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Soft Regular Generalized ω-Closed Sets and Soft ω-$T_{1/2}$ Spaces

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Abstract: Soft $rg\omega$-closed sets are introduced as a new class of soft sets that strictly contain the classes of soft rg-closed sets and soft $g\omega$-closed sets. Furthermore, the behavior of soft $rg\omega$-closed sets with respect to soft unions, soft intersections, and soft subspaces, as well as induced soft topologies are investigated. Moreover, soft ω-$T_{1/2}$ spaces which is a weaker form soft $T_{1/2}$ spaces is defined and investigated. In addition to these, the characterizations of soft rg-$T_{1/2}$ spaces and soft $rg\omega$-$T_{1/2}$ spaces are discussed. The work also looks at the relationship between our novel notions in soft topological spaces and their analogs in topological spaces.

Keywords: $g\omega$-closed sets; rg-closed sets; $T_{1/2}$ spaces; soft $g\omega$-closed sets; soft rg-closed sets; soft $T_{1/2}$ spaces

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1. Introduction and Preliminaries

Classical mathematical theories fail to tackle complex problems with uncertain data in a variety of fields, including engineering, the environment, economics, medical science, social science, and others. Probability theories, fuzzy sets [1], rough sets [2], intuitionistic fuzzy sets [3], and vague sets [4] are all considered mathematical tools for dealing with uncertainty. According to Molodtsov [5], each of these structures has its own set of difficulties. These difficulties are mostly due to shortcomings in the parameterization tool for theories. Molodtsov [5] defined soft sets to deal with uncertainty away from these difficulties. Several researchers have presented and investigated the theory of soft sets (see [6,7]). Soft sets were used in a range of applications by the authors [5,8], including operation research, game theory, Riemann integration, Perron integration, smoothness of function, probability, and measurement theory. In addition, the authors [9] used soft sets to solve decision-making difficulties.

Several researchers have used soft set theory to study various mathematical structures. Soft topology is one of the structures presented as a novel expansion of classical topology by Shabir and Naz [10]. Many classical topological notions have been developed and expanded in soft set contexts (see, [11–25]), but significant additions remain potential. Thus, the study of soft topology is a current trend among topological researchers.

We expect that by defining a reasonable new class of soft sets in soft topological spaces, we will pave the way for a lot of future research articles on the subject. Soft ω-open sets, for example, were presented in [26] as a generalization of soft open sets in soft topological spaces, and other related research articles, such as [14,24,25], have appeared. In this paper, we will combine the classes of soft regular open sets and soft ω-closed sets to define soft regular generalized ω-closed sets as a new class of sets that contains the classes of soft regular closed sets and soft regular generalized ω-closed sets. We will also introduce several results for some weaker forms of soft $T_{1/2}$ spaces.

In Section 2, we introduce the notion of soft regular generalized ω-closed sets and illustrate its relationships with each of the soft regular closed sets and soft regular generalized ω-closed sets as two well-known classes of soft sets. Additionally, we investigate the main properties of this new class of soft sets.

In Section 3, we introduce and investigate soft ω-$T_{1/2}$ spaces, soft regular generalized $T_{1/2}$, and soft regular generalized ω-$T_{1/2}$ as three new classes of STSs. We focus on their characterizations.

In this paper, STS and TS will be used in this study to signify soft topological space and topological space, respectively. We shall use the concepts and terminologies from [14,26,27] throughout this paper.

Now, we shall recollect several notions that will be employed in the sequel.

Definition 1 ([28])**.** *Let (Y, λ) be a TS, $D \subseteq Y$, and $y \in Y$. Then y is a condensation point of D if for each $U \in \lambda$ with $y \in U$, the set $U \cap D$ is uncountable. D is called an ω-closed set in (Y, λ) if it contains all its condensation points. D is called an ω-open set in (Y, λ) if $Y - D$ is an ω-closed set in (Y, λ). The collection of all ω-open sets in (Y, λ) is denoted by λ_ω.*

Definition 2. *Let (Y, λ) be a TS and let $V \subseteq Y$. Then V is called*

(a) Ref. [29] a generalized closed (briefly: g-closed) set in (Y, λ) if $Cl_\lambda(V) \subseteq U$ whenever $U \in \lambda$ and $V \subseteq U$. The collection of all g-closed sets in (Y, λ) is denoted by $GC(Y, \lambda)$.

(b) Ref. [29] a generalized open (briefly: g-open) set in (Y, λ) if $Y - V \in GC(Y, \lambda)$. The collection of all g-open sets in (Y, λ) is denoted by $GO(Y, \lambda)$.

(c) Ref. [30] a generalized ω-closed (briefly: gω-closed) set in (Y, λ) if $Cl_{\lambda_\omega}(V) \subseteq U$ whenever $U \in \lambda$ and $V \subseteq U$. The collection of all gω-closed sets in (Y, λ) is denoted by $G\omega C(Y, \lambda)$.

(d) Ref. [30] a generalized ω-open (briefly: gω-open) set in (Y, λ) if $Y - V \in G\omega C(Y, \lambda)$. The collection of all gω-open sets in (Y, λ) is denoted by $G\omega O(Y, \lambda)$.

(e) Ref. [31] a regular open set in (Y, λ) if $Int_\lambda(Cl_\lambda(V)) = V$. The collection of all regular open sets in (Y, λ) is denoted by $RO(Y, \lambda)$. Complements of regular open sets are called regular closed sets. The collection of all regular closed sets in (Y, λ) is denoted by $RC(Y, \lambda)$.

(f) Ref. [32] a regular generalized closed (briefly: rg-closed) set in (Y, λ) if $Cl_\lambda(V) \subseteq U$ whenever $U \in RO(Y, \lambda)$ and $V \subseteq U$. The collection of all rg-closed sets in (Y, λ) is denoted by $RGC(Y, \lambda)$. Complements of rg-closed sets are called rg-open sets. The collection of all rg-open sets in (Y, λ) is denoted by $RGO(Y, \lambda)$.

(g) Ref. [33] a regular generalized ω-closed (briefly: rgω-closed) set in (Y, λ) if $Cl_{\lambda_\omega}(V) \subseteq U$ whenever $U \in RO(Y, \lambda)$ and $V \subseteq U$. The collection of all rgω-closed sets in (Y, λ) is denoted by $RG\omega C(Y, \lambda)$. Complements of rgω-closed sets are called rgω-open sets. The collection of all rg-open sets in (Y, λ) is denoted by $RG\omega O(Y, \lambda)$.

Definition 3. *Let Y be a universal set and E be a set of parameters. Then $G \in SS(Y, E)$ defined by*

(a) Ref. [27] $G(e) = \begin{cases} Z & \text{if } e = b \\ \emptyset & \text{if } e \neq b \end{cases}$ *is denoted by b_Z.*

(b) Ref. [27] $G(e) = Z$ for every $e \in E$ is denoted by C_Z.

(c) Ref. [34] $G(e) = \begin{cases} \{y\} & \text{if } e = b \\ \emptyset & \text{if } e \neq b \end{cases}$ *is denoted by b_y and is called a soft point.*

The set of all soft points in $SS(Y, B)$ is denoted $SP(Y, B)$.

Definition 4 ([34])**.** *Let $H \in SS(Y, E)$ and $b_y \in SP(Y, E)$. Then b_y is said to belong to H (notation: $b_y \tilde{\in} H$) if $b_y \tilde{\subseteq} H$ or equivalently: $b_y \tilde{\in} H$ if and only if $y \in H(b)$.*

Theorem 1 ([5])**.** *Let (Y, γ, A) be a STS. Then for each $e \in E$, the collection $\{H(e) : H \in \gamma\}$ defines a topology on Y. This topology is called an induced topology on Y and is denoted by γ_e.*

Theorem 2 ([35])**.** *For any TS (Y, λ) and any set of parameters E. The collection*

$$\{K \in SS(Y, E) : K(e) \in \lambda \text{ for each } e \in E\}$$

forms a soft topology on Y relative to E. This soft topology is denoted by $\tau(\lambda)$.

Theorem 3 ([27])**.** *For any collection of TSs* $\{(Y,\lambda_e) : e \in E\}$*, the family*

$$\{K \in SS(Y,E) : K(e) \in \lambda_e \text{ for all } e \in E\}$$

forms a soft topology on Y relative to E. This soft topology is denoted by $\underset{e\in E}{\oplus}\lambda_e$*.*

Definition 5 ([27])**.** *Let* (Y,γ,E) *be a STS and let* $K \in SS(Y,E)$*. Then K is called a soft ω-open set in* (Y,γ,E) *if for each* $e_y\widetilde{\in}K$*, there exist* $G \in \gamma$ *and* $N \in CSS(Y,A)$ *such that* $e_y\widetilde{\in}G - N\widetilde{\subseteq}K$*. The family of all soft ω-open set in* (Y,γ,E) *is denoted by* γ_ω*.*

Definition 6. *Let* (Y,γ,E) *be a STS and let* $K \in SS(Y,E)$*. Then K is called*

(a) *Ref. [36] a soft generalized closed (briefly: soft g-closed) set in* (Y,γ,E) *if* $Cl_\gamma(K)\widetilde{\subseteq}H$ *whenever* $H \in \gamma$ *and* $K\widetilde{\subseteq}H$*. The collection of all soft g-closed sets in* (Y,γ,E) *is denoted by* $GC(Y,\gamma,E)$*.*

(b) *Ref. [36] a soft generalized open (briefly: soft g-open) set in* (Y,γ,E) *if* $1_E - K \in GC(Y,\gamma,E)$*. The collection of all soft g-open sets in* (Y,γ,E) *is denoted by* $GO(Y,\gamma,E)$*.*

(c) *Ref. [14] a soft generalized ω-closed (briefly: soft gω-closed) set in* (Y,γ,E) *if* $Cl_{\gamma_\omega}(K)\widetilde{\subseteq}H$ *whenever* $H \in \lambda$ *and* $K\widetilde{\subseteq}H$*. The collection of all soft gω-closed sets in* (Y,γ,E) *is denoted by* $G\omega C(Y,\gamma,E)$*.*

(d) *Ref. [37] a soft regular open set in* (Y,γ,E) *if* $Int_\gamma(Cl_\gamma(K)) = K$*. The collection of all soft regular open sets in* (Y,γ,E) *is denoted by* $RO(Y,\gamma,E)$*. Soft complements of soft regular open sets are called soft regular closed sets. The collection of all soft regular closed sets in* (Y,γ,E) *is denoted by* $RC(Y,\gamma,E)$*.*

(e) *Ref. [37] a soft regular generalized closed (briefly: soft rg-closed) set in* (Y,γ,E) *if* $Cl_\gamma(K)\widetilde{\subseteq}H$ *whenever* $H \in RO(Y,\gamma,E)$ *and* $K\widetilde{\subseteq}H$*. The collection of all soft rg-closed sets in* (Y,γ,E) *is denoted by* $RGC(Y,\gamma,E)$*. Complements of rg-closed sets are called rg-open sets. The collection of all rg-open sets in* (Y,γ,E) *is denoted by* $RGO(Y,\gamma,E)$*.*

2. Soft Regular Generalized ω-Closed Sets

Definition 7. *A soft subset H of a STS* (Y,γ,A) *is called a soft regular generalized ω-closed set (simply: soft rgω-closed) in* (Y,γ,A) *if* $Cl_{\gamma_\omega}(H)\widetilde{\subseteq}K$ *whenever* $K \in RO(Y,\gamma,A)$ *and* $H\widetilde{\subseteq}K$*. The collection of all soft rgω-closed sets in* (Y,γ,A) *will be denoted by* $RG\omega C(Y,\gamma,A)$*.*

Theorem 4. *For any STS* (Y,γ,A)*,* $RGC(Y,\gamma,A) \subseteq RG\omega C(Y,\gamma,A)$*.*

Proof. Let $K \in RGC(Y,\gamma,A)$ and let $H \in RO(Y,\gamma,A)$ such that $K\widetilde{\subseteq}H$. Then $Cl_\gamma(K)\widetilde{\subseteq}H$ and thus, $Cl_{\gamma_\omega}(K)\widetilde{\subseteq}Cl_\gamma(K)\widetilde{\subseteq}H$. Hence, $K \in RG\omega C(Y,\gamma,A)$. □

The following example shows that $RGC(Y,\gamma,A) \neq RG\omega C(Y,\gamma,A)$, in general:

Example 1. *Let* $Y = \{1,2,3,4\}$*,* $A = \{a\}$*, and*

$$\gamma = \{T \in SS(Y,A) \colon T(a) \in \{\emptyset, Y, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}\}.$$

Then $RO(Y,\gamma,A) = \{T \in SS(Y,A) \colon T(a) \in \{\emptyset, Y, \{1\}, \{2\}\}\}$*. Let* $K \in SS(Y,A)$ *with* $T(a) = \{1\}$*. Then* $Cl_{\gamma_\omega}(K) = K$ *and* $(Cl_\gamma(K))(a) = \{1,2,3\}$*. Since* $K \in RO(Y,\gamma,A)$ *and* $K\widetilde{\subseteq}K$ *while* $(Cl_\gamma(K))(a) = \{1,2,3\} \nsubseteq K(a) = \{1\}$*, then* $K \notin RGC(Y,\gamma,A)$*. To show that* $K \in RG\omega C(Y,\gamma,A)$*, let* $H \in RO(Y,\gamma,A)$ *such that* $K\widetilde{\subseteq}H$*. Then either* $H = K$ *or* $H(a) = Y$*. In both cases we must have* $Cl_{\gamma_\omega}(K) = K\widetilde{\subseteq}H$*. Hence,* $K \in RG\omega C(Y,\gamma,A) - RGC(Y,\gamma,A)$*.*

Theorem 5. *For any soft anti-locally countable STS* (Y,γ,A)*,* $RGC(Y,\gamma,A) = RG\omega C(Y,\gamma,A)$*.*

Proof. Let (Y,γ,A) be soft anti-locally countable. Then according to Theorem 4 we have $RGC(Y,\gamma,A) \subseteq RG\omega C(Y,\gamma,A)$. To show that $RG\omega C(Y,\gamma,A) \subseteq RGC(Y,\gamma,A)$, let $K \in RG\omega C(Y,\gamma,A)$ and let $H \in RO(Y,\gamma,A)$ such that $K\widetilde{\subseteq}H$. Then $Cl_{\gamma_\omega}(K)\widetilde{\subseteq}H$. On the other hand, according to Theorem 14 of [26], $Cl_{\gamma}(K) = Cl_{\gamma_{\omega}}(K)$. Therefore, $Cl_{\gamma}(K)\widetilde{\subseteq}H$. Hence, $K \in RG\omega C(Y,\gamma,A)$. □

Theorem 6. *For any STS (Y,γ,A), $G\omega C(Y,\gamma,A) \subseteq RG\omega C(Y,\gamma,A)$.*

Proof. Let $K \in G\omega C(Y,\gamma,A)$ and let $H \in RO(Y,\gamma,A) \subseteq \gamma$ such that $K\widetilde{\subseteq}H$. Since $RO(Y,\gamma,A) \subseteq \gamma$, then $H \in \gamma$ and so $Cl_{\gamma_{\omega}}(K)\widetilde{\subseteq}H$. Therefore, $K \in RG\omega C(Y,\gamma,A)$. □

The following example shows that $G\omega C(Y,\gamma,A) \neq RG\omega C(Y,\gamma,A)$, in general:

Example 2. *Let $Y = \mathbb{R}$, $A = \{a\}$, $K \in SS(Y,A)$ such that $K(a) = \mathbb{R} - \mathbb{Q}$, and $\gamma = \{0_A, 1_A, K\}$. Since $Int_{\gamma}(Cl_{\gamma}(K)) = Int_{\gamma}(1_A) = 1_A$, then $RO(Y,\gamma,A) = \{0_A, 1_A\}$. To show that $K \in RG\omega C(Y,\gamma,A)$, let $H \in RO(Y,\gamma,A)$ such that $K\widetilde{\subseteq}H$, then $H = 1_A$ and so $Cl_{\gamma_{\omega}}(K)\widetilde{\subseteq}H$. On the other hand, since $K \in \gamma$ and $K\widetilde{\subseteq}K$ but $Cl_{\gamma_{\omega}}(K) = 1_A\widetilde{\nsubseteq}K$, then $K \notin G\omega C(Y,\gamma,A)$.*

Theorem 7. *For any STS (Y,γ,A), $(\gamma_{\omega})^c \subseteq RG\omega C(Y,\gamma,A)$.*

Proof. Let $K \in (\gamma_{\omega})^c$ and let $H \in RO(Y,\gamma,A)$ such that $K\widetilde{\subseteq}H$. Since $K \in (\gamma_{\omega})^c$, then $Cl_{\gamma_{\omega}}(K) = K\widetilde{\subseteq}H$. Hence, $K \in RG\omega C(Y,\gamma,A)$. □

The following example will show that $(\gamma_{\omega})^c \neq RG\omega C(Y,\gamma,A)$, in general:

Example 3. *Let $Y = \mathbb{R}$, $A = \{a\}$, μ be the usual topology on $\mathbb{R}$, and $\mu = \{S \in SS(Y,A)\colon S(a) \in \mu\}$. Then $C_{[1,\infty)} \in RG\omega C(Y,\gamma,A) - (\gamma_{\omega})^c$.*

Theorem 8. *Let (Y,γ,A) be soft locally countable. Then $RG\omega C(Y,\gamma,A) = SS(Y,A)$.*

Proof. Follows from Theorem 6 and Theorem 4 of [14]. □

The following question is natural:

Question 1. *Let (Y,γ,A) be a STS. Is it true that $RGC(Y,\gamma_{\omega},A) \subseteq RGC(Y,\gamma,A)$?*

The following example provides a negative response to Question 1:

Example 4. *Let $Y = \mathbb{R}$, $A = \{a\}$, μ be the usual topology on $\mathbb{R}$, and $\gamma = \{S \in SS(Y,A) : S(a) \in \mu\}$. Let $K = C_{\mathbb{Q}\cap(2,\infty)}$. Since $K \in CSS(Y,A)$, then by Theorem 2 (d) of [26], $K \in (\gamma_{\omega})^c$, and so $Cl_{\gamma_{\omega}}(K) = K$. Thus, for every $H \in RO(Y,\gamma_{\omega},A)$ such that $K\widetilde{\subseteq}H$, we have $Cl_{\gamma_{\omega}}(K) = K\widetilde{\subseteq}H$. Hence, $K \in RGC(Y,\gamma_{\omega},A)$. Now, since $K\widetilde{\subseteq}C_{(2,\infty)} \in RO(Y,\gamma,A)$ but $Cl_{\gamma}(K) = C_{[2,\infty)}\widetilde{\nsubseteq}C_{(2,\infty)}$, then $K \notin RGC(Y,\gamma,A)$.*

Theorem 9. *Let (Y,γ,A) be a STS. If $K \in RG\omega C(Y,\gamma,A)$, then the only soft regular closed set in (Y,γ,A) which is soft contained in $Cl_{\gamma_{\omega}}(K) - K$ is 0_A.*

Proof. Let $K \in RG\omega C(Y,\gamma,A)$ and T be a soft regular closed set in (Y,γ,A) such that $T\widetilde{\subseteq}Cl_{\gamma_{\omega}}(K) - K$. Then $T\widetilde{\subseteq}Cl_{\gamma_{\omega}}(K)$ and $K\widetilde{\subseteq}1_A - T \in RO(Y,\gamma,A)$. Since $K \in RG\omega C(Y,\gamma,A)$ and $K\widetilde{\subseteq}1_A - T \in RO(Y,\gamma,A)$, then $Cl_{\gamma_{\omega}}(K)\widetilde{\subseteq}1_A - T$, and thus $Cl_{\gamma_{\omega}}(K)\widetilde{\cap}T = 0_A$. On the other hand, since $T\widetilde{\subseteq}Cl_{\gamma_{\omega}}(K)$, then $Cl_{\gamma_{\omega}}(K)\widetilde{\cap}T = T$. It follows that $T = 0_A$. □

Question 2. *Let (Y,γ,A) be a STS and let $K \in SS(Y,A)$ such that the only soft regular closed set in (Y,γ,A) which is soft contained in $Cl_{\gamma_{\omega}}(K) - K$ is 0_A. Is it true that $K \in RG\omega C(Y,\gamma,A)$?*

Theorem 10. *Let (Y,γ,A) be a STS. If $K \in RG\omega C(Y,\gamma,A)$ and $S \in SS(Y,A)$ such that $K\widetilde{\subseteq}S\widetilde{\subseteq}Cl_{\gamma_{\omega}}(K)$, then $S \in RG\omega C(Y,\gamma,A)$.*

Proof. Let $K \in RG\omega C(Y,\gamma,A)$ and $S \in SS(Y,A)$ such that $K\widetilde{\subseteq}S\widetilde{\subseteq}Cl_{\gamma_\omega}(K)$. Let $H \in RO(Y,\gamma,A)$ such that $S\widetilde{\subseteq}H$. Since $K\widetilde{\subseteq}S$, then $K\widetilde{\subseteq}H$. Since $K \in RG\omega C(Y,\gamma,A)$, then $Cl_{\gamma_\omega}(K)\widetilde{\subseteq}H$. Since $K\widetilde{\subseteq}S\widetilde{\subseteq}Cl_{\gamma_\omega}(K)$, then $Cl_{\gamma_\omega}(K)\widetilde{\subseteq}Cl_{\gamma_\omega}(S)\widetilde{\subseteq}Cl_{\gamma_\omega}(Cl_{\gamma_\omega}(K)) = Cl_{\gamma_\omega}(K)$ and thus $Cl_{\gamma_\omega}(K) = Cl_{\gamma_\omega}(S)$. Therefore, $Cl_{\gamma_\omega}(S)\widetilde{\subseteq}H$. Hence, $S \in RG\omega C(Y,\gamma,A)$. □

Theorem 11. *Let (Y,γ,A) be a STS and let $K,S \in RG\omega C(Y,\gamma,A)$. Then $K\widetilde{\cup}S \in RG\omega C(Y,\gamma,A)$.*

Proof. Let $K,S \in RG\omega C(Y,\gamma,A)$. Let $H \in RO(Y,\gamma,A)$ such that $K\widetilde{\cup}S\widetilde{\subseteq}H$. Then $K\widetilde{\subseteq}H$ and $S\widetilde{\subseteq}H$. So, $Cl_{\gamma_\omega}(K)\widetilde{\subseteq}H$ and $Cl_{\gamma_\omega}(S)\widetilde{\subseteq}H$. Thus, $Cl_{\gamma_\omega}(K)\widetilde{\cup}Cl_{\gamma_\omega}(S) = Cl_{\gamma_\omega}(K\widetilde{\cup}S)\widetilde{\subseteq}H$. Hence, $K\widetilde{\cup}S \in RG\omega C(Y,\gamma,A)$. □

For a STS (Y,γ,A), $RG\omega C(Y,\gamma,A)$ is not closed under arbitrary soft union:

Example 5. *Let $Y = \mathbb{R}$, $A = \{a\}$, μ be the usual topology on $\mathbb{R}$, and $\gamma = \{S \in SS(Y,A) : S(a) \in \mu\}$. For each $n \in \mathbb{N}$, let $K_n = C_{[\frac{1}{n+1},1]}$. Then $\{K_n : n \in \mathbb{N}\} \subseteq RG\omega C(Y,\gamma,A)$ and $\widetilde{\cup}\{K_n : n \in \mathbb{N}\} = C_{(0,1]}$. Since $C_{(0,2)} \in RO(Y,\gamma,A)$ such that $C_{(0,1]}\widetilde{\subseteq}C_{(0,2)}$ while $Cl_{\gamma_\omega}(C_{(0,1]}) = C_{[0,1]}\widetilde{\nsubseteq}C_{(0,1]}$, then $C_{(0,1]} \notin RG\omega C(Y,\gamma,A)$.*

Theorem 12. *Let (Y,γ,A) be a STS and let $\{K_t : t \in I\}$ be soft locally finite in (Y,γ,A) such that $\{K_t : t \in I\} \subseteq RG\omega C(Y,\gamma,A)$. $\widetilde{\cup}\{K_t : t \in I\}\widetilde{\in}RG\omega C(Y,\gamma,A)$.*

Proof. Let $H \in RO(Y,\gamma,A)$ with $\widetilde{\cup}\{K_t : t \in I\}\widetilde{\subseteq}H$. Then for each $t \in I$, $K_t\widetilde{\subseteq}H \in RO(Y,\gamma,A)$, and so $Cl_{\gamma_\omega}(K_t)\widetilde{\subseteq}H$. Since $\{K_t : t \in I\}$ is soft locally finite in (Y,γ,A), then by Lemma 3.5 of [24], $Cl_{\gamma_\omega}(\widetilde{\cup}\{K_t : t \in I\}) = \widetilde{\cup}\{Cl_{\gamma_\omega}(K_t) : t \in I\}\widetilde{\subseteq}K$. Hence, $\widetilde{\cup}\{K_t : t \in I\}\widetilde{\in}RG\omega C(Y,\gamma,A)$. □

For a given STS (Y,γ,A), the following example shows that $RG\omega C(Y,\gamma,A)$ is not closed under finite soft intersection:

Example 6. *Let $B = (0,2)$, $C = (2,4)$, $D = (4,6)$, $E = (6,8)$, $Y = B \cup C \cup D \cup E$, $A = \{a\}$, and $\gamma = \{S \in SS(Y,A) : S(a) \in \{\emptyset, Y, B, C, B \cup C, B \cup C \cup D\}\}$. Then $RO(Y,\gamma,A) = \{S \in SS(Y,A) : S(a) \in \{\emptyset, Y, C, D\}\}$. Let $K,T \in SS(Y,A)$ such that $K(a) = B \cup \{3\}$ and $T(a) = B \cup \{5\}$. If $H \in RO(Y,\gamma,A)$ such that $K\widetilde{\subseteq}H$, then $H = 1_A$ and $Cl_{\gamma_\omega}(K)\widetilde{\subseteq}H$. Thus, $K \in RG\omega C(Y,\gamma,A)$. Similarly, we can see that $T \in RG\omega C(Y,\gamma,A)$. Since $(K \cap T)(a) = A$, then $K \cap T \in RO(Y,\gamma,A)$ and $(Cl_{\gamma_\omega}(K \cap T))(a) = (Cl_\gamma(K \cap T))(a) = B \cup D \cup E$. Since $K \cap T\widetilde{\subseteq}K \cap T \in RO(Y,\gamma,A)$ but $Cl_{\gamma_\omega}(K \cap T)\widetilde{\nsubseteq}K \cap T$, then $K \cap T \notin RG\omega C(Y,\gamma,A)$.*

Theorem 13. *Let (Y,γ,A) be a STS and let Z be a non-empty subset of Y. Let $K \in SS(Z,A) \cap \gamma$. If $K \in RG\omega C(Y,\gamma,A)$, then $K \in RG\omega C(Z,\gamma_Z,A)$.*

Proof. Let $K \in RG\omega C(Y,\gamma,A)$ and let $H \in RO(Z,\gamma_Z,A)$ such that $K\widetilde{\subseteq}H$. Choose $S \in RO(Y,\gamma,A)$ such that $H = S\widetilde{\cap}C_Z$. Since $K \in RG\omega C(Y,\gamma,A)$ and $K\widetilde{\subseteq}S \in RO(Y,\gamma,A)$, then $Cl_{\gamma_\omega}(K)\widetilde{\subseteq}S$, and so $Cl_{(\gamma_\omega)_Z}(K) = Cl_{\gamma_\omega}(K)\widetilde{\cap}C_Z\widetilde{\subseteq}S\widetilde{\cap}C_Z = H$. On the other hand, by Theorem 15 of [26], $Cl_{(\gamma_\omega)_Z}(K) = Cl_{(\gamma_Z)_\omega}(K)$ and thus, $Cl_{(\gamma_Z)_\omega}(K)\widetilde{\subseteq}H$. Hence, $K \in RG\omega C(Z,\gamma_Z,A)$. □

Theorem 14. *Let (Y,γ,A) be a STS and let Z be a non-empty subset of Y such that $C_Z \in (\gamma_\omega)^c$. Let $K \in SS(Z,A)$. If $K \in RG\omega C(Z,\gamma_Z,A)$, then $K \in RG\omega C(Y,\gamma,A)$.*

Proof. Let $K \in RG\omega C(Z,\gamma_Z,A)$ and let $H \in RO(Y,\gamma,A)$ such that $K\widetilde{\subseteq}H$. Then $K\widetilde{\subseteq}H\widetilde{\cap}C_Z \in RO(Z,\gamma_Z,A)$. Since $K \in RG\omega C(Z,\gamma_Z,A)$, then $Cl_{(\gamma_Z)_\omega}(K)\widetilde{\subseteq}H\widetilde{\cap}C_Z$. Since by Theorem 15 of [26] we have $Cl_{(\gamma_\omega)_Z}(K) = Cl_{(\gamma_Z)_\omega}(K)$, then $Cl_{(\gamma_\omega)_Z}(K) = Cl_{\gamma_\omega}(K)\widetilde{\cap}C_Z\widetilde{\subseteq}H\widetilde{\cap}C_Z$. Also,

$$Cl_{\gamma_\omega}(K) = Cl_{\gamma_\omega}(K\widetilde{\cap}C_Z)\widetilde{\subseteq}Cl_{\gamma_\omega}(K)\widetilde{\cap}Cl_{\gamma_\omega}(C_Z) = Cl_{\gamma_\omega}(K)\widetilde{\cap}C_Z\widetilde{\subseteq}H\widetilde{\cap}C_Z\widetilde{\subseteq}H.$$

□

Therefore, $K \in RG\omega C(Y, \gamma, A)$.
The following are natural questions:

Question 3. *Let (Y, γ, A) be a STS and let $K \in RGC(Y, \gamma, A)$. Is it true that $K(a) \in RGC(Y, \gamma_a)$ for each $a \in A$?*

Question 4. *Let (Y, γ, A) be a STS and let $K \in RG\omega C(Y, \gamma, A)$. Is it true that $K(a) \in RG\omega C(Y, \gamma_a)$ for each $a \in A$?*

The next four results give partial answers of these two questions:

Theorem 15. *Let $\{(Y, \mu_a) : a \in A\}$ be an indexed family of TSs. Let $K \in SS(Y, A)$. Then $K \in RGC(Y, \oplus_{a\in A}\mu_a, A)$ if and only if $K(e) \in RGC(Y, \mu_a)$ for all $a \in A$.*

Proof. Necessity. Let $K \in RGC(Y, \oplus_{a\in A}\mu_a, A)$ and let $b \in A$. Let $O \in RO(Z, \mu_b)$ such that $K(b) \subseteq O$. Let $H \in SS(Y, A)$ defined by $H(b) = O$ and $H(a) = Z$ if $e \neq b$. Since $H(e) \in RO(Y, \mu_a)$ for every $a \in A$, then by Proposition 3.28 of [38], $H \in RO(Y, \oplus_{a\in A}\mu_a, A)$. Since $K \in RGC(Y, \oplus_{a\in A}\mu_a, A)$ and $K \tilde{\subseteq} H \in RO(Y, \oplus_{a\in A}\mu_a, A)$, then $Cl_{\oplus_{a\in A}\mu_a}(K) \tilde{\subseteq} H$ and so $\left(Cl_{\oplus_{a\in A}\mu_a}(K)\right)(b) \tilde{\subseteq} H(b) = O$. On the other hand, by Lemma 4.9 of [25], $Cl_{\mu_b}(K(b)) = \left(Cl_{\oplus_{a\in A}\mu_a}(K)\right)(b)$. This ends the proof.

Sufficiency. Let $K(e) \in RGC(Y, \mu_a)$ for all $a \in A$. Let $H \in RO(Y, \oplus_{a\in A}\mu_a, A)$ such that $K \tilde{\subseteq} H$. Then for every $a \in A$, $K(a) \subseteq H(a)$. Also, by Proposition 3.28 of [38], $H(a) \in RO(Y, \mu_a)$ for all $a \in A$. So, $Cl_{\mu_a}(K(a)) \subseteq H(a)$ for all $a \in A$. Thus, by Lemma 4.9 of [25], $\left(Cl_{\oplus_{a\in A}\mu_a}(K)\right)(e) \subseteq H(a)$ for all $a \in A$. Hence, $Cl_{\oplus_{a\in A}\mu_a}(K) \tilde{\subseteq} H$. Therefore, $K \in RGC(Y, \oplus_{a\in A}\mu_a, A)$. □

Corollary 1. *Let (Y, μ) be a TS and E be any set of parameters. Let $K \in SS(Y, A)$. Then $K \in RGC(Z, \tau(\mu), E)$ if and only if $K(e) \in RGC(Z, \mu)$ for all $a \in A$.*

Proof. For each $a \in A$, put $\mu_a = \mu$. Then $\tau(\mu) = \oplus_{a\in A}\mu_a$. Thus, by Theorem 15, we get the result. □

Theorem 16. *Let $\{(Y, \mu) : a \in A\}$ be an indexed family of TSs. Let $K \in SS(Y, A)$. Then $K \in RG\omega C(Y, \oplus_{a\in A}\mu_a, A)$ if and only if $K(e) \in RG\omega C(Y, \mu_a)$ for all $a \in A$.*

Proof. *Necessity.* Let $K \in RG\omega C(Y, \oplus_{a\in A}\mu_a, A)$ and let $b \in A$. Let $O \in RO(Y, \mu_b)$ such that $K(b) \subseteq O$. Let $H \in SS(Y, A)$ defined by $H(b) = O$ and $H(a) = Y$ if $a \neq b$. Since $H(a) \in RO(Y, \mu_a)$ for every $a \in A$, then by Proposition 3.28 of [38], $H \in RO(Y, \oplus_{a\in A}\mu_a, A)$. Since $K \in RG\omega C(Y, \oplus_{a\in A}\mu_a, A)$ and $K \tilde{\subseteq} H \in RO(Y, \oplus_{a\in A}\mu_a, A)$, then $Cl_{(\oplus_{a\in A}\mu_a)_\omega}(K) \tilde{\subseteq} H$ and so $\left(Cl_{(\oplus_{a\in A}\mu_a)_\omega}(K)\right)(b) \tilde{\subseteq} H(b) = O$. On the other hand, by Lemma 4.7 of [25], $Cl_{(\mu_b)_\omega}(K(b)) = \left(Cl_{(\oplus_{a\in A}\mu_a)_\omega}(K)\right)(b)$. This shows that $K(b) \in RG\omega C(Y, \mu_b)$.

Sufficiency. Let $K(a) \in RGC(Y, \mu_a)$ for all $a \in A$. Let $H \in RO(Y, \oplus_{a\in A}\mu_a, A)$ such that $K \tilde{\subseteq} H$. Then for every $a \in A$, $K(a) \subseteq H(a)$. Also, by Proposition 3.28 of [38], $H(a) \in RO(Y, \mu_a)$ for all $a \in A$. So, $Cl_{(\mu_a)_\omega}(K(a)) \subseteq H(a)$ for all $a \in A$. Thus, by Lemma 4.7 of [25], $\left(Cl_{(\oplus_{a\in A}\mu_a)_\omega}(K)\right)(a) \subseteq H(a)$ for all $a \in A$. Hence, $Cl_{(\oplus_{a\in A}\mu_a)_\omega}(K) \tilde{\subseteq} H$. Therefore, $K \in RG\tilde{C}(Y, \oplus_{a\in A}\mu_a, A)$. □

Corollary 2. *Let (Y, μ) be a TS and A be any set of parameters. Let $K \in SS(Y, A)$. Then $K \in RG\omega C(Y, \tau(\mu), A)$ if and only if $K(a) \in RG\omega C(Y, \mu)$ for all $a \in A$.*

Proof. For each $a \in A$, put $\mu_a = \mu$. Then $\tau(\mu) = \oplus_{a \in A} \mu_a$. Thus, by Theorem 16, we get the result. □

Definition 8. *Let (Y, γ, A) be a STS and let $K \in SS(Y, A)$. Then K is called a soft regular generalized ω-open (simply: soft rgω-open) set in (Y, γ, A) if $1_A - K \in RG\omega C(Y, \gamma, A)$. The family of all soft rgω-open sets in (Y, γ, A) will be denoted by $RG\omega O(Y, \gamma, A)$.*

Theorem 17. *Let (Y, γ, A) be a STS and let $K \in SS(Y, A)$. Then $K \in RG\omega O(Y, \gamma, A)$ if and only if $H \tilde{\subseteq} Int_{\gamma_\omega}(K)$ whenever $H \in RC(Y, \gamma, A)$ such that $H \tilde{\subseteq} K$.*

Proof. *Necessity.* Suppose that $K \in RG\omega O(Y, \gamma, A)$ and let $H \in RC(Y, \gamma, A)$ such that $H \tilde{\subseteq} K$. Then $1_A - K \in RG\omega C(Y, \gamma, A)$, $1_A - H \in RO(Y, \gamma, A)$, and $1_A - K \tilde{\subseteq} 1_A - H$. So, $Cl_{\gamma_\omega}(1_A - K) \tilde{\subseteq} 1_A - H$. But $Cl_{\gamma_\omega}(1_A - K) = 1_A - Int_{\gamma_\omega}(K)$. Thus, $1_A - Int_{\gamma_\omega}(K) \tilde{\subseteq} 1_A - H$ and hence $H \tilde{\subseteq} Int_{\gamma_\omega}(K)$.

Sufficiency. Suppose that $H \tilde{\subseteq} Int_{\gamma_\omega}(K)$ whenever $H \in RC(Y, \gamma, A)$ such that $H \tilde{\subseteq} K$. We will show that $1_A - K \in RG\omega C(Y, \gamma, A)$. Let $S \in RO(Y, \gamma, A)$ such that $1_A - K \tilde{\subseteq} S$. Then we have $1_A - S \in RC(Y, \gamma, A)$ such that $1_A - S \tilde{\subseteq} K$. Thus, $1_A - S \tilde{\subseteq} Int_{\gamma_\omega}(K)$ and hence $1_A - Int_{\gamma_\omega}(K) \tilde{\subseteq} S$. But $1_A - Int_{\gamma_\omega}(K) = Cl_{\gamma_\omega}(1_A - K)$. It follows that $1_A - K \in RG\omega C(Y, \gamma, A)$. □

Theorem 18. *Let (Y, γ, A) be a STS and let $K, S \in RG\omega O(Y, \gamma, A)$. Then $K \tilde{\cap} S \in RG\omega O(Y, \gamma, A)$.*

Proof. Let $K, S \in RG\omega O(Y, \gamma, A)$. Then $1_A - K, 1_A - S \in RG\omega C(Y, \gamma, A)$. So by Theorem 11, $(1_A - K) \tilde{\cup} (1_A - S) = 1_A - (K \tilde{\cap} S) \in RG\omega C(Y, \gamma, A)$. Hence, $(K \tilde{\cap} S) \in RG\omega O(Y, \gamma, A)$. □

For a STS (Y, γ, A), $RG\omega O(Y, \gamma, A)$ is not closed under arbitrary soft intersection:

Example 7. *Let $Y = \mathbb{R}$, $A = \{a\}$, μ be the usual topology on $\mathbb{R}$, and $\gamma = \{S \in SS(Y, A) : S(a) \in \mu\}$. For each $n \in \mathbb{N}$, let $K_n = 1_A - C_{[\frac{1}{n+1}, 1]}$. Then $\{K_n : n \in \mathbb{N}\} \subseteq RG\omega O(Y, \gamma, A)$ and $\tilde{\cap}\{K_n : n \in \mathbb{N}\} = 1_A - C_{(0,1]}$. Since by Example 5, $C_{(0,1]} \notin RG\omega C(Y, \gamma, A)$, then $1_A - C_{(0,1]} \notin RG\omega O(Y, \gamma, A)$.*

Theorem 19. *Let (Y, γ, A) be a STS and let $\{K_t : t \in I\} \subseteq RG\omega O(Y, \gamma, A)$ such that $\{1_A - K_t : t \in I\}$ is soft locally finite in (Y, γ, A), then $\tilde{\cap}\{K_t : t \in I\} \tilde{\in} RG\omega O(Y, \gamma, A)$.*

Proof. Since $\{K_t : t \in I\} \subseteq RG\omega O(Y, \gamma, A)$, then $\{1_A - K_t : t \in I\} \subseteq RG\omega C(Y, \gamma, A)$. Then by Theorem 12, $\tilde{\cup}\{1_A - K_t : t \in I\} \tilde{\in} RG\omega C(Y, \gamma, A)$. But $\tilde{\cup}\{1_A - K_t : t \in I\} = 1_A - \tilde{\cap}\{K_t : t \in I\}$. It follows that $\tilde{\cap}\{K_t : t \in I\} \tilde{\in} RG\omega O(Y, \gamma, A)$. □

For a given STS (Y, γ, A), the following example shows that $RG\omega O(Y, \gamma, A)$ is not closed under finite soft union:

Example 8. *Let $B = (0, 2)$, $C = (2, 4)$, $D = (4, 6)$, $E = (6, 8)$, $Y = B \cup C \cup D \cup E$, $A = \{a\}$, and $\gamma = \{S \in SS(Y, A) : S(a) \in \{\emptyset, Y, B, C, B \cup C, B \cup C \cup D\}\}$. Let $K, T \in SS(Y, A)$ such that $K(a) = B \cup \{3\}$ and $T(a) = B \cup \{5\}$. Since by Example 6, $K, T \in RG\omega C(Y, \gamma, A)$ but $K \tilde{\cap} T \notin RG\omega C(Y, \gamma, A)$, then $1_A - K, 1_A - T \in RG\omega O(Y, \gamma, A)$ but $(1_A - K) \tilde{\cup} (1_A - T) = 1_A - (K \tilde{\cap} T) \notin RG\omega O(Y, \gamma, A)$.*

Theorem 20. *Let (Y, γ, A) be a STS. If $K, T \in \gamma \cap RG\omega O(Y, \gamma, A)$ such that the pair (K, T) is soft separation relative to (Y, γ, A), then $K \tilde{\cup} T \in RG\omega O(Y, \gamma, A)$.*

Proof. Let $H \in RC(Y, \gamma, A)$ such that $H \tilde{\subseteq} K \tilde{\cup} T$. Then

$$\begin{array}{rcl} H\tilde{\cap}Cl_\gamma(K) & \tilde{\subseteq} & (K\tilde{\cup}T)\tilde{\cap}Cl_\gamma(K) \\ & = & (K\tilde{\cap}Cl_\gamma(K))\tilde{\cup}(T\tilde{\cap}Cl_\gamma(K)) \\ & = & K\tilde{\cup}0_A \\ & = & K \end{array}$$

and

$$\begin{array}{rcl} H\tilde{\cap}Cl_\gamma(T) & \tilde{\subseteq} & (K\tilde{\cup}T)\tilde{\cap}Cl_\gamma(T) \\ & = & (T\tilde{\cap}Cl_\gamma(T))\tilde{\cup}(K\tilde{\cap}Cl_\gamma(T)) \\ & = & T\tilde{\cup}0_A \\ & = & T. \end{array} \qquad \square$$

So, by Theorem 17, we have $H\tilde{\cap}Cl_\gamma(K)\tilde{\subseteq}Int_{\gamma_\omega}(K)$ and $H\tilde{\cap}Cl_\gamma(T)\tilde{\subseteq}Int_{\gamma_\omega}(T)$. Now,

$$\begin{array}{rcl} H & = & H\tilde{\cap}(K\tilde{\cup}T) \\ & = & (H\tilde{\cap}Cl_\gamma(K))\tilde{\cup}(H\tilde{\cap}Cl_\gamma(T)) \\ & \tilde{\subseteq} & Int_{\gamma_\omega}(K)\tilde{\cup}Int_{\gamma_\omega}(T) \\ & \tilde{\subseteq} & Int_{\gamma_\omega}(K\tilde{\cup}T). \end{array}$$

Thus, by Theorem 17, $K\tilde{\cup}T \in RG\omega O(Y,\gamma,A)$.

Theorem 21. *Let (Y,γ,A) be a STS and let $K \in RG\omega O(Y,\gamma,A)$ and $H \in RO(Y,\gamma,A)$. If $Int_{\gamma_\omega}(K)\tilde{\cup}(1_A - K)\tilde{\subseteq}H$, then $H = 1_A$.*

Proof. Suppose that $K \in RG\omega O(Y,\gamma,A)$ and $H \in RO(Y,\gamma,A)$ such that $Int_{\gamma_\omega}(K)\tilde{\cup}(1_A - K)\tilde{\subseteq}H$. Then $1_A - H\tilde{\subseteq}(1_A - Int_{\gamma_\omega}(K))\tilde{\cap}K = K - Int_{\gamma_\omega}(K)$. Since $1_A - H \in RC(Y,\gamma,A)$, then by Theorem 9, $1_A - H = 0_A$. Hence, $H = 1_A$. □

3. Separation Axioms

Definition 9. *A TS (Y,μ) is called ω-$T_{1/2}$ if $G\omega C(Y,\mu) \subseteq (\mu_\omega)^c$.*

Example 9. *Let $Y = \mathbb{R}$ and $\mu = \{\emptyset,\mathbb{R}\}$. Since $\mathbb{R} - \{0\} \in G\omega C(Y,\mu) - (\mu_\omega)^c$, then (Y,μ) is not ω-$T_{1/2}$.*

Theorem 22. *A TS (Y,μ) is ω-$T_{1/2}$ if and only if for each $y \in Y$, $\{y\} \in \mu_\omega \cup \mu^c$.*

Proof. *Necessity.* Let (Y,μ) be ω-$T_{1/2}$ and let $y \in Y$. Suppose that $\{y\} \notin \mu^c$. Then $Y - \{y\} \notin \mu$. We are going to show that $Y - \{y\} \in GC(Y,\mu)$. Let $U \in \mu$ such that $Y - \{y\} \subseteq U$. Then either $U = Y - \{y\}$ or $U = Y$. Since $Y - \{y\} \notin \mu$, then $U = Y$ and thus $Cl_\mu(Y - \{y\}) \subseteq U = Y$. Therefore, $Y - \{y\} \in GC(Y,\mu) \subseteq G\omega C(Y,\mu)$. Since (Y,μ) is ω-$T_{1/2}$, then $Y - \{y\} \in (\mu_\omega)^c$. Hence, $\{y\} \in \mu_\omega$.

Sufficiency. Let $\{y\} \in \mu_\omega \cup \mu^c$ for every $y \in Y$. Suppose to the contrary that there exists $V \in G\omega C(Y,\mu) - (\mu_\omega)^c$. Then $V \in G\omega C(Y,\mu)$ and there exists $y \in Cl_{\mu_\omega}(V) - V$. By assumption, $\{y\} \in \mu_\omega \cup \mu^c$. If $\{y\} \in \mu_\omega$, then $\{y\} \cap V \neq \emptyset$ and thus $y \in V$. But $y \notin V$. Thus, we must have $\{y\} \in \mu^c$ and hence $Y - \{y\} \in \mu$. Since $y \notin V$, then $V \subseteq Y - \{y\}$. Since $V \in G\omega C(Y,\mu)$, then $Cl_{\mu_\omega}(V) \subseteq Y - \{y\}$. But $y \in Cl_{\mu_\omega}(V)$, a contradiction. □

Theorem 23. *Every locally countable TS is ω-$T_{1/2}$.*

Proof. Let (Y,μ) be locally countable. Then by Corollary 2.4 of [39], (Y,μ) is a discrete TS and thus $\{y\} \in \mu_\omega \subseteq \{y\} \in \mu_\omega \cup \mu^c$ for every $y \in Y$. Thus, by Theorem 22, (Y,μ) is ω-$T_{1/2}$. □

The following example demonstrates that the contrary of Theorem 23 does not have to be true in general:

Example 10. *Let $Y = \mathbb{R}$ and μ be the topology on $\mathbb{R}$ having $\{U \subseteq \mathbb{R} - \mathbb{N} : (\mathbb{R} - \mathbb{N}) - U \text{ is finite}\} \cup \{\mathbb{N}\}$ as a base. It is clear that (Y,μ) is not locally countable. To see that (Y,μ) is ω-$T_{1/2}$. Let $y \in Y$. If $y \in \mathbb{N}$, then we have $\mathbb{N} \in \mu$ with $y \in \mathbb{N}$ and $\mathbb{N} - \{y\}$ is a countable set, and hence $\{y\} \in \mu_\omega \subseteq \mu_\omega$*

$\cup\mu^c$. If $y \in \mathbb{R}-\mathbb{N}$, then $\mathbb{R}-\{y\} = ((\mathbb{R}-\mathbb{N})-\{y\}) \cup \mathbb{N}$ with $((\mathbb{R}-\mathbb{N})-\{y\}) \in \mu$ and $\mathbb{N} \in \mu$, and so $\mathbb{R}-\{y\} \in \mu$, hence, $\{y\} \in \mu^c \subseteq \mu_\omega \cup\mu^c$. Thus, by Theorem 22, (Y,μ) is ω-$T_{1/2}$.

Theorem 24. *Every $T_{1/2}$ TS is ω-$T_{1/2}$.*

Proof. Suppose that (Y,μ) is $T_{1/2}$ and let $y \in Y$, then by Theorem 2.5 of [39], $\{y\} \in \mu \cup\mu^c \subseteq \mu_\omega \cup\mu^c$. Therefore, by Theorem 22, (Y,μ) is ω-$T_{1/2}$. □

Example 11. *Let (Y,μ) be as in Example 10. We proved that (Y,μ) is ω-$T_{1/2}$. It is clear that $\{1\} \notin \mu$. If $\mathbb{R}-\{1\} \in \mu$, then there exists $U \in \mu$ such that $2 \in U \subseteq \mathbb{R}-\{1\}$. Thus, $\mathbb{N} \subseteq U \subseteq \mathbb{R}-\{1\}$ which is impossible. Hence, $\mathbb{R}-\{1\} \notin \mu$. It follows that $\{1\} \notin \mu \cup\mu^c$ and so (Y,μ) is not $T_{1/2}$.*

Theorem 25. *For a locally countable TS (Y,μ), the following are equivalent:*

(a) (Y,μ) *is* T_1.
(b) (Y,μ) *is* $T_{1/2}$.
(c) (Y,μ) *is* ω-$T_{1/2}$.

Proof. (a) $\Longrightarrow$ (b): Obvious.
(b) $\Longrightarrow$ (c): Follows from Theorem 24.
(c) $\Longrightarrow$ (a): Follows from Definition 3.1 and Corollary 2.4 of [39].
□

Theorem 26. *Let (Y,μ) be a TS and let Z be a non-empty subset of Y. If (Y,μ) is ω-$T_{1/2}$, then (Z,μ_Z) is ω-$T_{1/2}$.*

Proof. Suppose that (Y,μ) is ω-$T_{1/2}$. Let $z \in Z$. Since (Y,μ) is ω-$T_{1/2}$, then $\{z\} \in \mu_\omega \cup\mu^c$. Thus, $\{z\} \in (\mu_\omega)_Z \cup(\mu_Z)^c$. But by Theorem 15 of [26], $(\mu_\omega)_Z = (\mu_Z)_\omega$. It follows that (Z,μ_Z) is ω-$T_{1/2}$. □

Definition 10. *A STS (Y,γ,A) is called soft ω-$T_{1/2}$ if $G\omega C(Y,\gamma,A) \subseteq (\gamma_\omega)^c$.*

Theorem 27. *A STS (Y,γ,A) is soft ω-$T_{1/2}$ if and only if for each $a_y \in SP(Y,A)$, $a_y \in \gamma_\omega \cup \gamma^c$.*

Proof. *Necessity.* Let (Y,γ,A) be soft ω-$T_{1/2}$. Let $a_y \in SP(Y,A)$. Suppose that $a_y \notin \gamma^c$. Then $1_A - a_y \notin \gamma$. Now we will show that $1_A - a_y \in G\omega C(Y,\gamma,A)$. Let $T \in \gamma$ such that $1_A - a_y \widetilde{\subseteq} T$. Since $1_A - a_y \notin \gamma$, then $T = 1_A$ and thus $Cl_{\gamma_\omega}(1_A - a_y) \widetilde{\subseteq} T = 1_A$. Hence, $1_A - a_y \in G\omega C(Y,\gamma,A)$. Since (Y,γ,A) is soft ω-$T_{1/2}$, then $1_A - a_y \in (\gamma_\omega)^c$. Therefore, $a_y \in \gamma_\omega$.

Sufficiency. Let $a_y \in \gamma_\omega \cup \gamma^c$ for every $a_y \in SP(Y,A)$. Suppose to the contrary that there exists $K \in G\omega C(Y,\gamma,A) - (\gamma_\omega)^c$. Then $K \in G\omega C(Y,\gamma,A)$ and there exists $a_y \widetilde{\in} Cl_{\mu_\omega}(K) - K$. By assumption, $a_y \in \gamma_\omega \cup \gamma^c$. If $a_y \in \gamma_\omega$, then $a_y \widetilde{\cap} K \neq 0_A$ and thus $a_y \widetilde{\in} K$. But $a_y \widetilde{\notin} K$. Thus, we must have $a_y \in \gamma^c$ and hence $1_A - a_y \in \gamma$. Since $a_y \widetilde{\notin} K$, then $K \widetilde{\subseteq} 1_A - a_y$. Since $K \in G\omega C(Y,\gamma,A)$, then $Cl_{\mu_\omega}(K) \widetilde{\subseteq} 1_A - a_y$. But $a_y \widetilde{\in} Cl_{\mu_\omega}(K)$, a contradiction. □

Theorem 28. *If (Y,γ,A) is a soft ω-$T_{1/2}$ STS, then (Y,γ_a) is ω-$T_{1/2}$ for every $a \in A$.*

Proof. Let (Y,γ,A) be soft ω-$T_{1/2}$ and let $a \in A$. Let $y \in Y$. Since (Y,γ,A) is soft ω-$T_{1/2}$, then by Theorem 27, $a_y \in \gamma_\omega \cup \gamma^c$, and so $\{y\} \in (\gamma_\omega)_a \cup (\gamma_a)^c$. But by Theorem 7 of [26], $(\gamma_\omega)_a = (\gamma_a)_\omega$. Hence, by Theorem 22, (Y,γ_a) is ω-$T_{1/2}$. □

Theorem 29. *Let $\{(Y,\mu_a) : a \in A\}$ be an indexed family of TSs. Then $(Y, \oplus_{a\in A}\mu_a, A)$ is soft ω-$T_{1/2}$ if and only if (Y,μ) is ω-$T_{1/2}$ for every $a \in A$.*

Proof. *Necessity.* Let $(Y, \oplus_{a\in A}\mu_a, A)$ be soft ω-$T_{1/2}$ and let $a \in A$. Then by Theorem 28, $(Z, (\oplus_{a\in A}\mu_a)_a)$ is ω-$T_{1/2}$. Also, by Theorem 3.7 of [27], $(\oplus_{a\in A}\mu_a)_a = \mu_a$. Therefore, (Y, μ) is ω-$T_{1/2}$.

Sufficiency. Let (Y, μ) be ω-$T_{1/2}$ for every $a \in A$. Let $a_z \in SP(Y, A)$. Since (Y, μ) is ω-$T_{1/2}$, then by Theorem 22, $\{y\} \in (\mu_a)_\omega \cup \mu^c$. Thus, $a_y \in (\oplus_{a\in A}(\mu_a)_\omega) \cup (\oplus_{a\in A}\mu_a)^c$. But by Theorem 8 of [26], $(\oplus_{a\in A}(\mu_a)_\omega) = (\oplus_{a\in A}\mu_a)_\omega$. Hence, by Theorem 27, $(Y, \oplus_{a\in A}\mu_a, A)$ is soft ω-$T_{1/2}$. □

Theorem 30. *Let (Y, μ) be a TS and A be any set of parameters. Then $(Y, \tau(\mu), A)$ is soft ω-$T_{1/2}$ if and only if (Y, μ) is ω-$T_{1/2}$.*

Proof. For every $a \in A$, let $\mu_a = \mu$. Then $\tau(\mu) = \oplus_{a\in A}\mu_a$. This, by Theorem 29 we get the result. □

Theorem 31. *Every soft $T_{1/2}$ STS is soft ω-$T_{1/2}$.*

Proof. Follows from Definition 3.10 and Theorem 28 of [14]. □

Example 11 of [14] shows that Theorem 31 is not reversible, in general.

Theorem 32. *For a soft anti-locally countable STS (Y, γ, A), the following are equivalent:*

(a) (Y, γ, A) *is soft* T_1.
(b) (Y, γ, A) *is soft* $T_{1/2}$.
(c) (Y, γ, A) *is soft* ω-$T_{1/2}$.

Proof. (a) $\Longrightarrow$ (b): Obvious.
(b) $\Longrightarrow$ (c): Follows from Theorem 31.
(c) $\Longrightarrow$ (a): Follows from Definition 10 of this paper and Theorem 29 of [14]. □

Theorem 33. *Let (Y, γ, A) be a STS and let Z be a non-empty subset of Y. If (Y, γ, A) is soft ω-$T_{1/2}$, then (Z, γ_Z, A) is soft ω-$T_{1/2}$.*

Proof. Suppose that (Y, γ, A) is soft ω-$T_{1/2}$. Let $a_z \in SP(Z, A)$. Then $a_z \in SP(Y, A)$. Since (Y, γ, A) is soft ω-$T_{1/2}$, then by Theorem 3.11, $a_z \in \gamma_\omega \cup \gamma^c$. Thus, $a_z \in (\gamma_\omega)_Z \cup (\gamma_Z)^c$. But by Theorem 15 of [26], $(\gamma_\omega)_Z = (\gamma_Z)_\omega$. Hence, by Theorem 27, (Z, γ_Z, A) is soft ω-$T_{1/2}$. □

Theorem 34. *Every soft locally countable STS is soft ω-$T_{1/2}$.*

Proof. Let (Y, γ, A) be soft locally countable. Then by Corollary 5 of [26], (Y, γ_ω, A) is a discrete STS and so $a_y \in \gamma_\omega \subseteq \gamma_\omega \cup \gamma^c$ for every $a_y \in SP(Y, A)$. Thus, by Theorem 27, (Y, γ, A) is soft ω-$T_{1/2}$. □

The converse of Theorem 34 need not be true, in general:

Example 12. *Let (Y, μ) be as in Example 10. Consider the STS $(Y, \tau(\mu), A)$. Then clearly that $(Y, \tau(\mu), A)$ is not soft locally countable. On the other hand, since (Y, μ) is ω-$T_{1/2}$, then by Theorem 30, $(Y, \tau(\mu), A)$ is soft ω-$T_{1/2}$.*

Definition 11. *A STS (Y, γ, A) is called soft regular generalized $T_{1/2}$ (simply rg-$T_{1/2}$) if RGC $(Y, \gamma, A) \subseteq \gamma^c$.*

Theorem 35. *A STS (Y, γ, A) is soft rg-$T_{1/2}$ if and only if for each $a_y \in SP(Y, A)$, $a_y \in \gamma \cup RC(Y, \gamma, A)$.*

Proof. *Necessity.* Let (Y,γ,A) be soft rg-$T_{1/2}$. Let $a_y \in SP(Y,A)$. Suppose that $a_y \notin RC(Y,\gamma,A)$. Then $1_A - a_y \notin RO(Y,\gamma,A)$. Now we will show that $1_A - a_y \in RGC(Y,\gamma,A)$. Let $T \in RO(Y,\gamma,A)$ such that $1_A - a_y \widetilde{\subseteq} T$. Since $1_A - a_y \notin RO(Y,\gamma,A)$, then $T = 1_A$ and thus $Cl_\gamma(1_A - a_y) \widetilde{\subseteq} T = 1_A$. Hence, $1_A - a_y \in RGC(Y,\gamma,A)$. Since (Y,γ,A) is soft rg-$T_{1/2}$, then $1_A - a_y \in \gamma^c$. Therefore, $a_y \in \gamma$.

Sufficiency. Let $a_y \in \gamma \cup RC(Y,\gamma,A)$ for every $a_y \in SP(Y,A)$. Suppose to the contrary that there exists $K \in R(Y,\gamma,A) - \gamma^c$. Then $K \in RGC(Y,\gamma,A)$ and there exists $a_y \widetilde{\in} Cl_\mu(K) - K$. By assumption, $a_y \in \gamma \cup RC(Y,\gamma,A)$. If $a_y \in \gamma$, then $a_y \widetilde{\cap} K \neq 0_A$ and thus $a_y \widetilde{\in} K$. But $a_y \widetilde{\notin} K$. Thus, we must have $a_y \in RC(Y,\gamma,A)$ and hence $1_A - a_y \in RO(Y,\gamma,A)$. Since $a_y \widetilde{\notin} K$, then $K \widetilde{\subseteq} 1_A - a_y$. Since $K \in RGC(Y,\gamma,A)$, then $Cl_\mu(K) \widetilde{\subseteq} 1_A - a_y$. But $a_y \widetilde{\in} Cl_\mu(K)$, a contradiction. □

Theorem 36. *For a STS (Y,γ,A), the following are equivalent:*

(a) (Y,γ,A) *is soft rg-$T_{1/2}$.*
(b) (Y,γ,A) *is a soft discrete STS.*

Proof. (a) $\Longrightarrow$ (b): We will show that $SP(Y,A) \subseteq \gamma$. Suppose to the contrary that there exists $a_y \in SP(Y,A) - \gamma$. Then by Theorem 36, $a_y \in RC(Y,\gamma,A)$ and so $Cl_\gamma(Int_\gamma(a_y)) = a_y$. Thus, $Int_\gamma(a_y) = a_y$ and hence $a_y \in \gamma$, a contradiction.

(b) $\Longrightarrow$ (a): Obvious. □

Definition 12. *A STS (Y,γ,A) is called soft regular generalized ω-$T_{1/2}$ (simply $rg\omega$-$T_{1/2}$) if $RG\omega C(Y,\gamma,A) \subseteq (\gamma_\omega)^c$.*

Theorem 37. *A STS (Y,γ,A) is soft $rg\omega$-$T_{1/2}$ if and only if for each $a_y \in SP(Y,A)$, $a_y \in \gamma_\omega \cup RC(Y,\gamma,A)$.*

Proof. *Necessity.* Let (Y,γ,A) be soft $rg\omega$-$T_{1/2}$. Let $a_y \in SP(Y,A)$. Suppose that $a_y \notin RC(Y,\gamma,A)$. Then $1_A - a_y \notin RO(Y,\gamma,A)$. Now we will show that $1_A - e_y \in RG\omega C(Y,\gamma,A)$. Let $T \in RO(Y,\gamma,A)$ such that $1_A - a_y \widetilde{\subseteq} T$. Since $1_A - a_y \notin RO(Y,\gamma,A)$, then $T = 1_A$ and thus $Cl_{\gamma_\omega}(1_A - a_y) \widetilde{\subseteq} T = 1_A$. Hence, $1_A - a_y \in RG\omega C(Y,\gamma,A)$. Since (Y,γ,A) is soft $rg\omega$-$T_{1/2}$, then $1_A - a_y \in (\gamma_\omega)^c$. Therefore, $a_y \in \gamma_\omega$.

Sufficiency. Let $a_y \in \gamma_\omega \cup RC(Y,\gamma,A)$ for every $a_y \in SP(Y,A)$. Suppose to the contrary that there exists $K \in R(Y,\gamma,A) - (\gamma_\omega)^c$. Then $K \in RG\omega C(Y,\gamma,A)$ and there exists $a_y \widetilde{\in} Cl_{\mu_\omega}(K) - K$. By assumption, $a_y \in \gamma_\omega \cup RC(Y,\gamma,A)$. If $a_y \in \gamma_\omega$, then $a_y \widetilde{\cap} K \neq 0_A$ and thus $a_y \widetilde{\in} K$. But $a_y \widetilde{\notin} K$. Thus, we must have $a_y \in RC(Y,\gamma,A)$ and hence $1_A - a_y \in RO(Y,\gamma,A)$. Since $a_y \widetilde{\notin} K$, then $K \widetilde{\subseteq} 1_A - a_y$. Since $K \in RG\omega C(Y,\gamma,A)$, then $Cl_{\mu_\omega}(K) \widetilde{\subseteq} 1_A - a_y$. But $a_y \widetilde{\in} Cl_{\mu_\omega}(K)$, a contradiction. □

Lemma 1. *A STS (Y,γ,A) is soft locally countable if and only if (Y,γ_ω,A) is a soft discrete STS.*

Proof. Straightforward. □

Theorem 38. *For a STS (Y,γ,A), the following are equivalent:*

(a) (Y,γ,A) *is soft $rg\omega$-$T_{1/2}$.*
(b) (Y,γ_ω,A) *is a soft discrete STS.*
(c) (Y,γ,A) *is soft locally countable.*

Proof. (a) $\Longrightarrow$ (b): We will show that $SP(Y,A) \subseteq \gamma_\omega$. Suppose to the contrary that there exists $a_y \in SP(Y,A) - \gamma_\omega$. Then by Theorem 37, $a_y \in RC(Y,\gamma,A)$ and so $Cl_\gamma(Int_\gamma(a_y)) = a_y$. Thus, $Int_\gamma(a_y) = a_y$ and hence $a_y \in \gamma \subseteq \gamma_\omega$, a contradiction.

(b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (b) follow from Lemma 1.

(b) $\Longrightarrow$ (a): Obvious.

□

4. Conclusions

After the work of Shabir and Naz [10], several types of classes of soft sets have been analyzed. For example, soft semi-open sets, soft pre-open sets, soft regular open sets, soft generalized closed sets, soft regular generalized closed sets, soft ω-open sets, soft generalized ω-closed sets, and so on. In this paper's first part, we continued working in the same direction by studying the class of soft regular generalized ω-closed sets. With the help of examples, we have studied the behavior of soft regular generalized 3c9-closed sets with respect to soft unions, soft intersections, and soft subspaces.

In the second part of this paper, we have introduced soft ω-$T_{1/2}$ spaces, soft regular generalized $T_{1/2}$, and soft regular generalized ω-$T_{1/2}$ as three new types of STSs. We have focused on their characterizations and correspondence with the analog concepts in TSs.

Future research could focus on the following topics: (1) investigating the concepts related to $rg\omega$-closed sets such as soft continuity, (2) investigating the behavior of soft $rg\omega$-closed sets under product STSs, or (3) solving three open questions raised in this paper.

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