V_1}\phi)V_2 = 0, \tag{7}$$

for any vector fields V_1, V_2 on $\mathcal{M}_1$, where ∇ stands for the Riemannian connection of the metric g_1 on $\mathcal{M}_1$. For a cosymplectic manifold, we have

$$\nabla_{V_1}\xi = 0 \tag{8}$$

for any vector field V_1 on $\mathcal{M}_1$.

O'Neill's tensors [35] $\mathcal{T}$ and $\mathcal{A}$ are given by

$$\mathcal{A}_{F_1}F_2 = \mathcal{H}\nabla_{\mathcal{H}F_1}\mathcal{V}F_2 + \mathcal{V}\nabla_{\mathcal{H}F_1}\mathcal{H}F_2, \tag{9}$$

$$\mathcal{T}_{F_1}F_2 = \mathcal{H}\nabla_{\mathcal{V}F_1}\mathcal{V}F_2 + \mathcal{V}\nabla_{\mathcal{V}F_1}\mathcal{H}F_2, \tag{10}$$

for any F_1, F_2 on $\mathcal{M}_1$. It is easy to see that $\mathcal{T}_{F_1}$ and $\mathcal{A}_{F_1}$ are skew-symmetric operators on the tangent bundle of $\mathcal{M}_1$ reversing the vertical and the horizontal distributions. In addition, for any vertical vector fields X_1 and X_2, the tensor field $\mathcal{T}$ has the symmetry property, i.e.,

$$\mathcal{T}_{X_1}X_2 = \mathcal{T}_{X_2}X_1, \tag{11}$$

while for horizontal vector fields Z_1, Z_2, the tensor field $\mathcal{A}$ has alternation property, i.e.,

$$\mathcal{A}_{Z_1}Z_2 = -\mathcal{A}_{Z_2}Z_1. \tag{12}$$

From Equations (9) and (10), we have

$$\nabla_{U_1}U_2 = \mathcal{T}_{U_1}U_2 + \mathcal{V}\nabla_{U_1}U_2, \tag{13}$$

$$\nabla_{U_1}W_1 = \mathcal{T}_{U_1}W_1 + \mathcal{H}\nabla_{U_1}W_1, \tag{14}$$

$$\nabla_{W_1}U_1 = \mathcal{A}_{W_1}U_1 + \mathcal{V}\nabla_{W_1}U_1, \tag{15}$$

$$\nabla_{W_1}W_2 = \mathcal{H}\nabla_{W_1}W_2 + \mathcal{A}_{W_1}W_2 \tag{16}$$

for all $U_1, U_2 \in \Gamma(\ker \Pi_*)$ and $W_1, W_2 \in \Gamma(\ker \Pi_*)^{\perp}$, where $\mathcal{H}\nabla_{U_1}W_1 = \mathcal{A}_{W_1}U_1$ and W_1 is basic. It can be easily seen that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of the distribution.

It is noticed that for $p \in \mathcal{M}_1$, $Z_1 \in \mathcal{V}_p$ and $X_1 \in \mathcal{H}_p$ the linear operators

$$\mathcal{A}_{X_1}, \ \mathcal{T}_{Z_1} : T_p\mathcal{M}_1 \to T_p\mathcal{M}_1$$

are skew-symmetric, i.e.,

$$g_1(\mathcal{A}_{X_1}F_1, F_2) = -g_1(F_1, \mathcal{A}_{X_1}F_2) \text{ and } g_1(\mathcal{T}_{Z_1}F_1, F_2) = -g_1(F_1, \mathcal{T}_{Z_1}F_2) \tag{17}$$

for each $F_1, F_2 \in T_P\mathcal{M}_1$. Since $\mathcal{T}_{Z_1}$ is skew-symmetric, we observe that Π has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

The map Π between two Riemannian manifolds is totally geodesic if

$$(\nabla\Pi_*)(V_1, V_2) = 0 \ \ \forall V_1, V_2 \in \Gamma(T\mathcal{M}_1).$$

A totally umbilical map is a Riemannian map with totally umbilical fibers [36] if

$$\mathcal{T}_{Y_1}Y_2 = g_1(Y_1, Y_2)H \tag{18}$$

for all $Y_1, Y_2 \in \Gamma(\ker \Pi_*)$, where H denotes the mean curvature vector field of fibers.

The map Π_* can be observed as a section of the bundle $Hom(T\mathcal{M}_1, \Pi^{-1}T\mathcal{M}_2) \longrightarrow \mathcal{M}_1$, where $\Pi^{-1}T\mathcal{M}_2$ is the bundle which has fibers $(\Pi^{-1}T\mathcal{M}_2)_x = T_{\Pi(x)}\mathcal{M}_2$ and has a connection ∇ induced from the Riemannian connection $\nabla^{\mathcal{M}_1}$ and the pullback connection ∇^{Π}, then the second fundamental form of Π is given by

$$(\nabla\Pi_*)(W_1, W_2) = \nabla^{\Pi}_{W_1}\Pi_*(W_2) - \Pi_*(\nabla^{\mathcal{M}_1}_{W_1}W_2) \tag{19}$$

for the vector fields $W_1, W_2 \in \Gamma(T\mathcal{M}_1)$. We know that the second fundamental form is symmetric.

Now, we have the following lemma [2]:

Lemma 1. *Let $\Pi : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$ be a map between Riemannian manifolds. Then*

$$g_2((\nabla\Pi_*)(Y_1, Y_2), \Pi_*(Y_3)) = 0 \ \ \forall Y_1, Y_2, Y_3 \in \Gamma(\ker \Pi_*)^{\perp}. \tag{20}$$

As a result of above Lemma, we obtain

$$(\nabla\Pi_*)(Z_1, Z_2) \in (\Gamma(range\Pi_*)^\perp) \ \ \forall Z_1, Z_2 \in \Gamma(\ker\Pi_*)^\perp. \tag{21}$$

3. CSIR Map from Cosymplectic Manifolds

Let S be a revolution surface in R^3 with rotation axis L. For any $q \in S$, we denote the distance from q to L by $r(q)$. Given a geodesic $\alpha : I \subset R \to S$ on S, let $\theta(t)$ be the angle between $\alpha(t)$ and the meridian curve through $\alpha(t)$, $t \in I$. A well-known Clairaut's theorem says that for any geodesic α on S, the product $r\sin\theta(t)$ is constant along α, i.e., it is independent of t.

Recently, Sahin [30] initiated the study of Clairaut Riemannian maps. He defined a map $\Pi : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$ called a Clairaut Riemannian map if there exists a positive function r on $\mathcal{M}_1$, such that for any geodesic α on $\mathcal{M}_1$, the function $(r \circ \alpha)\sin\theta$ is constant, where for any t, $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$. Moreover, he obtained the following necessary and sufficient condition for a Riemannian map to be a Clairaut Riemannian map:

Theorem 1 ([30])**.** *Let $\Pi : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2)$ be a Riemannian map with connected fibers. Then, Π is a Clairaut Riemannian map with $r = e^f$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla f$, where ∇f is the gradient of the function f with respect to g_1.*

Definition 1 ([3])**.** *Let Π be a Riemannian map from an almost contact metric manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$. Then, we say that Π is an SIR map if there is a distribution $\mathfrak{D}_1 \subseteq \ker\Pi_*$ such that*

$$\ker\Pi_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2, \ \ \phi(\mathfrak{D}_1) = \mathfrak{D}_1, \ \ \phi(\mathfrak{D}_2) \subseteq (\ker\Pi_*)^\perp,$$

where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are mutually orthogonal distributions in $(\ker\Pi_*)$.

Let μ denote the complementary orthogonal subbundle to $\phi(\mathfrak{D}_2)$ in $(\ker\Pi_*)^\perp$. Then, we have

$$(\ker\Pi_*)^\perp = \phi(\mathfrak{D}_2) \oplus \mu.$$

Obviously, μ is an invariant subbundle of $(\ker\Pi_*)^\perp$ with respect to the contact structure ϕ. We say that an SIR map $\Pi : \mathcal{M}_1 \to \mathcal{M}_2$ admits a vertical Reeb vector field ξ if it is tangent to $(\ker\Pi_*)$ and it admits a horizontal Reeb vector field ξ if it is normal to $(\ker\Pi_*)$. It is easy to see that μ contains the Reeb vector field in case the Riemannian map admits horizontal Reeb vector field.

Now, we define the notion of the CSIR map in contact manifolds as follows:

Definition 2. *An SIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ is called a CSIR map if it satisfies the condition of a Clairaut Riemannian map.*

For any vector field $Z_1 \in \Gamma(\ker\Pi_*)$, we input

$$Z_1 = PZ_1 + QZ_1, \tag{22}$$

where P and Q are projection morphisms [36] of $\ker\Pi_*$ onto $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.

For any $V_1 \in (\ker\Pi_*)$, we obtain

$$\phi V_1 = \psi V_1 + \omega V_1, \tag{23}$$

where $\psi V_1 \in \Gamma(\mathfrak{D}_1)$ and $\omega V_1 \in \Gamma(\phi\mathfrak{D}_2)$. In addition, for $V_2 \in \Gamma(\ker\Pi_*)^\perp$, we have

$$\phi V_2 = BV_2 + CV_2, \tag{24}$$

where $BV_2 \in \Gamma(\mathfrak{D}_2)$ and $CV_2 \in \Gamma(\mu)$.

Definition 3 ([14]). *Let Π be an SIR map from an almost contact metric manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$. If $\mu = \{0\}$ or $\mu =< \xi >$, i.e., $(\ker \Pi_*)^\perp = \phi(\mathfrak{D}_2)$ or $(\ker \Pi_*)^\perp = \phi(\mathfrak{D}_2) \oplus < \xi >$, respectively. Then we call ϕ a Lagrangian Riemannian map. In this case, for any horizontal vector field V_1, we have*

$$BV_1 = \phi V_1 \text{ and } CV_1 = 0. \tag{25}$$

Lemma 2. *Let Π be an SIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then, we obtain*

$$\mathcal{V}\nabla_{Y_1}\psi Y_2 + \mathcal{T}_{Y_1}\omega Y_2 = B\mathcal{T}_{Y_1}Y_2 + \psi\mathcal{V}\nabla_{Y_1}Y_2, \tag{26}$$

$$\mathcal{T}_{Y_1}\psi Y_2 + \mathcal{H}\nabla_{Y_1}\omega Y_2 = C\mathcal{T}_{Y_1}Y_2 + \omega\mathcal{V}\nabla_{Y_1}Y_2, \tag{27}$$

$$\mathcal{V}\nabla_{V_1}BV_2 + \mathcal{A}_{V_1}CV_2 = B\mathcal{H}\nabla_{V_1}V_2 + \psi\mathcal{A}_{V_1}V_2, \tag{28}$$

$$\mathcal{A}_{V_1}BV_2 + \mathcal{H}\nabla_{V_1}CV_2 = C\mathcal{H}\nabla_{V_1}V_2 + \omega\mathcal{A}_{V_1}V_2, \tag{29}$$

$$\mathcal{V}\nabla_{Y_1}BV_1 + \mathcal{T}_{Y_1}CV_1 = \psi\mathcal{T}_{Y_1}V_1 + B\mathcal{H}\nabla_{Y_1}V_1, \tag{30}$$

$$\mathcal{T}_{Y_1}BV_1 + \mathcal{H}\nabla_{Y_1}CV_1 = \omega\mathcal{T}_{Y_1}V_1 + C\mathcal{H}\nabla_{Y_1}V_1, \tag{31}$$

$$\mathcal{V}\nabla_{V_1}\psi Y_1 + \mathcal{A}_{V_1}\omega Y_1 = B\mathcal{A}_{V_1}Y_1 + \psi\mathcal{V}\nabla_{V_1}Y_1, \tag{32}$$

$$\mathcal{A}_{V_1}\psi Y_1 + \mathcal{H}\nabla_{V_1}\omega Y_1 = C\mathcal{A}_{V_1}Y_1 + \omega\mathcal{V}\nabla_{V_1}Y_1, \tag{33}$$

where $Y_1, Y_2 \in \Gamma(\ker \Pi_*)$ and $V_1, V_2 \in \Gamma(\ker \Pi_*)^\perp$.

Proof. Using Equations (7), (13)–(16), (23) and (24), we obtain Lemma 2. □

Corollary 1. *Let Π be a Lagrangian Riemannian map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then, we obtain*

$$\mathcal{V}\nabla_{X_1}\psi X_2 + \mathcal{T}_{X_1}\omega X_2 = B\mathcal{T}_{X_1}X_2 + \psi\mathcal{V}\nabla_{X_1}X_2, \mathcal{T}_{X_1}\psi X_2 + \mathcal{H}\nabla_{X_1}\omega X_2 = \omega\mathcal{V}\nabla_{X_1}X_2,$$

$$\mathcal{V}\nabla_{Z_1}BZ_2 = B\mathcal{H}\nabla_{Z_1}Z_2 + \psi\mathcal{A}_{Z_1}Z_2, \mathcal{A}_{Z_1}BZ_2 = \omega\mathcal{A}_{Z_1}Z_2,$$

$$\mathcal{V}\nabla_{X_1}BZ_1 = \psi\mathcal{T}_{X_1}Z_1 + B\mathcal{H}\nabla_{X_1}Z_1, \mathcal{T}_{X_1}BZ_1 = \omega\mathcal{T}_{X_1}Z_1,$$

$$\mathcal{V}\nabla_{Z_1}\psi X_1 + \mathcal{A}_{Z_1}\omega X_1 = B\mathcal{A}_{Z_1}X_1 + \psi\mathcal{V}\nabla_{Z_1}X_1, \mathcal{A}_{Z_1}\psi X_1 + \mathcal{H}\nabla_{Z_1}\omega X_1 = \omega\mathcal{V}\nabla_{Z_1}X_1,$$

where $X_1, X_2 \in \Gamma(\ker \Pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker \Pi_*)^\perp$.

Lemma 3. *Let Π be an SIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then, we have*

$$\mathcal{T}_{U_1}\xi = 0, \tag{34}$$

$$\mathcal{A}_{U_2}\xi = 0 \tag{35}$$

for $U_1 \in \Gamma(\ker \Pi_*)^\perp$ and $U_2 \in \Gamma(\ker \Pi_*)^\perp$.

Proof. Using Equations (8), (14) and (16), we obtain Lemma 3. □

Lemma 4. *Let Π be an SIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$. If $\alpha : I_2 \subset R \to \mathcal{M}_1$ is a regular curve and $Y_1(t)$ and $Y_2(t)$ are the vertical* *and horizontal components of the tangent vector field* $\dot{\alpha} = E$ *of* $\alpha(t)$*, respectively, then* α *is a geodesic if and only if along* α *the following relations hold:*

$$\mathcal{V}\nabla_{\dot{\alpha}}BY_2 + \mathcal{V}\nabla_{\dot{\alpha}}\psi Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})CY_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})\omega Y_1 = 0, \tag{36}$$

$$\mathcal{H}\nabla_{\dot{\alpha}}CY_2 + \mathcal{H}\nabla_{\dot{\alpha}}\omega Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})BY_2 + (\mathcal{A}_{Y_2} + \mathcal{T}_{Y_1})\psi Y_1 = 0. \tag{37}$$

Proof. Let $\alpha : I_2 \to \mathcal{M}_1$ be a regular curve on $\mathcal{M}_1$. Since $Y_1(t)$ and $Y_2(t)$ are the vertical and horizontal parts of the tangent vector field $\dot{\alpha}(t)$, i.e., $\dot{\alpha}(t) = Y_1(t) + Y_2(t)$, from Equations (2), (7), (13)–(16), (23) and (24), we obtain

$$\begin{aligned} \phi\nabla_{\dot{\alpha}}\dot{\alpha} &= \nabla_{\dot{\alpha}}\phi\dot{\alpha} \\ &= \nabla_{Y_1}\psi Y_1 + \nabla_{Y_1}\omega Y_1 + \nabla_{Y_2}\psi Y_1 + \nabla_{Y_2}\omega Y_1 \\ &\quad + \nabla_{Y_1}BY_2 + \nabla_{Y_1}CY_2 + \nabla_{Y_2}BY_2 + \nabla_{Y_2}CY_2 \\ &= \mathcal{V}\nabla_{\dot{\alpha}}BY_2 + \mathcal{V}\nabla_{\dot{\alpha}}\psi Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})CY_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})\omega Y_1 \\ &\quad + \mathcal{H}\nabla_{\dot{\alpha}}CY_2 + \mathcal{H}\nabla_{\dot{\alpha}}\omega Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})BY_2 + (\mathcal{A}_{Y_2} + \mathcal{T}_{Y_1})\psi Y_1. \end{aligned}$$

Taking the vertical and horizontal components in the above equation, we have

$$\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = \mathcal{V}\nabla_{\dot{\alpha}}BY_2 + \mathcal{V}\nabla_{\dot{\alpha}}\psi Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})CY_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})\omega Y_1,$$

$$\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = \mathcal{H}\nabla_{\dot{\alpha}}CY_2 + \mathcal{H}\nabla_{\dot{\alpha}}\omega Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})BY_2 + (\mathcal{A}_{Y_2} + \mathcal{T}_{Y_1})\psi Y_1.$$

Thus, α is a geodesic on $\mathcal{M}_1$ if and only if $\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = 0$ and $\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = 0$; this completes the proof. □

Theorem 2. *Let* Π *be an SIR map from a cosymplectic manifold* $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ *to a Riemannian manifold* $(\mathcal{M}_2, g_2)$*. Then,* Π *is a CSIR map with* $r = e^f$ *if and only if*

$$\begin{aligned} & g_1(\nabla f, V_2)||V_1||^2 \\ = \; & g_1(\mathcal{V}\nabla_{\dot{\alpha}}BV_2, \psi V_1) + g_1(\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) \\ & + g_1(\mathcal{H}\nabla_{\dot{\alpha}}CV_2, \omega V_1) + g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1), \end{aligned}$$

where $\alpha : I_2 \to \mathcal{M}_1$ *is a geodesic on* $\mathcal{M}_1$*;* $V_1(t)$ *and* $V_2(t)$ *are vertical and horizontal components of* $\dot{\alpha}(t)$*, respectively.*

Proof. Let $\alpha : I_2 \to \mathcal{M}_1$ be a geodesic on $\mathcal{M}_1$ with $V_1(t) = \mathcal{V}\dot{\alpha}(t)$ and $V_2(t) = \mathcal{H}\dot{\alpha}(t)$. We denote the angle in $[0, \pi]$ between $\dot{\alpha}(t)$ and $V_2(t)$ by $\theta(t)$. Assuming $v = ||\dot{\alpha}(t)||^2$, then we obtain

$$g_1(V_1(t), V_1(t)) = v\sin^2\theta(t), \tag{38}$$

$$g_1(V_2(t), V_2(t)) = v\cos^2\theta(t). \tag{39}$$

Now, differentiating (38), we obtain

$$\begin{aligned} \frac{d}{dt}g_1(V_1(t), V_1(t)) &= 2v\sin\theta(t)\cos\theta(t)\frac{d\theta}{dt}, \\ g_1(\nabla_{\dot{\alpha}}V_1(t), V_1(t)) &= v\sin\theta(t)\cos\theta(t)\frac{d\theta}{dt}. \end{aligned}$$

Using Equations (4) and (7) in the above equation, we obtain

$$g_1(\nabla_{\dot{\alpha}}\phi V_1(t), \phi V_1(t)) = v\sin\theta(t)\cos\theta(t)\frac{d\theta}{dt}. \tag{40}$$

Thus, we obtain

$$\begin{aligned} & g_1(\nabla_{\dot{\alpha}}\phi V_1, \phi V_1) \\ = \; & g_1(\mathcal{V}\nabla_{\dot{\alpha}}\psi V_1, \psi V_1) + g_1(\mathcal{H}\nabla_{\dot{\alpha}}\omega V_1, \omega V_1) \\ & + g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})\psi V_1, \omega V_1) + g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})\omega V_1, \psi V_1). \end{aligned} \tag{41}$$

Using Equations (36) and (37) in (41), we have

$$\begin{aligned} & g_1(\nabla_{\dot{\alpha}}\phi V_1, \phi V_1) \\ = \; & -g_1(\mathcal{V}\nabla_{\dot{\alpha}}BV_2, \psi V_1) - g_1(\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) \\ & -g_1(\mathcal{H}\nabla_{\dot{\alpha}}CV_2, \omega V_1) - g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1). \end{aligned} \tag{42}$$

From Equations (40) and (42), we have

$$\begin{aligned} & v \sin\theta(t) \cos\theta(t) \frac{d\theta}{dt} \\ = \; & -g_1(\mathcal{V}\nabla_{\dot{\alpha}}BV_2, \psi V_1) - g_1(\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) \\ & -g_1(\mathcal{H}\nabla_{\dot{\alpha}}CV_2, \omega V_1) - g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1). \end{aligned} \tag{43}$$

Moreover, π is a Clairaut semi-invariant Riemannian map with $r = e^f$ if and only if $\frac{d}{dt}(e^{f\circ\alpha}\sin\theta) = 0$, i.e., $e^{f\circ\alpha}(\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{df}{dt}) = 0$, which, by multiplying with nonzero factor $v\sin\theta$, gives

$$\begin{aligned} -v\cos\theta\sin\theta\frac{d\theta}{dt} &= v\sin^2\theta\frac{df}{dt} \\ v\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_1(V_1, V_1)\frac{df}{dt} \\ v\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_1(\nabla f, \dot{\alpha})||V_1||^2 \\ v\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_1(\nabla f, V_2)||V_1||^2. \end{aligned} \tag{44}$$

Thus, from Equations (43) and (44), we have

$$\begin{aligned} & g_1(\nabla f, V_2)||V_1||^2 \\ = \; & g_1(\mathcal{V}\nabla_{\dot{\alpha}}BV_2, \psi V_1) + g_1(\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) \\ & + g_1(\mathcal{H}\nabla_{\dot{\alpha}}CV_2, \omega V_1) + g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1). \end{aligned}$$

Hence, Theorem 2 is proved. □

Corollary 2. *Let Π be an SIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ admitting horizontal Reeb vector field. Then, we obtain*

$$g_1(\nabla f, \xi) = 0.$$

Theorem 3. *Let Π be a CSIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ with $r = e^f$, then at least one of the following statement is true:*

(i) f is constant on $\phi(\mathfrak{D}_2)$;
(ii) The fibers are one-dimensional;
(iii) $\overset{\Pi}{\nabla}_{\phi U_1}\Pi_*(Z_1) = -Z_1(f)\Pi_*(\phi U_1)$, for all $U_1 \in \Gamma(\mathfrak{D}_2), Z_1 \in \Gamma(\mu)$ and $\xi \neq Z_1$.

Proof. Let Π be a CSIR map from a cosymplectic manifold to a Riemannian manifold. Then, for $V_1, V_2 \in \Gamma(\mathfrak{D}_2)$, using Equation (18) and Theorem 1, we obtain

$$\mathcal{T}_{V_1}V_2 = -g_1(V_1, V_2) grad f. \tag{45}$$

Taking the inner product of Equation (45) with ϕU_1, we have

$$g_1(\mathcal{T}_{V_1}V_2, \phi U_1) = -g_1(V_1, V_2)g_1(gradf, \phi U_1) \tag{46}$$

for all $U_1 \in \Gamma(\mathfrak{D}_2)$.

From Equations (4), (7), (13) and (46), we obtain

$$g_1(\nabla_{V_1}\phi V_2, U_1) = g_1(V_1, V_2)g_1(gradf, \phi U_1).$$

Since ∇ is a metric connection, by using Equations (14) and (45) in the above equation, we obtain

$$g_1(V_1, U_1)g_1(gradf, \phi V_2) = g_1(V_1, V_2)g_1(gradf, \phi U_1). \tag{47}$$

Taking $U_1 = V_2$ and interchanging the role of V_1 and V_2, we obtain

$$g_1(V_2, V_2)g_1(gradf, \phi V_1) = g_1(V_1, V_2)g_1(gradf, \phi V_2). \tag{48}$$

From Equations (47) and (48), we obtain

$$g_1(gradf, \phi V_1) = \frac{(g_1(V_1, V_2))^2}{||V_1||^2||V_2||^2}g_1(gradf, \phi V_1). \tag{49}$$

If $gradf \in \Gamma(\phi(\mathfrak{D}_2))$, then Equation (49) and the condition of equality in the Schwarz inequality imply that either f is constant on $\phi(\mathfrak{D}_2)$ or the fibers are one-dimensional. This implies the proof of (i) and (ii).

Now, from Equations (13) and (45), we obtain

$$g_1(\nabla_{V_1}U_1, Z_1) = -g_1(V_1, U_1)g_1(gradf, Z_1), \tag{50}$$

for all $Z_1 \in \Gamma(\mu)$ and $\xi \neq Z_1$. Using Equations (4), (7), and (50), we have

$$g_1(\nabla_{V_1}\phi U_1, \phi Z_1) = -g_1(V_1, U_1)g_1(gradf, Z_1),$$

which implies

$$g_1(\nabla_{\phi U_1}V_1, \phi Z_1) = -g_1(V_1, U_1)g_1(gradf, Z_1). \tag{51}$$

Since ∇ is a metric connection, then by using Equations (47) and (51), we have

$$g_1(\mathcal{H}\nabla_{\phi U_1}Z_1, \phi V_1) = -g_1(\phi V_1, \phi U_1)g_1(\ gradf, Z_1).$$

In addition, for the Riemannian map Π, we have

$$g_2(\Pi_*(\nabla^{\mathcal{M}_1}_{\phi U_1}Z_1), \Pi_*(\phi V_1)) = -g_2(\Pi_*(\phi V_1), \Pi_*(\phi U_1))g_1(gradf, Z_1). \tag{52}$$

Again, using Equations (19), (21) and (52), we obtain

$$g_2(\overset{\Pi}{\nabla}_{\phi U_1}\Pi_*(Z_1), \Pi_*(\phi V_1)) = -g_2(\Pi_*(\phi V_1), \Pi_*(\phi U_1))g_1(gradf, Z_1),$$

which implies.

$$\overset{\Pi}{\nabla}_{\phi U_1}\Pi_*(Z_1) = -Z_1(f)\Pi_*(\phi U_1). \tag{53}$$

If $gradf \in \Gamma(\mu)\backslash\{\xi\}$, then (53) implies (iii). This completes the proof. □

Corollary 3. *Let Π be a CSIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ with $r = e^f$ and $\dim(D_2) > 1$. Then, the fibers of Π are totally geodesic if and only if $\overset{\Pi}{\nabla}_{\phi U_1}\Pi_*(Z_1) = 0\ \forall U_1 \in \Gamma(\mathfrak{D}_2)$ and $Z_1 \in \Gamma(\mu)$.*

Lemma 5. *Let Π be a CSIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ with $r = e^f$ and $\dim(D_2) > 1$. Then, $\overset{\Pi}{\nabla}_{Z_1} \Pi_*(\phi X_1) = Z_1(f)\Pi_*(\phi X_1)$ $\forall X_1 \in \Gamma(\mathfrak{D}_2)$ and $Z_1 \in \Gamma(\ker \Pi_*)^\perp \backslash \{\xi\}$.*

Proof. Let Π be a CSIR map from a cosymplectic manifold to a Riemannian manifold. From Theorem 1, fibers are totally umbilical with mean curvature vector field $H = -gradf$, then we have

$$\begin{aligned} -g_1(\nabla_{X_1} Z_1, X_2) &= g_1(\nabla_{X_1} X_2, Z_1) \\ -g_1(\nabla_{X_1} Z_1, X_2) &= -g_1(X_1, X_2) g_1(gradf, Z_1) \end{aligned}$$

for all $X_1, X_2 \in \Gamma(\mathfrak{D}_2)$ and $Z_1 \in \Gamma(\ker \Pi_*)^\perp \backslash \{\xi\}$.

Using Equations (4) and (7) in the above equation, we obtain

$$g_1(\nabla_{Z_1} \phi X_1, \phi X_2) = g_1(\phi X_1, \phi X_2) g_1(gradf, Z_1). \tag{54}$$

Since Π is an SIR map, by using Equation (54), we have

$$g_2(\Pi_*(\nabla^{\Pi}_{Z_1} \phi X_1), \Pi_*(\phi X_2)) = g_2(\Pi_*(\phi X_1), \Pi_*(\phi X_2)) g_1(gradf, Z_1). \tag{55}$$

From (19) and (55), we obtain

$$g_2(\overset{\Pi}{\nabla}_{Z_1} \Pi_*(\phi X_1), \Pi_*(\phi X_2)) = g_2(\Pi_*(\phi X_1), \Pi_*(\phi X_2)) g_1(gradf, Z_1), \tag{56}$$

which implies $\overset{\Pi}{\nabla}_{Z_1} \Pi_*(\phi X_1) = Z_1(f)\Pi_*(\phi X_1)$ $\forall X_1 \in \Gamma(\mathfrak{D}_2)$ and $Z_1 \in \Gamma(\ker \Pi_*)^\perp \backslash \{\xi\}$. □

Corollary 4. *Let Π be a CSIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ with $r = e^f$ and $\dim(D_2) > 1$. Then, $\overset{\Pi}{\nabla}_{Z_1} \Pi_*(\phi X_1) = 0$ $\forall X_1 \in \Gamma(\mathfrak{D}_2)$ and $Z_1 = \xi \in \Gamma(\ker \Pi_*)^\perp$.*

Theorem 4. *Let Π be a CSIR map with $r = e^f$ from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$. If $\mathcal{T}$ is not identically zero, then the invariant distribution $\mathfrak{D}_1$ does not define a totally geodesic foliation on $\mathcal{M}_1$.*

Proof. For $V_1, V_2 \in \Gamma(\mathfrak{D}_1)$ and $Z_1 \in \Gamma(\mathfrak{D}_2)$, using Equations (4), (7), (13) and (18), we obtain

$$\begin{aligned} g_1(\nabla_{V_1} V_2, Z_1) &= g_1(\nabla_{V_1} \phi V_2, \phi Z_1) \\ &= g_1(\mathcal{T}_{V_1} \phi V_2, \phi Z_1) \\ &= -g_1(V_1, \phi V_2) g_1(gradf, \phi Z_1). \end{aligned}$$

Thus, the assertion can be seen from the above equation and the fact that $gradf \in \phi(\mathfrak{D}_2)$. □

Theorem 5. *Let Π be a CSIR map with $r = e^f$ from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$. Then, the fibers of Π are totally geodesic, or anti-invariant distribution $\mathfrak{D}_2$ is one-dimensional.*

Proof. If the fibers of Π are totally geodesic, it is obvious. For the second one, since Π is a Clairaut proper semi-invariant Riemannian map, then either $\dim(\mathfrak{D}_2) = 1$ or $\dim(\mathcal{D}_2) > 1$.

If $\dim(\mathcal{D}_2) > 1$, then we can choose $V_1, V_2 \in \Gamma(\mathfrak{D}_2)$ such that $\{V_1, V_2\}$ is orthonormal. From Equations (14), (23) and (24), we obtain

$$\begin{aligned} \mathcal{T}_{V_1}\phi V_2 + \mathcal{H}\nabla_{V_1}\phi V_2 &= \nabla_{V_1}\phi V_2 \\ \mathcal{T}_{V_1}\phi V_2 + \mathcal{H}\nabla_{V_1}\phi V_2 &= B\mathcal{T}_{V_1}V_2 + C\mathcal{T}_{V_1}V_2 + \psi\mathcal{V}\nabla_{V_1}V_2 + \omega\mathcal{V}\nabla_{V_1}V_2. \end{aligned}$$

Taking the inner product of the above equation with V_1, we obtain

$$g_1(\mathcal{T}_{V_1}\phi V_2, V_1) = g_1(B\mathcal{T}_{V_1}V_2, V_1) + g_1(\psi\mathcal{V}\nabla_{V_1}V_2, V_1). \quad (57)$$

From Equation (7), we have

$$g_1(\mathcal{T}_{V_1}V_1, \phi V_2) = -g_1(\mathcal{T}_{V_1}\phi V_2, V_1) = g_1(\mathcal{T}_{V_1}V_2, \phi V_1). \quad (58)$$

Now, using Equations (18) and (58), we obtain

$$g_1(\mathcal{T}_{V_1}V_1, \phi V_2) = -g_1(gradf, \phi V_2). \quad (59)$$

From Equations (18), (58) and (59), we obtain

$$-g_1(gradf, \phi V_2) = g_1(\mathcal{T}_{V_1}V_1, \phi V_2) = -g_1(\mathcal{T}_{V_1}\phi V_2, V_1) = g_1(\mathcal{T}_{V_1}V_2, \phi V_1), \quad (60)$$

from which we obtain

$$\begin{aligned} g_1(gradf, \phi V_2) &= -g_1(\mathcal{T}_{V_1}V_2, \phi V_1) \\ g_1(gradf, \phi V_2) &= g_1(V_1, V_2)g_1(gradf, \phi V_1) \\ g_1(gradf, \phi V_2) &= 0. \end{aligned}$$

Thus, we obtain

$$gradf \perp \phi(\mathfrak{D}_2).$$

Therefore, the dimension of $\mathfrak{D}_2$ must be one. □

Theorem 6. *Let Π be a CSIR map from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ with $r = e^f$ and $\dim(D_2) > 1$. Then, we obtain*

$$\sum_{\kappa=1}^{\omega} g_1(\mathcal{A}_{X_1}x_\kappa, \mathcal{A}_{X_1}x_\kappa) = \sum_{\kappa=1}^{\omega} g_2(\nabla^{\Pi}_{X_1}\Pi_*(\phi x_\kappa), \nabla^{\Pi}_{X_1}\Pi_*(\phi x_\kappa)), \quad (61)$$

$$\sum_{i=1}^{\beta+\int} g_2((\nabla\Pi_*)(F_i, X_1), (\nabla\Pi_*)(X_1, F_i)) = \sum_{l=1}^{\int} g_2((\nabla\Pi_*)(\vartheta_l, X_1), (\nabla\Pi_*)(X_1, \vartheta_l)), \quad (62)$$

$$\sum_{j=1}^{\beta} g_1(\mathcal{A}_{X_1}w_j, \mathcal{A}_{X_1}w_j) = (X_1(f))^2 \sum_{j=1}^{\beta} g_1(w_j, w_j), \quad (63)$$

$\forall\ X_1 \in \Gamma(\ker\Pi_*)^{\perp}\backslash\{\xi\}$, where $\{x_1, x_2, ..., x_\omega\}, \{w_1, w_2, ..., w_\beta\}, \{F_1, F_2, ..., F_{\beta+\int}\}$ and $\{\vartheta_1, \vartheta_2, ...\vartheta_\int\}$ are orthonormal frames of $\mathfrak{D}_1, \mathfrak{D}_2, \phi(\mathfrak{D}_2)^{\perp} \oplus \mu$ and μ, respectively.

Proof. Let $\Pi : (\mathcal{M}_1, \phi, \xi, \eta, g_1) \to (\mathcal{M}_2, g_2)$ be a CSIR map, then for all $X_1 \in \Gamma(\ker\Pi_*)^{\perp}\backslash\{\xi\}$, we have

$$\sum_{\kappa=1}^{\omega} g_1(\mathcal{A}_{X_1}x_\kappa, \mathcal{A}_{X_1}x_\kappa) = \sum_{\kappa=1}^{\omega} g_1(\mathcal{H}\nabla_{X_1}\phi x_\kappa, \mathcal{H}\nabla_{X_1}\phi x_\kappa). \quad (64)$$

Since Π is a Riemannian map, in view of Equation (19), Equation (64) transforms to

$$\sum_{\kappa=1}^{\omega} g_1(\mathcal{A}_{X_1}x_\kappa, \mathcal{A}_{X_1}x_\kappa) = \sum_{\kappa=1}^{\omega} g_2(\Pi_*(\nabla^{\mathcal{M}_1}_{X_1}\phi x_\kappa), \Pi_*(\nabla^{\mathcal{M}_1}_{X_1}\phi x_\kappa))$$
$$= \sum_{\kappa=1}^{\omega} g_2(\nabla^{\Pi}_{X_1}\Pi_*(\phi x_\kappa), \nabla^{\Pi}_{X_1}\Pi_*(\phi x_\kappa)).$$

Now, for all $X_1 \in \Gamma(\ker\Pi_*)^\perp \backslash \{\xi\}$, we obtain

$$\sum_{i=1}^{\beta+\int} g_2((\nabla\Pi_*)(F_i, X_1), (\nabla\Pi_*)(X_1, F_i))$$
$$= \sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\Pi_*)(\phi w_j + \vartheta_l, X_1), (\nabla\Pi_*)(X_1, \phi w_j + \vartheta_l)).$$

Since $\phi w_j \in \Gamma(\ker\Pi_*)^\perp$ and $(\nabla\Pi_*)$ is linear, from the above equation, we have

$$\sum_{i=1}^{\beta+\int} g_2((\nabla\Pi_*)(F_i, X_1), (\nabla\Pi_*)(F_i, X_1)) \tag{65}$$
$$= \sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), (\nabla\Pi_*)(X_1, \phi w_j))$$
$$+ \sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\Pi_*)(\vartheta_l, X_1), (\nabla\Pi_*)(X_1, \phi w_j))$$
$$+ \sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\Pi_*)(\phi w_j, X_1), (\nabla\Pi_*)(X_1, \vartheta_l))$$
$$+ \sum_{l=1}^{\int} g_2((\nabla\Pi_*)(\vartheta_l, X_1), (\nabla\Pi_*)(X_1, \vartheta_l)).$$

Thus, (61) holds.
On the other side, using (19) in the first term of the right-hand side of (65), we have

$$\sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), (\nabla\Pi_*)(X_1, \phi w_j))$$
$$= \sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), \overset{\Pi}{\nabla}_{X_1}\Pi_*(\phi w_j) - \Pi_*(\nabla^{\mathcal{M}_1}_{X_1}\phi w_j)),$$

which, by using Equations (4), (7), and (65), turns into

$$\sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), (\nabla\Pi_*)(X_1, \phi w_j)) \tag{66}$$
$$= \sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), \overset{\Pi}{\nabla}_{X_1}\Pi_*(\phi w_j)).$$

Now, by using Lemma 4 in (66), we obtain

$$\sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), (\nabla\Pi_*)(X_1, \phi w_j)) = \sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), X_1(f)\Pi_*(\phi w_j)).$$

This implies that

$$\sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), (\nabla\Pi_*)(X_1, \phi w_j)) = \sum_{j=1}^{\beta} X_1(f) g_2((\nabla\Pi_*)(\phi w_j, X_1), \Pi_*(\phi w_j)).$$

By using Equation (20) in the above equation, it follows that

$$\sum_{j=1}^{\beta} g_2((\nabla\Pi_*)(\phi w_j, X_1), (\nabla\Pi_*)(X_1, \phi w_j)) = 0. \tag{67}$$

Similarly, we find

$$\sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\Pi_*)(\vartheta_l, X_1), (\nabla\Pi_*)(X_1, \phi w_j)) = 0, \tag{68}$$

$$\sum_{j=1}^{\beta}\sum_{l=1}^{\int} g_2((\nabla\Pi_*)(\phi w_j, X_1), (\nabla\Pi_*)(X_1, \vartheta_l)) = 0. \tag{69}$$

Thus, by using Equations (67)–(69) in Equation (65), we obtain (62).
Further, for $X_1 \in \Gamma(\ker\Pi_*)^{\perp} \backslash \{\xi\}$, we obtain

$$\begin{aligned}\sum_{j=1}^{\beta} g_1(\mathcal{A}_{X_1} w_j, \mathcal{A}_{X_1} w_j) &= \sum_{j=1}^{\beta} g_1(\mathcal{H}\nabla_{X_1} w_j, \mathcal{H}\nabla_{X_1} w_j) \\ &= \sum_{j=1}^{\beta} g_1(\mathcal{H}\nabla_{X_1}\phi w_j, \mathcal{H}\nabla_{X_1}\phi w_j).\end{aligned}$$

Since Π is a Riemannian map, in view of Equation (19), the above equation becomes

$$\begin{aligned}&\sum_{j=1}^{\beta} g_1(\mathcal{A}_{X_1} w_j, \mathcal{A}_{X_1} w_j) \\ =& \sum_{j=1}^{\beta}\{g_2((\nabla\Pi_*)(X_1, \phi w_j), (\nabla\Pi_*)(X_1, \phi w_j)) \\ &-2g_2((\nabla\Pi_*)(X_1, \phi w_j), \overset{\Pi}{\nabla}_{X_1}\Pi_*(\phi w_j)) \\ &+g_2(\overset{\Pi}{\nabla}_{X_1}\Pi_*(\phi w_j), \overset{\Pi}{\nabla}_{X_1}\Pi_*(\phi w_j))\},\end{aligned} \tag{70}$$

which, by using Lemma 4 and Equations (21) and (67) in (70), we obtain

$$\begin{aligned}\sum_{j=1}^{\beta} g_1(\mathcal{A}_{X_1} w_j, \mathcal{A}_{X_1} w_j) &= \sum_{j=1}^{\beta} g_2(X_1(f)\Pi_*(\phi w_j), X_1(f)\Pi_*(\phi w_j)) \\ &= (X_1(f))^2 \sum_{j=1}^{\beta} g_2(\Pi_*(\phi w_j), \Pi_*(\phi w_j)).\end{aligned} \tag{71}$$

Since $\phi w_j \in \Gamma(\ker\Pi_*)^{\perp}$ and Π is a Riemannian map, from (71) we obtain

$$\sum_{j=1}^{\beta} g_1(\mathcal{A}_{X_1} w_j, \mathcal{A}_{X_1} w_j) = (X_1(f))^2 \sum_{j=1}^{\beta} g_1(w_j, w_j). \tag{72}$$

Thus, from Equations (4) and (72), we obtain (63). □

4. Example

Let $\mathcal{M}_1$ be a differentiable manifold given by $\mathcal{M}_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in R^7 : x_7 > 0\}$. We define the Riemannian metric g_1 on $\mathcal{M}_1$ by $g_1 = e^{2x_7}dx_1^2 + e^{2x_7}dx_2^2 + e^{2x_7}dx_3^2 + e^{2x_7}dx_4^2 + e^{2x_7}dx_5^2 + e^{2x_7}dx_6^2 + dx_7^2$, and the cosymplectic structure (ϕ, ξ, η, g_1) on $\mathcal{M}_1$ is defined as

$$\phi(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial x_5} + x_6\frac{\partial}{\partial x_6} + x_7\frac{\partial}{\partial x_7})$$
$$= (x_4\frac{\partial}{\partial x_1} + x_5\frac{\partial}{\partial x_2} + x_6\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_4} - x_2\frac{\partial}{\partial x_5} - x_3\frac{\partial}{\partial x_6}),$$

$\xi = \frac{\partial}{\partial x_7}$, $\eta = dx_7$, and g_1 was earlier defined.

Let $\mathcal{M}_2 = \{(v_1, v_2, v_3, v_4) \in R^4\}$ be a Riemannian manifold with Riemannian metric g_2 on $\mathcal{M}_2$ given by $g_2 = e^{2x_7}dv_1^2 + e^{2x_7}dv_2^2 + e^{2x_7}dv_3^2 + dv_4^2$. Define a map $\Pi : R^7 \to R^4$ by

$$\Pi(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (\frac{x_2 - x_5}{\sqrt{2}}, 101, x_6, x_7).$$

Then, we have

$$\ker \Pi_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2,$$

where

$$\mathfrak{D}_1 =< V_1 = e_1, V_2 = e_4 >, \mathfrak{D}_2 =< V_3 = e_2 + e_5, V_4 = e_3 >,$$

and

$$(\ker \Pi_*)^{\perp} =< H_1 = e_2 - e_5, H_2 = e_6, H_3 = e_7 >,$$

where $\{e_1 = e^{-x_7}\frac{\partial}{\partial x_1}, e_2 = e^{-x_7}\frac{\partial}{\partial x_2}, e_3 = e^{-x_7}\frac{\partial}{\partial x_3}, e_4 = e^{-x_7}\frac{\partial}{\partial x_4}, e_5 = e^{-x_7}\frac{\partial}{\partial x_5}, e_6 = e^{-x_7}\frac{\partial}{\partial x_6}, e_7 = \frac{\partial}{\partial x_7}\}$, $\{e_1^* = \frac{\partial}{\partial v_1}, e_2^* = \frac{\partial}{\partial v_2}, e_3^* = \frac{\partial}{\partial v_3}, e_4^* = \frac{\partial}{\partial v_4}\}$ are bases on $T_p\mathcal{M}_1$ and $T_{\Pi(p)}\mathcal{M}_2$, respectively, for all $p \in \mathcal{M}_1$. By direct computations, we can see that $\Pi_*(H_1) = \sqrt{2}e^{-x_7}e_1^*$, $\Pi_*(H_2) = e^{-x_7}e_2^*, \Pi_*(H_3) = e_3^*$ and $g_1(H_i, H_j) = g_2(\Pi_* H_i, \Pi_* H_j)$ for all $H_i, H_j \in \Gamma(\ker \Pi_*)^{\perp}$, $i, j = 1, 2, 3$. Thus, Π is a Riemannian map with $(range\Pi_*)^{\perp} =< e_4^* >$. Moreover, it is easy to see that $\phi V_3 = H_1$ and $\phi V_4 = H_2$. Therefore, Π is an SIR map.

Now, we will find the smooth function f on $\mathcal{M}_1$ satisfying $T_V V = g_1(V, V)\nabla f$ $\forall V \in \Gamma(\ker \Pi_*)$. The covariant derivative for the vector fields $E = E_i\frac{\partial}{\partial x_i}, F = F_j\frac{\partial}{\partial x_j}$ on $\mathcal{M}_1$ is defined as

$$\nabla_E F = E_i F_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + E_i \frac{\partial F_j}{\partial x_i}\frac{\partial}{\partial x_j}, \tag{73}$$

where the covariant derivative of basis vector fields $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial x_i}$ is defined by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \tag{74}$$

and Christoffel symbols are defined by

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\frac{\partial g_{1jl}}{\partial x_i} + \frac{\partial g_{1il}}{\partial x_j} - \frac{\partial g_{1ij}}{\partial x_l}). \tag{75}$$

Thus, we obtain

$$g_{1ij} = \begin{bmatrix} e^{2x_7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2x_7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2x_7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2x_7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2x_7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2x_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{76}$$

$$g_1^{ij} = \begin{bmatrix} e^{-2x_7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-2x_7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2x_7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2x_7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-2x_7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2x_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By using Equations (75) and (76), we find

$$\begin{aligned}
\Gamma^1_{11} &= \Gamma^2_{11} = \Gamma^3_{11} = \Gamma^4_{11} = \Gamma^5_{11} = \Gamma^6_{11} = 0, \Gamma^7_{11} = -e^{2x_7}, \\
\Gamma^1_{22} &= \Gamma^2_{22} = \Gamma^3_{22} = \Gamma^4_{22} = \Gamma^5_{22} = \Gamma^6_{22} = 0, \Gamma^7_{22} = -e^{2x_7}, \\
\Gamma^1_{33} &= \Gamma^2_{33} = \Gamma^3_{33} = \Gamma^4_{33} = \Gamma^5_{33} = \Gamma^6_{33} = 0, \Gamma^7_{33} = -e^{2x_7}, \\
\Gamma^1_{44} &= \Gamma^2_{44} = \Gamma^3_{44} = \Gamma^4_{44} = \Gamma^5_{44} = \Gamma^6_{44} = 0, \Gamma^7_{44} = -e^{2x_7}, \\
\Gamma^1_{55} &= \Gamma^2_{55} = \Gamma^3_{55} = \Gamma^4_{55} = \Gamma^5_{55} = \Gamma^6_{55} = 0, \Gamma^7_{55} = -e^{2x_7}, \\
\Gamma^1_{12} &= \Gamma^2_{12} = \Gamma^3_{12} = \Gamma^4_{12} = \Gamma^5_{12} = \Gamma^6_{12} = \Gamma^7_{12} = 0, \\
\Gamma^1_{21} &= \Gamma^2_{21} = \Gamma^3_{21} = \Gamma^4_{21} = \Gamma^5_{21} = \Gamma^6_{21} = \Gamma^7_{21} = 0, \\
\Gamma^1_{13} &= \Gamma^2_{13} = \Gamma^3_{13} = \Gamma^4_{13} = \Gamma^5_{13} = \Gamma^6_{13} = \Gamma^7_{13} = 0, \\
\Gamma^1_{31} &= \Gamma^2_{31} = \Gamma^3_{31} = \Gamma^4_{31} = \Gamma^5_{31} = \Gamma^6_{31} = \Gamma^7_{31} = 0, \\
\Gamma^1_{14} &= \Gamma^2_{14} = \Gamma^3_{14} = \Gamma^4_{14} = \Gamma^5_{14} = \Gamma^6_{14} = \Gamma^7_{14} = 0, \\
\Gamma^1_{41} &= \Gamma^2_{41} = \Gamma^3_{41} = \Gamma^4_{41} = \Gamma^5_{41} = \Gamma^6_{41} = \Gamma^7_{41} = 0, \\
\Gamma^1_{15} &= \Gamma^2_{15} = \Gamma^3_{15} = \Gamma^4_{15} = \Gamma^5_{15} = \Gamma^6_{15} = \Gamma^7_{15} = 0, \\
\Gamma^1_{51} &= \Gamma^2_{51} = \Gamma^3_{51} = \Gamma^4_{51} = \Gamma^5_{51} = \Gamma^6_{51} = \Gamma^7_{51} = 0, \\
\Gamma^1_{23} &= \Gamma^2_{23} = \Gamma^3_{23} = \Gamma^4_{23} = \Gamma^5_{23} = \Gamma^6_{23} = \Gamma^7_{23} = 0, \\
\Gamma^1_{32} &= \Gamma^2_{32} = \Gamma^3_{32} = \Gamma^4_{32} = \Gamma^5_{32} = \Gamma^6_{32} = \Gamma^7_{32} = 0, \\
\Gamma^1_{24} &= \Gamma^2_{24} = \Gamma^3_{24} = \Gamma^4_{24} = \Gamma^5_{24} = \Gamma^6_{24} = \Gamma^7_{24} = 0, \\
\Gamma^1_{42} &= \Gamma^2_{42} = \Gamma^3_{42} = \Gamma^4_{42} = \Gamma^5_{42} = \Gamma^6_{42} = \Gamma^7_{42} = 0, \\
\Gamma^1_{25} &= \Gamma^2_{25} = \Gamma^3_{25} = \Gamma^4_{25} = \Gamma^5_{25} = \Gamma^6_{25} = \Gamma^7_{25} = 0, \\
\Gamma^1_{52} &= \Gamma^2_{52} = \Gamma^3_{52} = \Gamma^4_{52} = \Gamma^5_{52} = \Gamma^6_{52} = \Gamma^7_{52} = 0, \\
\Gamma^1_{34} &= \Gamma^2_{34} = \Gamma^3_{34} = \Gamma^4_{34} = \Gamma^5_{34} = \Gamma^6_{34} = \Gamma^7_{34} = 0, \\
\Gamma^1_{43} &= \Gamma^2_{43} = \Gamma^3_{43} = \Gamma^4_{43} = \Gamma^5_{43} = \Gamma^6_{43} = \Gamma^7_{43} = 0, \\
\Gamma^1_{35} &= \Gamma^2_{35} = \Gamma^3_{35} = \Gamma^4_{35} = \Gamma^5_{35} = \Gamma^6_{35} = \Gamma^7_{35} = 0, \\
\Gamma^1_{53} &= \Gamma^2_{53} = \Gamma^3_{53} = \Gamma^4_{53} = \Gamma^5_{53} = \Gamma^6_{53} = \Gamma^7_{53} = 0, \\
\Gamma^1_{45} &= \Gamma^2_{45} = \Gamma^3_{45} = \Gamma^4_{45} = \Gamma^5_{45} = \Gamma^6_{45} = \Gamma^7_{45} = 0, \\
\Gamma^1_{54} &= \Gamma^2_{54} = \Gamma^3_{54} = \Gamma^4_{54} = \Gamma^5_{54} = \Gamma^6_{54} = \Gamma^7_{54} = 0.
\end{aligned} \tag{77}$$

Using Equations (73), (74) and (77), we calculate

$$\begin{aligned}
\nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = -\frac{\partial}{\partial x_7}, \\
\nabla_{e_1} e_2 &= \nabla_{e_1} e_3 = \nabla_{e_1} e_4 = \nabla_{e_2} e_1 = \nabla_{e_2} e_3 = \nabla_{e_2} e_4 = 0, \\
\nabla_{e_3} e_1 &= \nabla_{e_3} e_2 = \nabla_{e_3} e_4 = \nabla_{e_4} e_1 = \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = 0.
\end{aligned} \tag{78}$$

Thus, we find

$$\nabla_{V_1}V_1 = \nabla_{e_1}e_1 = -\frac{\partial}{\partial x_7}, \nabla_{V_2}V_2 = \nabla_{e_4}e_4 = -\frac{\partial}{\partial x_7}, \quad (79)$$
$$\nabla_{V_3}V_3 = \nabla_{e_2+e_5}e_2+e_5 = -2\frac{\partial}{\partial x_7}, \nabla_{V_4}V_4 = \nabla_{e_3}e_3 = -\frac{\partial}{\partial x_7}.$$

Now, from

$$\mathcal{T}_V V = \mathcal{T}_{\lambda_1 V_1+\lambda_2 V_2+\lambda_3 V_3+\lambda_4 V_4}\lambda_1 V_1+\lambda_2 V_2+\lambda_3 V_3+\lambda_4 V_4, \lambda_1,\lambda_2,\lambda_3,\lambda_4 \in R,$$

we lead to

$$\begin{aligned}\mathcal{T}_V V &= \lambda_1^2\mathcal{T}_{V_1}V_1+\lambda_2^2\mathcal{T}_{V_2}V_2+\lambda_3^2\mathcal{T}_{V_3}V_3+\lambda_4^2\mathcal{T}_{V_4}V_4 \\ &+2\lambda_1\lambda_2\mathcal{T}_{V_1}V_2+2\lambda_1\lambda_3\mathcal{T}_{V_1}V_3+2\lambda_1\lambda_4\mathcal{T}_{V_1}V_4+2\lambda_2\lambda_3\mathcal{T}_{V_2}V_3 \\ &+2\lambda_2\lambda_4\mathcal{T}_{V_2}V_4+2\lambda_3\lambda_4\mathcal{T}_{V_3}V_4.\end{aligned} \quad (80)$$

From Equations (13) and (79), we obtain

$$\mathcal{T}_{V_1}V_1 = -\frac{\partial}{\partial x_7}, \mathcal{T}_{V_2}V_2 = -\frac{\partial}{\partial x_7}, \mathcal{T}_{V_3}V_3 = -2\frac{\partial}{\partial x_7}, \mathcal{T}_{V_4}V_4 = -\frac{\partial}{\partial x_7}, \quad (81)$$
$$\mathcal{T}_{V_1}V_2 = 0, \mathcal{T}_{V_1}V_3 = 0, \mathcal{T}_{V_1}V_4 = 0, \mathcal{T}_{V_2}V_3 = 0, \mathcal{T}_{V_2}V_4 = 0, \mathcal{T}_{V_3}V_4 = 0.$$

Thus, by using Equations (80) and (81), we obtain

$$\mathcal{T}_V V = -(\lambda_1^2+\lambda_2^2+2\lambda_3^2+\lambda_1^4)\frac{\partial}{\partial x_7}. \quad (82)$$

Since $V = \lambda_1 V_1+\lambda_2 V_2+\lambda_3 V_3+\lambda_4 V_4$, $g_1(\lambda_1 V_1+\lambda_2 V_2+\lambda_3 V_3+\lambda_4 V_4, \lambda_1 V_1+\lambda_2 V_2+\lambda_3 V_3+\lambda_4 V_4) = \lambda_1^2+\lambda_2^2+2\lambda_3^2+\lambda_1^4$. For any smooth function f on R^7, the gradient of f with respect to the metric g_1 is given by $\nabla f = \sum_{i,j=1}^{7} g_1^{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$. Hence, $\nabla f = e^{-2x_7}\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + e^{-2x_7}\frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_2} + e^{-2x_7}\frac{\partial f}{\partial x_3}\frac{\partial}{\partial x_3} + e^{-2x_7}\frac{\partial f}{\partial x_4}\frac{\partial}{\partial x_4} + e^{-2x_7}\frac{\partial f}{\partial x_5}\frac{\partial}{\partial x_5} + e^{-2x_7}\frac{\partial f}{\partial x_6}\frac{\partial}{\partial x_6} + \frac{\partial f}{\partial x_7}\frac{\partial}{\partial x_7}$. Hence, $\nabla f = \frac{\partial}{\partial x_7}$ for the function $f = x_7$. Then, it is easy to see that $\mathcal{T}_V V = -g_1(V,V)\nabla f$; thus, by Theorem 1, Π is a CSIR map from cosymplectic manifold onto Riemannian manifold.

5. Conclusions

In the last few years, Riemannian maps have been extensively studied between different kinds of the manifolds. Recently, a special type of Riemannian map, namely, the "Clairaut Riemannian map" was introduced and studied by Sahin [30]; moreover, he, in [37], gave an open problem to find characterizations for Clairaut Riemannian maps. As a continuation of this study, we tried to study Clairaut semi-invariat Riemannian maps in contact geometry. Here, we investigated the various most fundamental geometric properties on the fibers and distributions of these maps. In the future, we plan to focus on studying Clairaut's semi-slant Riemannian maps, Clairaut's hemi-slant Riemannian maps, and Clairaut's bi-slant Riemannian maps between different kinds of the manifolds.

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