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Article

Construction of Eigenfunctions to One Nonlocal Second-Order Differential Operator with Double Involution

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Abstract: In this paper, we study the eigenfunctions to one nonlocal second-order differential operator with double involution. We give an explicit form of the eigenfunctions to the boundary value problem in the unit ball with Dirichlet conditions on the boundary. For the problem under consideration, the completeness of the system of eigenfunctions is established.

Keywords: nonlocal differential operator; double involution; boundary value problem; eigenfunctions; eigenvalues

MSC: 35J25; 35J05; 35P10

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1. Introduction and the Problem Statement

Among the nonlocal differential equations, which are the subject of many works, a special place is occupied by equations with involutive deviations of the argument. An involution is a mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S^2x = x$. It should be noted monographs [1–3] from a variety of research papers in this direction. Refs. [4–15] are devoted to the questions of solvability of boundary and initial-boundary value problems for differential equations with involution. Spectral questions of differential equations with involution are studied in [16–25]. For example, in [22], the following boundary value problem

$$y''(t) + ay''(-t) = \lambda y(t), \quad -\pi < t < \pi,$$
$$y(-\pi) = y(\pi) = 0$$

is studied. The eigenfunctions and eigenvalues of this problem are given explicitly. The system of eigenfunctions is complete in $L_2[-\pi, \pi]$.

In [10], the following nonlocal analog of the Laplace operator is introduced

$$L_l[u](x) \equiv a_0\Delta u(x) + a_1\Delta u(Sx) + \ldots + a_{l-1}\Delta u(S^{l-1}x),$$

where Δ is the Laplace operator, a_i for $i = 0, 1, \ldots, l-1$ are real numbers, S is an $n \times n$ orthogonal matrix for which there exists a number $l \in \mathbb{N}$ such that $S^l = I$ and I is the identity matrix. In the paper [10] cited above, for the corresponding nonlocal Poisson equation $L_l[u] = f(x)$ in the unit ball Ω, the solvability questions for some boundary value problems with different boundary conditions are studied. The corresponding spectral problem for the Dirichlet boundary value problem is studied in [14]. In that work, as in the case of the one-dimensional problem from [22], the eigenfunctions and eigenvalues of the considered problem are obtained explicitly. A theorem on the completeness of the system of eigenfunctions in the space $L_2(\Omega)$ is proved.

Furthermore, in [24], a nonlocal Laplace operator with multiple involution of the following form is introduced:

$$L_n[u](x) \equiv \sum_{(i_n \dots i_1)_2=0}^{2^n-1} a_{(i_n \dots i_1)_2} \Delta u\left(S_n^{i_n} \dots S_1^{i_1} x\right),$$

where $a_{(i_n \dots i_1)_2}$ are real numbers, $(i_n \dots i_1)_2$ is a representation of the index i in the binary number system, S_i are orthogonal $n \times n$ matrices satisfying the condition $S_i^2 = I$, $i = 1, \dots, n$. In this paper [24], the explicit form of the eigenfunctions and eigenvalues of the corresponding Dirichlet problem

$$L_n[u](x) + \lambda u(x) = 0,\ x \in \Omega; \quad u(x) = 0,\ x \in \partial\Omega$$

is given and the completeness of the system of eigenfunctions in the space $L_2(\Omega)$ is proved.

In [26], one boundary value problem for the biharmonic equation is studied. This problem contains modified Hadamard integrodifferential operators in the boundary conditions.

In the present paper, continuing the above studies of the solvability of boundary value problems for harmonic and biharmonic equations with both ordinary involution and multiple involution, we are going to investigate similar issues for the Laplace operator with double involution of arbitrary orders. Special form matrices arising in the considered problem are investigated in Theorems 1–3 of Section 2. Then, in Section 3 (see Theorems 4 and 5), with the help of Lemma 1, the existence of eigenfunctions and eigenvalues of the problem under consideration is investigated. In Section 4 (see Theorems 6 and 7), with the help of Lemma 2, the eigenfunctions and eigenvalues of the considered nonlocal differential equation are constructed. These eigenfunctions are presented explicitly. The completeness of the resulting system of eigenfunctions in $L_2(\Omega)$ is established. All new concepts and results obtained are illustrated by seven examples.

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$, $n \geq 2$, and $\partial\Omega = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere. Let also S_1, S_2 be two real commutative orthogonal $n \times n$ matrices such that $S_i^{l_i} = I$, $l_i \in \mathbb{N}$, $i = 1, 2$. Note that, since $|x|^2 = (S_i^T S_i x, x) = (S_i x, S_i x) = |S_i x|^2$, then $x \in \Omega \Rightarrow S_i x \in \Omega$ and $y \in \partial\Omega \Rightarrow S_i y \in \partial\Omega$. For example, the matrix S_i can be an orthogonal matrix of the following form:

$$S = \begin{pmatrix} I_k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos\alpha & -\sin\alpha & \mathbf{0} \\ \mathbf{0} & \sin\alpha & \cos\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{n-k-2} \end{pmatrix},$$

where $\alpha = \frac{2\pi}{l}$, $0 \leq k \leq n-2$, and $\mathbf{0}$ are zero matrices of appropriate size. It is clear that $S^l = I$.

Let $l_1, l_2 \in \mathbb{N}_0$ and $a_0, \dots, a_{l_1-1}, a_{l_1}, \dots, a_{2l_1-1}, \dots, a_{(l_2-1)l_1-1}, \dots, a_{l_2 l_1 - 1}$ be a sequence of real numbers which we denote by $\mathbf{a}$. If we represent the index i in the form $i = (i_2, i_1) \equiv i_2 \cdot l_1 + i_1$, where $i_k = 0, 1, \dots l_k - 1$ for $k = 1, 2$, then the elements of $\mathbf{a}$ can be represented as $a_{(0,0)}, \dots, a_{(0,l_1-1)}, a_{(1,0)}, \dots, a_{(1,l_1-1)}, \dots, a_{(l_2-2,l_1-1)}, \dots, a_{(l_2-1,l_1-1)}$. It is clear that, if $0 \leq i < l_1 l_2$, then $i_1 = \{i/l_1\}$, $i_2 = [i/l_1]$, where $[\cdot]$ and $\{\cdot\}$ are integer and fractional parts of a number. Furthermore, we consider the sequence $\mathbf{a}$ also as a vector.

We introduce a new nonlocal differential operator formed by the sequence $\mathbf{a}$ and the Laplace operator Δ

$$L_{\mathbf{a}} u \equiv \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)} \Delta u(S_2^{i_2} S_1^{i_1} x)$$

and formulate a natural boundary value problem with $L_{\mathbf{a}}$.

Problem S_2. *Find a non-zero function $u(x)$ such that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ and which satisfied the equations*

$$L_{\mathbf{a}} u(x) + \lambda u(x) = 0, \ x \in \Omega, \tag{1}$$
$$u(x) = 0, \ x \in \partial\Omega, \tag{2}$$

where $\lambda \in \mathbb{R}$.

In a special case $l_1 = l_2 = 2$, this problem coincides with the spectral boundary value problem studied in [24].

2. Auxiliary Results

In order to start studying the above problem (1) and (2), we need some auxiliary assertions. We introduce the function

$$v(x) = \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)} u(S_2^{i_2} S_1^{i_1} x), \tag{3}$$

where the summation is carried out over the index $i = (i_2, i_1) \equiv i_2 \cdot l_1 + i_1$ in the form $(0,0), \dots, (0, l_1 - 1), (1,0), \dots, (1, l_1 - 1), \dots, (l_2 - 2, l_1 - 1), \dots, (l_2 - 1, l_1 - 1)$. From equality (3), taking into account that $S_2^{l_2} = S_1^{l_1} = I$, it can be concluded that the following functions $v(S_2^{j_2} S_1^{j_1} x)$, where $j = 0, \dots, l_2 l_1 - 1$ are expressed as a linear combinations of the functions $u(S_2^{i_2} S_1^{i_1} x)$. Let us introduce the following vectors:

$$U(x) = \left(u(S_2^{i_2} S_1^{i_1} x)\right)^T_{i=0,\dots,l_2 l_1 - 1}, \quad V(x) = \left(v(S_2^{i_2} S_1^{i_1} x)\right)^T_{i=0,\dots,l_2 l_1 - 1}$$

of order $l_2 l_1$. Then, the dependence $V(x)$ on $U(x)$ can be presented in the matrix form:

$$V(x) = A_{(2)} U(x), \tag{4}$$

where $A_{(2)} = \left(a_{i,j}\right)_{i,j=0,\dots,l_2 l_1 - 1}$ is some matrix of order $l_2 l_1 \times l_2 l_1$.

Let us investigate the structure of matrices of the form $A_{(2)}$. For this, we introduce a new operation on indices of matrix coefficients as follows: $i \oplus j = (i_2, i_1) \oplus (j_2, j_1) \equiv ((i_2 + j_2 \bmod l_2), (i_1 + j_1 \bmod l_1))$, where (i_2, i_1) is a representation of the index i as mentioned above. It is clear that $\oplus$ is a commutative and associative operation on $i \in \{0, \dots, l_2 l_1 - 1\}$ and $(i_2, i_1) \oplus (0,0) = (i_2, i_1)$. Since $(i_2, i_1) \oplus (j_2, j_1) = (0,0) \Leftrightarrow i_2 + j_2 = 0 \bmod l_2, i_1 + j_1 = 0 \bmod l_1 \Leftrightarrow j_2 = l_2 - i_2, j_1 = l_1 - i_1$, then we can write $\ominus i = (l_2 - i_2, l_1 - i_1)$. For example, if $l_1 = 2$, $l_2 = 3$, then $\ominus(2,1) = (1,1)$ or $\ominus 5 = 3$. If we assume that $(-i_2, -i_1) \equiv \ominus i = (l_2 - i_2, l_1 - i_1)$, then we have

$$\begin{aligned} (-i_2, -i_1) \oplus (j_2, j_1) &= (l_2 - i_2, l_1 - i_1) \oplus (j_2, j_1) \\ &= (l_2 - i_2 + j_2 \bmod l_2, l_1 - i_1 + j_1 \bmod l_1) = (-i_2 + j_2 \bmod l_2, -i_1 + j_1 \bmod l_1) \end{aligned}$$

i.e., the operation $\oplus$ is formally applicable to numbers of the form $(-i_2, -i_1)$. We assume that

$$i \ominus j \equiv i \oplus (\ominus j) = (i_2 - j_2 \bmod l_2, i_1 - j_1 \bmod l_1).$$

We extend the operations $\oplus$ and $\ominus$ to all numbers of the form (i_2, i_1) by setting $(i_2, i_1) \equiv (i_2 \bmod l_2, i_1 \bmod l_1)$. For example, if $l_1 = 2$, $l_2 = 3$, then $(1, -1) = (1, 1)$ and $(5, -3) = (2, 1)$.

Theorem 1. *The matrix $A_{(2)}$ defined by the equality (4) is represented as*

$$A_{(2)} \equiv \left(a_{i,j}\right)_{i,j=0,\dots,l_2 l_1 - 1} = \left(a_{j \ominus i}\right)_{i,j=0,\dots,l_2 l_1 - 1}. \tag{5}$$

The sum of matrices of the form (5) is a matrix of the same form.

Proof. Consider the function $v(S_2^{i_2}S_1^{i_1}x)$ whose coefficients at $u(S_2^{j_2}S_1^{j_1})$ make up the $i \equiv (i_2, i_1)$th row of the matrix $A_{(2)}$

$$v(S_2^{i_2}S_1^{i_1}x) = \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}u(S_2^{j_2}S_1^{j_1}S_2^{i_2}S_1^{i_1}x)$$

$$= \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}u(S_2^{j_2+i_2 \bmod l_2}S_1^{j_1+i_1 \bmod l_1}x). \quad (6)$$

Here, the following properties $S_2^{l_2}x = x$, $S_1^{l_1}x = x$ and $S_1S_2x = S_2S_1x$ of matrices S_1 and S_2 have been used. If we replace the index j by the index k using the equality $k = i \oplus j$, then $k \ominus i = i \oplus j \ominus i = j$, and we have $j \leftrightarrow k$. Substitution $j \to k$ changes the order of summation in (6). For instance, if $l_1 = 2$, $l_2 = 3$ and $i = (0,1)$, then $j : 0,1,2,3,4,5$ goes to $k = 1 \oplus j : 1,0,3,2,5,4$. After changing the index, we have

$$v(S_2^{i_2}S_1^{i_1}x) = \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{k\ominus i}u(S_2^{k_2}S_1^{k_1}x).$$

Comparing the resulting equality with (4), we make sure that (5) is true $a_{i,k} = a_{k\ominus i}$. There is no doubt that, if $\alpha, \beta \in \mathbb{R}$, then

$$\alpha\left(a_{j\ominus i}\right)_{i,j=0,\ldots,l_2l_1-1} + \beta\left(b_{j\ominus i}\right)_{i,j=0,\ldots,l_2l_1-1} = \left(\alpha a_{j\ominus i} + \beta b_{j\ominus i}\right)_{i,j=0,\ldots,l_2l_1-1},$$

which completes the proof of the theorem. □

Example 1. *For example, let us write the matrix $A_{(2)}$ for $l_2 = 3$ and $l_1 = 2$*

$$A_{(2)} = \begin{pmatrix} a_{(0,0)\ominus(0,0)} & a_{(0,1)\ominus(0,0)} & a_{(1,0)\ominus(0,0)} & a_{(1,1)\ominus(0,0)} & a_{(2,0)\ominus(0,0)} & a_{(2,1)\ominus(0,0)} \\ a_{(0,0)\ominus(0,1)} & a_{(0,1)\ominus(0,1)} & a_{(1,0)\ominus(0,1)} & a_{(1,1)\ominus(0,1)} & a_{(2,0)\ominus(0,1)} & a_{(2,1)\ominus(0,1)} \\ a_{(0,0)\ominus(1,0)} & a_{(0,1)\ominus(1,0)} & a_{(1,0)\ominus(1,0)} & a_{(1,1)\ominus(1,0)} & a_{(2,0)\ominus(1,0)} & a_{(2,1)\ominus(1,0)} \\ a_{(0,0)\ominus(1,1)} & a_{(0,1)\ominus(1,1)} & a_{(1,0)\ominus(1,1)} & a_{(1,1)\ominus(1,1)} & a_{(2,0)\ominus(1,1)} & a_{(2,1)\ominus(1,1)} \\ a_{(0,0)\ominus(2,0)} & a_{(0,1)\ominus(2,0)} & a_{(1,0)\ominus(2,0)} & a_{(1,1)\ominus(2,0)} & a_{(2,0)\ominus(2,0)} & a_{(2,1)\ominus(2,0)} \\ a_{(0,0)\ominus(2,1)} & a_{(0,1)\ominus(2,1)} & a_{(1,0)\ominus(2,1)} & a_{(1,1)\ominus(2,1)} & a_{(2,0)\ominus(2,1)} & a_{(2,1)\ominus(2,1)} \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 \\ a_4 & a_5 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_0 & a_1 \\ a_3 & a_2 & a_5 & a_4 & a_1 & a_0 \end{pmatrix}.$$

Let us present some corollaries of Theorem 1.

Corollary 1. *The matrix $A_{(2)}$ is uniquely determined by its first row $\mathbf{a} = \left(a_0, a_1, \ldots, a_{l_2l_1-1}\right)$.*

It is not difficult to see that the i-th row of the matrix $A_{(2)}$ is represented via its 1st row as $\left(a_{0\ominus i}, a_{1\ominus i}, \ldots, a_{(l_2l_1-1)\ominus i}\right)$. We indicate this property of the matrix $A_{(2)}$ by the equality $A_{(2)} \equiv A_{(2)}(\mathbf{a})$. For example, the matrix $A_{(2)}$ from Example 1 can be written as $A_{(2)}(\mathbf{a})$, where $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$.

In Ref. [10], matrices of the following form are studied:

$$A_{(1)}(a_0,\dots,a_{l-1}) = (a_{j-i \bmod l})_{i,j=0,l-1} = \begin{pmatrix} a_0 & a_1 & \dots & a_{l-1} \\ a_{l-1} & a_0 & \dots & a_{l-2} \\ & \dots & & \\ a_1 & a_2 & \dots & a_0 \end{pmatrix}, \tag{7}$$

which coincide with the matrix $A_{(2)}$ in the case $l_2 = 1$, $l_1 = l$ since, in this case, $(i_2, i_1) = (0, i_1) = i_1$ and $i_1 = 0,\dots,l-1$.

Corollary 2. *The matrix $A_{(2)}$ has the structure of a matrix consisting of $l_2 \times l_2$ square blocks, each of which is an $l_1 \times l_1$ matrix of type $A_{(1)}$. If we represent the sequence $\mathbf{a}$ as $\mathbf{a} = (\mathbf{a}_0,\dots,\mathbf{a}_{l_2-1})$, where $\mathbf{a}_{j_2} = (a_{j_2 l_1},\dots,a_{(j_2+1)l_1-1})$ and denote $A_{(1)}^{(j_2)} = A_{(1)}(\mathbf{a}_{j_2})$, then the equality is true*

$$A_{(2)}(\mathbf{a}) = A_{(1)}\left(A_{(1)}^{(0)}, A_{(1)}^{(1)},\dots,A_{(1)}^{(l_2-1)}\right) \equiv \begin{pmatrix} A_{(1)}^{(0)} & A_{(1)}^{(1)} & \dots & A_{(1)}^{(l_2-1)} \\ A_{(1)}^{(l_2-1)} & A_{(1)}^{(0)} & \dots & A_{(1)}^{(l_2-2)} \\ & \dots & & \\ A_{(1)}^{(1)} & A_{(1)}^{(2)} & \dots & A_{(1)}^{(0)} \end{pmatrix}. \tag{8}$$

Proof. Obviously, the block matrix on the right side of (8) has size $l_2 l_1 \times l_2 l_1$. Denote its arbitrary element as $a_{i,j}$, where $i, j = 0,\dots,l_2 l_1 - 1$. If we write $i = (i_2, i_1)$, $j = (j_2, j_1)$, then we have $a_{i,j} = a_{(i_2,i_1),(j_2,j_1)}$. This means that the element $a_{i,j}$ is in the j_2th block column and in the i_2th block row of this block matrix. Therefore, in accordance with the structure of the matrix $A_{(1)}$, we have $a_{i,j} \in A_{(1)}^{(j_2-i_2 \bmod l_2)}$. If we now take into account the values of the indices j_1 and i_1, which mean that the element $a_{i,j}$ is in the j_1-th column and in the i_1-th row of the matrix $A_{(1)}^{(j_2-i_2 \bmod l_2)}$, then $a_{i,j} = b_{j_1-i_1 \bmod l_1}$, where b_k is the element of the 1st row of the matrix $A_{(1)}^{(j_2-i_2 \bmod l_2)}$. Since from the definition of $A_{(1)}^{(j_2)}$ it follows that $b_k = a_{j_2 l_1 + k}$, $k = 0,\dots,l_1-1$, then we have

$$a_{i,j} = b_{j_1-i_1 \bmod l_1} = a_{(j_2-i_2 \bmod l_2)l_1 + (j_1-i_1 \bmod l_1)} = a_{((j_2-i_2 \bmod l_2),(j_1-i_1 \bmod l_1))} = a_{j \ominus i}.$$

Taking into account equality (5), from Theorem 1, this implies the equality of matrices (8). This proves the corollary. □

Example 2. *We can see the property (8) of matrices of the form $A_{(2)}$ by the matrix $A_{(2)}(\mathbf{a})$ from Example 1, where $l_2 = 3$, $l_1 = 2$, $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$. If denote $\mathbf{a}_0 = (a_0, a_1)^T$, $\mathbf{a}_1 = (a_2, a_3)^T$, $\mathbf{a}_2 = (a_4, a_5)^T$ and*

$$A_{(1)}(\mathbf{a}_0) = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} = A_{(1)}^{(0)}, A_{(1)}(\mathbf{a}_1) = \begin{pmatrix} a_2 & a_3 \\ a_3 & a_2 \end{pmatrix} = A_{(1)}^{(1)}, A_{(1)}(\mathbf{a}_2) = \begin{pmatrix} a_4 & a_5 \\ a_5 & a_4 \end{pmatrix} = A_{(1)}^{(2)},$$

then $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)^T$ and the matrix $A_{(2)}(\mathbf{a})$ is written as

$$A_{(2)}(\mathbf{a}) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 \\ a_4 & a_5 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_0 & a_1 \\ a_3 & a_2 & a_5 & a_4 & a_1 & a_0 \end{pmatrix} = \begin{pmatrix} A_{(1)}^{(0)} & A_{(1)}^{(1)} & A_{(1)}^{(2)} \\ A_{(1)}^{(2)} & A_{(1)}^{(0)} & A_{(1)}^{(1)} \\ A_{(1)}^{(1)} & A_{(1)}^{(2)} & A_{(1)}^{(0)} \end{pmatrix} = A_{(1)}\left(A_{(1)}^{(0)}, A_{(1)}^{(1)}, A_{(1)}^{(2)}\right).$$

Corollary 3. *The transposed matrix $A^T_{(2)}(\mathbf{a})$ has the structure of the matrix $A^T_{(2)}$, and besides $A^T_{(2)}(\mathbf{a}) = A_{(2)}(\mathbf{b})$, where $\mathbf{b} = (a_{(-j_2,-j_1)})_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$, and both index components $-j_2$ and $-j_1$ are taken by $\bmod l_2$ and by $\bmod l_1$, respectively.*

Proof. By Theorem 1, we write

$$A^T_{(2)}(\mathbf{a}) = (a_{j\ominus i})^T_{i,j=0,\ldots,(l_2-1,l_1-1)}$$
$$= (a_{i\ominus j})_{i,j=0,\ldots,(l_2-1,l_1-1)} = \left(a_{\ominus(j\ominus i)}\right)_{i,j=0,\ldots,(l_2-1,l_1-1)} = (b_{j\ominus i})_{i,j=0,\ldots,(l_2-1,l_1-1)}.$$

This is why $\mathbf{b} = (a_{\ominus j})_{j=0,\ldots,l_2l_1-1} = (a_{(-j_2,-j_1)})_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$. The corollary is proved. □

Example 3. *For the matrix $A^T_{(2)}(\mathbf{a})$ from Example 2, we have*

$$\mathbf{b} = (a_{\ominus j})_{j=0,\ldots,5} = (a_0, a_{-1}, a_{-2}, a_{-3}, a_{-4}, a_{-5})$$
$$= (a_0, a_{(0,-1)}, a_{(-1,0)}, a_{(-1,-1)}, a_{(-2,0)}, a_{(-2,-1)}) = (a_0, a_{(0,1)}, a_{(2,0)}, a_{(2,1)}, a_{(1,0)}, a_{(1,1)})$$
$$= (a_0, a_1, a_4, a_5, a_2, a_3).$$

Is the multiplication of matrices of the form (5) again such a matrix?

Theorem 2. *The product of matrices of form (5) is again the same matrix and multiplication is commutative.*

Proof. Let the matrices $A(2)(\mathbf{a})$ and $B(2)(\mathbf{b})$ be two matrices of the form (5). Then,

$$A_{(2)}(\mathbf{a})B_{(2)}(\mathbf{b}) = (a_{j\ominus i})_{i,j=0,\ldots l_2l_1-1}(b_{j\ominus i})_{i,j=0,\ldots l_2l_1-1} = \left(\sum_{k=0}^{l_2l_1-1} a_{k\ominus i}b_{j\ominus k}\right)_{i,j=0,\ldots l_2l_1-1}.$$

As in the proof of Theorem 1, let us change the index $k \to s$; in the sum above, in accordance with the equation $k \ominus i = s$. Then, $k = k \ominus i \oplus i = s \oplus i$ and therefore the correspondence $k \leftrightarrow s$ is one-to-one. Thus, the index replacement $k \to s$ changes only the summation order. Due to the commutativity and associativity of $\oplus$, we obtain

$$A_{(2)}(\mathbf{a})B_{(2)}(\mathbf{b}) = \left(\sum_{s=0}^{l_2l_1-1} a_s b_{j\ominus(s\oplus i)}\right)_{i,j=0,\ldots l_2l_1-1} = \left(\sum_{s=0}^{l_2l_1-1} a_s b_{(j\ominus i)\ominus s}\right)_{i,j=0,\ldots l_2l_1-1}. \quad (9)$$

The elements of the first row of the resulting matrix have the form $c_j = \sum_{s=0}^{l_2l_1-1} a_s b_{j\ominus s}$ and hence $A_{(2)}(\mathbf{a})B_{(2)}(\mathbf{b}) = (c_{j\ominus i})_{i,j=0,\ldots l_2l_1-1}$. Therefore, matrix $A_{(2)}(\mathbf{a})B_{(2)}(\mathbf{b})$ has the form (5).

The commutativity of the product $A_{(2)}B_{(2)}$ can be easily obtained from the equality (9). Replacing $(j \ominus i) \ominus s \to k$ in the last sum from (9) and hence $s = (j \ominus i) \ominus k$, we obtain

$$\sum_{s=0}^{l_2l_1-1} a_s b_{(j\ominus i)\ominus s} = \sum_{k=0}^{l_2l_1-1} a_{(j\ominus i)\ominus k} b_k = \sum_{k=0}^{l_2l_1-1} b_k a_{(j\ominus i)\ominus k}.$$

The last sum in this equality is a common element of the matrix $(B_{(2)}A_{(2)})_{i,j}$. Hence, (9) means that $A_{(2)}B_{(2)} = B_{(2)}A_{(2)}$. The theorem is proved. □

Let $\lambda_k = \exp(i2\pi k/l)$, $k = 0, \ldots, l-1$ be the lth root of unity. In [10], it is shown that, for the matrix of the form

$$A_{(1)}(\mathbf{a}) = \begin{pmatrix} a_0 & a_1 & \dots & a_{l-1} \\ a_{l-1} & a_0 & \dots & a_{l-2} \\ & \dots & & \\ a_1 & a_2 & \dots & a_0 \end{pmatrix},$$

where $\mathbf{a} = (a_0, \dots, a_{l-1})$, eigenvectors and eigenvalues have the form

$$\mathbf{b}_k = (1, \lambda_k, \dots, \lambda_k^{l-1})^T, \quad \mu_k = \sum_{s=0}^{l-1} a_s \lambda_k^s = \mathbf{a} \cdot \mathbf{b}_k. \tag{10}$$

Note that the eigenvectors of the matrices $A_{(1)}(\mathbf{a})$ do not depend on the vector $\mathbf{a}$. Is this true for matrices of type $A_{(2)}(\mathbf{a})$? We will see below.

We present a theorem that clarifies questions about eigenvectors and eigenvalues of matrices $A_{(2)}$ from (5).

Theorem 3. *Eigenvectors of the matrix $A_{(2)}(\mathbf{a})$ are written as*

$$\mathbf{a}_{(i_2,i_1)} = \left(I_{l_1}, \lambda_{i_2} I_{l_1}, \dots, \lambda_{i_2}^{l_2-1} I_{l_1}\right)^T \mathbf{b}_{i_1}, \tag{11}$$

where I_{l_1} is identity $l_1 \times l_1$ matrix, vector $\mathbf{b}_{i_1}$ is taken from (10) at $l = l_1$ and λ_{i_2} is the l_2th root of unity, $i_1 = 0, \dots, l_1 - 1$, $i_2 = 0, \dots, l_2 - 1$. The eigenvectors of the matrix $A_{(2)}(\mathbf{a})$ do not depend on the vector $\mathbf{a}$.

Proof. In accordance with Theorem 1, represent the matrix $A_{(2)}$ as

$$A_{(2)}(a_0, \dots, a_{l_2 l_1 - 1}) = \begin{pmatrix} A_{(1)}^{(0)} & A_{(1)}^{(1)} & \dots & A_{(1)}^{(l_2-1)} \\ A_{(1)}^{(l_2-1)} & A_{(1)}^{(0)} & \dots & A_{(1)}^{(l_2-2)} \\ & \dots & & \\ A_{(1)}^{(1)} & A_{(1)}^{(2)} & \dots & A_{(1)}^{(0)} \end{pmatrix}, \tag{12}$$

where $A_{(1)}^{(j_2)} = A_{(1)}(a_{j_2 l_1}, \dots, a_{(j_2+1)l_1 - 1})$, $j_2 = 0, \dots, l_2 - 1$. Consider the block multiplication of a $l_2 l_1 \times l_2 l_1$ matrix by a $l_2 l_1 \times l_1$ matrix of the form

$$\begin{pmatrix} A_{(1)}^{(0)} & A_{(1)}^{(1)} & \dots & A_{(1)}^{(l_2-1)} \\ A_{(1)}^{(l_2-1)} & A_{(1)}^{(0)} & \dots & A_{(1)}^{(l_2-2)} \\ & \dots & & \\ A_{(1)}^{(1)} & A_{(1)}^{(2)} & \dots & A_{(1)}^{(0)} \end{pmatrix} \cdot \begin{pmatrix} I_{l_1} \\ \lambda_{i_2} I_{l_1} \\ \dots \\ \lambda_{i_2}^{l_2-1} I_{l_1} \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ \dots \\ B_{l_2-1} \end{pmatrix},$$

where square blocks B_{i_2} have size $l_1 \times l_1$. Let us extend the values of the upper indices of the matrices $A_{(1)}^{(i_2)}$ to $\mathbb{Z}$ and calculate them by $\bmod l_2$. Then, similarly to (7), the mth block row of the matrix $A_{(2)}$ we represent as $\left(A_{(1)}^{(-m \bmod l_2)}, A_{(1)}^{(1-m \bmod l_2)} \dots, A_{(1)}^{(l_2-m \bmod l_2)}\right)$. Since the exponent of $\lambda_{i_2}^k$ can also be calculated by $\bmod l_2$, then we write

$$B_{i_2} = \sum_{k=0}^{l_2-1} A_{(1)}^{(k-m \bmod l_2)} I_{l_1} \lambda_{i_2}^k = \sum_{k=0}^{l_2-1} \lambda_{i_2}^k A_{(1)}^{(k-m \bmod l_2)} = \sum_{s=0}^{l_2-1} \lambda_{i_2}^{s+m} A_{(1)}^{(s)} = \lambda_{i_2}^m B_0.$$

Here, the substitution $s = k - m \bmod l_2$ of the index has been made. Thus, we have

$$A_{(2)}(a_0,\ldots,a_{l_2l_1-1})\cdot\begin{pmatrix} I_{l_1}\\ \lambda_{i_2}I_{l_1}\\ \cdots\\ \lambda_{i_2}^{l_2-1}I_{l_1}\end{pmatrix}=\begin{pmatrix} B_0\\ B_1\\ \cdots\\ B_{l_2-1}\end{pmatrix}=\begin{pmatrix} I_{l_1}\\ \lambda_{i_2}I_{l_1}\\ \cdots\\ \lambda_{i_2}^{l_2-1}I_{l_1}\end{pmatrix}B_0.$$

It is easy to see that, in the obtained equality, the matrix $B_0=B_0(\lambda_{i_2})=\sum\limits_{s=0}^{l_2-1}\lambda_{i_2}^s A_{(1)}^{(s)}$ has a type of $A_{(1)}$ and hence the vectors $\mathbf{b}_{i_1}$, for $i_1=0,\ldots,l_1-1$ are eigenvectors of B_0. We multiply the above matrix equality on the right by the vector $\mathbf{b}_{i_1}$. Then, we obtain

$$A_{(2)}\begin{pmatrix} I_{l_1}\\ \lambda_{i_2}I_{l_1}\\ \cdots\\ \lambda_{i_2}^{l_2-1}I_{l_1}\end{pmatrix}\mathbf{b}_{i_1}=\begin{pmatrix} I_{l_1}\\ \lambda_{i_2}I_{l_1}\\ \cdots\\ \lambda_{i_2}^{l_2-1}I_{l_1}\end{pmatrix}B_0\mathbf{b}_{i_1}=\lambda_{(i_2,i_1)}\begin{pmatrix} I_{l_1}\\ \lambda_{i_2}I_{l_1}\\ \cdots\\ \lambda_{i_2}^{l_2-1}I_{l_1}\end{pmatrix}\mathbf{b}_{i_1},$$

where $\lambda_{(i_2,i_1)}$ is the eigenvalue of the matrix $B_0(\lambda_{i_2})$ corresponding to the eigenvector $\mathbf{b}_{i_1}$. If we now recall the notation (11), then we obtain

$$A_{(2)}(\mathbf{a})\mathbf{a}_{(i_2,i_1)}=\lambda_{(i_2,i_1)}\mathbf{a}_{(i_2,i_1)},$$

i.e., $\mathbf{a}_{(i_2,i_1)}$ is an eigenvector of $A_{(2)}(\mathbf{a})$. This completes the proof. □

Now, present some corollaries from Theorem 3 that make it possible to construct eigenvectors and eigenvalues of the matrix $A_{(2)}(\mathbf{a})$.

Corollary 4. 1^0. *The eigenvector of $A_{(2)}(\mathbf{a})$ numbered by (i_2,i_1) one can write as*

$$\mathbf{a}_{(i_2,i_1)}=\left(\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)},\tag{13}$$

where λ_2 is the l_2th root of unity and λ_1 is the l_1th root of unity. The eigenvalue corresponding to this eigenvector can be written in a similar form

$$\lambda_{(i_2,i_1)}=\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}a_{(j_2,j_1)}\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}=\mathbf{a}\cdot\mathbf{a}_{(i_2,i_1)}.\tag{14}$$

2^0. *The eigenvectors of the matrix $A^T_{(2)}(\mathbf{a})$ coincide with the eigenvectors $\mathbf{a}_{(i_2,i_1)}$, and the corresponding eigenvalues have the form $\lambda^t_{(i_2,i_1)}=\bar{\lambda}_{(i_2,i_1)}=\lambda_{(-i_2,-i_1)}$.*

Proof. It is easy to see that the Equality (11) can be written as

$$\begin{aligned}\mathbf{a}_{(i_2,i_1)}&=\left(\mathbf{b}^T_{i_1},\lambda_{i_2}\mathbf{b}^T_{i_1},\ldots,\lambda_{i_2}^{l_2-1}\mathbf{b}^T_{i_1}\right)^T\\ &=\left(1,\lambda_{i_1},\ldots,\lambda_{i_1}^{l_1-1},\lambda_{i_2},\lambda_{i_2}\lambda_{i_1},\ldots\lambda_{i_2}\lambda_{i_1}^{l_1-1},\ldots,\lambda_{i_2}^{l_2-1},\lambda_{i_2}^{l_2-1}\lambda_{i_1},\ldots,\lambda_{i_2}^{l_2-1}\lambda_{i_1}^{l_1-1}\right)^T\\ &=\left(\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)},\end{aligned}$$

where the order of the vector elements corresponds to the order established for numbers (j_2,j_1). This proves equality (13).

Furthermore, from Theorem 3, it follows that the eigenvalue $\lambda_{(i_2,i_1)}$ of the matrix $A_{(2)}(\mathbf{a})$ corresponding to the eigenvector $\mathbf{a}_{(i_2,i_1)}$ is the same as the eigenvalue of the matrix

$$B_0(\lambda_{i_2})=\sum_{j_2=0}^{l_2-1}\lambda_{i_2}^{j_2}A_{(1)}^{(j_2)},$$

corresponding to the eigenvector $\mathbf{b}_{i_1}$. Here, $A_{(1)}^{(j_2)} = A_{(1)}(a_{j_2l_1}, \ldots, a_{(j_2+1)l_1-1})$. Since the matrix $B_0(\lambda_{i_2})$ is of type $A_{(1)}$, then, in accordance with (10), we find the vector representing the first row of the matrix $B_0(\lambda_{i_2})$. Denote this vector as $b_0(\lambda_{i_2})$

$$b_0(\lambda_{i_2}) = \left(\sum_{j_2=0}^{l_2-1} \lambda_{i_2}^{j_2} a_{j_2l_1}, \ldots, \sum_{j_2=0}^{l_2-1} \lambda_{i_2}^{j_2} a_{j_2l_1+l_1-1}\right).$$

Using the formula (10), we find

$$\lambda_{(i_2,i_1)} = \sum_{j_1=0}^{l_1-1} \lambda_{i_1}^{j_1} \sum_{j_2=0}^{l_2-1} a_{j_2l_1+i_1} \lambda_{i_2}^{j_2} = \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1},$$

which is the same as (14). Statement 1^0 is proved.

By Corollary 3 and Theorem 3, the eigenvectors of $A_{(2)}^T(\mathbf{a})$ coincide with the eigenvectors of $A_{(2)}(\mathbf{a})$. Let us find the eigenvalue corresponding to the vector $\mathbf{a}_{(i_2,i_1)}$. According to Corollary 3 $A_{(2)}^T(\mathbf{a}) = A_{(2)}(\mathbf{b})$, where $\mathbf{b} = (a_{(-j_2,-j_1)})_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$. This is why

$$\lambda_{(i_2,i_1)}^t = \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(-j_2,-j_1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} = \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{(k_2,k_1)} \lambda_{i_2}^{-k_2} \lambda_{i_1}^{-k_1}$$
$$= \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{(k_2,k_1)} \bar{\lambda}_{i_2}^{k_2} \bar{\lambda}_{i_1}^{k_1} = \bar{\lambda}_{(i_2,i_1)}.$$

It can be seen from the last equality that $\bar{\lambda}_{(i_2,i_1)} = \lambda_{(-i_2,-i_1)}$. Statement 2^0 is proved and hence the corollary is proved. □

Another property of the eigenvectors of the matrix $A_{(2)}(\mathbf{a})$ is given later in Corollary 7.

Remark 1. *The expression $\lambda_{i_2}\lambda_{i_1}$ is an ordered pair, and the first place in it is λ_{i_2} – the l_2-th root of unity, and the second place in it is λ_{i_1} – the l_1th root of unity. Therefore, in the general case, $\lambda_1\lambda_1 \neq \lambda_1^2$.*

Remark 2. *In ([24], Corollary 3), eigenvectors and eigenvectors of matrices, which for $n = 2$ are a special case of matrices $A_{(2)}$, are obtained. The eigenvectors were written as*

$$\mathbf{a}_2^{(i_2,i_1)} = \left((-1)^{j_2i_2+j_1i_1}\right)_{(j_2,j_1)=0,\ldots,3},$$

which is the same as (13) for $l_1 = l_2 = 2$. Indeed, in this case $\lambda_{i_2} = (-1)^{i_2}$, $\lambda_{i_1} = (-1)^{i_1}$, $(l_2-1, l_1-1) = (1,1) = 3$ and hence the common term of the eigenvector from (13) has the form

$$\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1} = (-1)^{i_2j_2+i_1j_1},$$

which coincides with the common term of the vector $\mathbf{a}_2^{(i_2,i_1)}$. The eigenvalues obtained in (14) also coincide with those found in [24] for $n = 2$.

Example 4. *For the matrix $A_{(2)}(\mathbf{a})$ from Example 2, we have $\lambda_{i_2} = \lambda^{i_2}$, $\lambda_{i_1} = (-1)^{i_1}$, where $\lambda = \exp(i\frac{2\pi}{3})$. Therefore, according to the formula $\mathbf{a}_{(i_2,i_1)} = \left(\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}\right)_{(j_2,j_1)=0,\ldots,(2,1)}^T$ (13), we obtain*

$$\mathbf{a}_{(0,0)} = (1,1,1,1,1,1), \quad \mathbf{a}_{(0,0)} = (1,-1,1,-1,1,-1), \quad \mathbf{a}_{(1,0)} = (1,1,\lambda,\lambda,\bar{\lambda},\bar{\lambda}),$$
$$\mathbf{a}_{(1,1)} = (1,-1,\lambda,-\lambda,\bar{\lambda},-\bar{\lambda}), \quad \mathbf{a}_{(2,0)} = (1,1,\bar{\lambda},\bar{\lambda},\lambda,\lambda), \quad \mathbf{a}_{(2,1)} = (1,-1,\bar{\lambda},-\bar{\lambda},\lambda,-\lambda),$$

and, using the formula $\lambda_{(i_2,i_1)} = \mathbf{a}\cdot\mathbf{a}_{(i_2,i_1)}$ (14), *we calculate*

$$\lambda_{(0,0)} = a_0 + a_1 + a_2 + a_3 + a_4 + a_5, \qquad \lambda_{(0,1)} = a_0 - a_1 + a_2 - a_3 + a_4 - a_5,$$
$$\lambda_{(1,0)} = a_0 + a_1 + \lambda(a_2 + a_3) + \bar{\lambda}(a_4 + a_5), \quad \lambda_{(1,1)} = a_0 - a_1 + \lambda(a_2 - a_3) + \bar{\lambda}(a_4 - a_5),$$
$$\lambda_{(2,0)} = a_0 + a_1 + \bar{\lambda}(a_2 + a_3) + \lambda(a_4 + a_5), \quad \lambda_{(2,1)} = a_0 - a_1 + \bar{\lambda}(a_2 - a_3) + \lambda(a_4 - a_5).$$

3. The Problem $\mathbf{S}_2$

To consider Problem $\mathbf{S}_2$, we need the following statement.

Lemma 1. *([10], Lemma 3.1) Let S be an orthogonal matrix, then the operator $I_S u(x) = u(Sx)$ and the Laplace operator Δ satisfy the equality $\Delta I_S u(x) = I_S \Delta u(x)$ for $u \in C^2(\Omega)$. The operator $\Lambda = \sum_{i=1}^{n} x_i u_{x_i}(x)$ and operator I_S also satisfy the equality $\Lambda I_S u(x) = I_S \Lambda u(x)$ for $u \in C^1(\bar{\Omega})$.*

Corollary 5. *Equation (1) generates a matrix equation which is equivalent to it*

$$A_{(2)}(\mathbf{a})\Delta U(x) + \lambda U(x) = 0, \tag{15}$$

where $U(x) = \left(u(S_2^{i_2} S_1^{i_1} x)\right)^T_{(i_2,i_1)=0,\ldots,(l_2-1,l_1-1)}$ *and* $\lambda \in \mathbb{R}$.

Proof. Let the function $u(x)$ be a solution to equation (1). Let us denote

$$v(x) = \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)} u(S_2^{i_2} S_1^{i_1} x).$$

and $V(x) = \left(v(S_2^{j_2} S_1^{j_1} x)\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$. The function $v(x)$ generates equality (4). Let us apply the Laplace operator Δ to (4). Since the matrices of the form $S_2^{i_2} S_1^{i_1}$ are orthogonal, by Lemma 1, we obtain

$$\begin{aligned}
\Delta V(x) &= \left(\Delta I_{S_2^{j_2} S_1^{j_1}} v(x)\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} = \left(I_{S_2^{j_2} S_1^{j_1}} \Delta v(x)\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} \\
&= \left(I_{S_2^{j_2} S_1^{j_1}} \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)} I_{S_2^{i_2} S_1^{i_1}} \Delta u(x)\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} \\
&= \left(\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} a_{(i_2,i_1)} I_{S_2^{j_2+i_2 \bmod l_2} S_1^{j_1+i_1 \bmod l_1}} \Delta u(x)\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} \\
&= \left(\sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{k\ominus j} I_{S_2^{k_2} S_1^{k_1}} \Delta u(x)\right)^T_{(k_2,k_1)=0,\ldots,(l_2-1,l_1-1)} \\
&= \left(\sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} a_{k\ominus j} \Delta u(S_2^{k_2} S_1^{k_1} x)\right)^T_{(k_2,k_1)=0,\ldots,(l_2-1,l_1-1)} = A_{(2)}(\mathbf{a})\Delta U(x).
\end{aligned}$$

In the transformations made, the replacement of the summation index $j \oplus i = k$ was used. Hence, using the equality $\Delta v(S_2^{k_2} S_1^{k_1} x) + \lambda u(S_2^{k_2} S_1^{k_1} x) = 0$ (see (1)), which implies that $\Delta V(x) + \lambda U(x) = 0$, we easily obtain (15). Finally, note that the first equation in (15) is the same as (1). This completes the proof. □

Using Lemma 1, we are going to state the existence of the eigenvalues of Problem $\mathbf{S}_2$.

Theorem 4. *Assume that the non-zero function $u(x)$ is an eigenfunction of Problem $\mathbf{S}_2$, and λ is its eigenvalue corresponding to $u(x)$. The function*

$$w(x) = U(x) \cdot \mathbf{a}_{(i_2,i_1)},$$

where $U(x) = u\left(S_2^{j_2} S_1^{j_1} x\right)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}$ and $\mathbf{a}_{(i_2,i_1)}$ is an eigenvector of the matrix $A_{(2)}(\mathbf{a})$ such that $\lambda_{(i_2,i_1)} \neq 0$ is a solution to the boundary value problem

$$\Delta w(x) + \mu w(x) = 0,\ x \in \Omega, \tag{16}$$

$$w(x) = 0,\ x \in \partial\Omega, \tag{17}$$

where $\mu = \lambda / \bar{\lambda}_{(i_2,i_1)}$.

Proof. Let us take $u(x)$ a non-zero eigenfunction of Problem $\mathbf{S}_2$ and the corresponding eigenvalue λ. In accordance with Corollary 5, equality (15) holds. If we multiply this equality by the vector $\mathbf{a}_{(i_2,i_1)}$ scalarly, then we obtain

$$A_{(2)}(\mathbf{a}) \Delta U(x) \cdot \mathbf{a}_{(i_2,i_1)} + \lambda U(x) \cdot \mathbf{a}_{(i_2,i_1)} = 0,$$

where we find

$$\Delta\Big(U(x) \cdot A^T_{(2)}(\mathbf{a}) \mathbf{a}_{(i_2,i_1)}\Big) + \lambda U(x) \cdot \mathbf{a}_{(i_2,i_1)} = 0.$$

Since, due to Corollary 4, the vector $\mathbf{a}_{(i_2,i_1)}$ is also an eigenvector of the matrix $A^T_{(2)}$, and $\bar{\lambda}_{(i_2,i_1)}$ is its eigenvalue, then we have

$$\bar{\lambda}_{(i_2,i_1)} \Delta w(x) + \lambda w(x) = 0$$

and, since $\lambda = \bar{\lambda}_{(i_2,i_1)} \mu$, we obtain

$$0 = \bar{\lambda}_{(i_2,i_1)} (\Delta w(x) + \mu w(x)),$$

where, because $\lambda_{(i_2,i_1)} \neq 0$, we obtain the equality (16)

$$\Delta w(x) + \mu w(x) = 0, \quad x \in \Omega.$$

Lastly, because $u(x) = 0$, for $x \in \partial\Omega$, and $x \in \partial\Omega \Rightarrow S_2^{j_2} S_1^{j_1} x \in \partial\Omega$, then we have $U(x) = 0$ for $x \in \partial\Omega$. Therefore, we obtain $w(x) = U(x) \cdot \mathbf{a}_{(i_2,i_1)} = 0$, for $x \in \partial\Omega$. This completes the proof. □

Let us prove the assertion converse to Theorem 4. It provides an opportunity to find solutions to the main Problem $\mathbf{S}_2$.

Theorem 5. *Assume that the non-zero function $u(x)$ is a solution of the boundary value problem (16) and (17) for some $\mu > 0$*

$$\Delta w(x) + \mu w(x) = 0,\ x \in \Omega, \quad w(x) = 0,\ x \in \partial\Omega,$$

then the function $u_{(i_2,i_1)}(x)$ determined from the equality

$$u_{(i_2,i_1)}(x) = W(x) \cdot \mathbf{a}_{(i_2,i_1)}, \tag{18}$$

where

$$W(x) = \Big(w(S_2^{j_2} S_1^{j_1} x)\Big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}$$

and the vector $\mathbf{a}_{(i_2,i_1)}$ *from* (13) *is an eigenvector of the matrix* $A_{(2)}(\mathbf{a})$ *with an eigenvalue* $\lambda_{(i_2,i_1)} \neq 0$*, which is a solution to Problem* $\mathbf{S}_2$ *for* $\lambda = \mu\bar{\lambda}_{(i_2,i_1)}$.

Proof. Let $w(x) \neq 0$ be a solution to the problem (16) and (17). Consider the vector $W(x) = \left(w(S_2^{j_2}S_1^{j_1}x)\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$ and compose the function

$$u_{(i_2,i_1)}(x) = W(x)\cdot\mathbf{a}_{(i_2,i_1)},$$

where $x \in \Omega$. It is not difficult to see that, according to Corollary 4, we have in Ω

$$\begin{aligned}
u_{(i_2,i_1)}(S_2^{j_2}S_1^{j_1}x) &= W(S_2^{j_2}S_1^{j_1}x)\cdot\mathbf{a}_{(i_2,i_1)} \\
&= \left(w(S_2^{j_2+k_2}S_1^{j_1+k_1}x)\right)^T_{(k_2,k_1)=0,\ldots,(l_2-1,l_1-1)}\cdot\left(\lambda_{i_2}^{k_2}\lambda_{i_1}^{k_1}\right)^T_{(k_2,k_1)=0,\ldots,(l_2-1,l_1-1)} \\
&= \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} w(S_2^{j_2+k_2}S_1^{j_1+k_1}x)\lambda_{i_2}^{k_2}\lambda_{i_1}^{k_1} = \sum_{(m_2,m_1)=0}^{(l_2-1,l_1-1)} w(S_2^{m_2}S_1^{m_1}x)\lambda_{i_2}^{m_2-j_2}\lambda_{i_1}^{m_1-j_1} \\
&= \left(\sum_{(m_2,m_1)=0}^{(l_2-1,l_1-1)} w(S_2^{m_2}S_1^{m_1}x)\lambda_{i_2}^{m_2}\lambda_{i_1}^{m_1}\right)\lambda_{i_2}^{-j_2}\lambda_{i_1}^{-j_1} = \bar{\lambda}_{i_2}^{j_2}\bar{\lambda}_{i_1}^{j_1}W(x)\cdot\mathbf{a}_{(i_2,i_1)}.
\end{aligned}$$

Therefore, again by Corollary 4,

$$\begin{aligned}
U_{(i_2,i_1)}(x) &= \left(u_{(i_2,i_1)}(S_2^{j_2}S_1^{j_1}x)\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} \\
&= W(x)\cdot\mathbf{a}_{(i_2,i_1)}\left(\bar{\lambda}_{i_2}^{j_2}\bar{\lambda}_{i_1}^{j_1}\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} = u_{(i_2,i_1)}(x)\bar{\mathbf{a}}_{(i_2,i_1)}.
\end{aligned}$$

Here, we used the substitution of indexes $j\oplus k = m \Leftrightarrow k = m\ominus j$. Thus,

$$\Delta U_{(i_2,i_1)}(x) = \Delta u_{(i_2,i_1)}(x)\bar{\mathbf{a}}_{(i_2,i_1)}$$

and therefore because, by Lemma 1,

$$\begin{aligned}
\Delta W(x) &= \left(\Delta w(S_2^{j_2}S_1^{j_1}x)\right)^T_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} \\
&= \left(-\mu w(S_2^{j_2}S_1^{j_1}x)\right)_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)} = -\mu W(x),
\end{aligned}$$

we obtain

$$\begin{aligned}
A_{(2)}(\mathbf{a})\Delta U_{(i_2,i_1)}(x) &= \Delta u_{(i_2,i_1)}(x)A_{(2)}(\mathbf{a})\bar{\mathbf{a}}_{(i_2,i_1)} = \Delta u_{(i_2,i_1)}(x)\overline{A_{(2)}(\mathbf{a})\mathbf{a}}_{(i_2,i_1)} \\
&= (\Delta W(x)\cdot\mathbf{a}_{(i_2,i_1)})\bar{\lambda}_{(i_2,i_1)}\bar{\mathbf{a}}_{(i_2,i_1)} = -\mu(W(x)\cdot\mathbf{a}_{(i_2,i_1)})\bar{\lambda}_{(i_2,i_1)}\bar{\mathbf{a}}_{(i_2,i_1)} \\
&= -\mu\bar{\lambda}_{(i_2,i_1)}u_{(i_2,i_1)}(x)\bar{\mathbf{a}}_{(i_2,i_1)} = -\mu\bar{\lambda}_{(i_2,i_1)}U_{(i_2,i_1)}(x).
\end{aligned}$$

Considering the first component of this equality, we obtain

$$\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}\Delta u_{(i_2,i_1)}(S_2^{j_2}S_1^{j_1}x) = -\mu\bar{\lambda}_{(i_2,i_1)}u_{(i_2,i_1)}(x), \quad x\in\Omega,$$

which means that the function $u_{(i_2,i_1)}(x)$ satisfies the equation (1).

Let us make sure that the boundary conditions (2) are met. Since $x \in \partial\Omega \Rightarrow S_2^{j_2} S_1^{j_1} x \in \partial\Omega$, then, for $x \in \partial\Omega$, we have

$$u_{(i_2,i_1)}(x) = W(x) \cdot \mathbf{a}_{(i_2,i_1)} = \left(w(S_2^{j_2} S_1^{j_1} x) \right)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} \cdot \mathbf{a}_{(i_2,i_1)} = \mathbf{0} \cdot \mathbf{a}_{(i_2,i_1)} = 0.$$

Thus, the function $u_{(i_2,i_1)}(x)$ is a solution to Problem $\mathbf{S}_2$. This completes the proof. □

Example 5. *Consider the problem* (1) *and* (2) *with* $l_2 = 3$, $l_1 = 2$ *and* $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$. *Let us use Theorem 5. To do this, take the eigenvectors of the matrix* $A_{(2)}(\mathbf{a})$ *in the form* (13) *from Example 4. Let* μ *be an eigenvalue of the boundary value problem* (16) *and* (17) *and* $w_\mu(x)$ *be a corresponding eigenfunction. Then, the eigenfunctions of the problem* (1) *and* (2) *corresponding to* μ *can be taken in the form* $u_{(i_2,i_1)}(x) = W(x) \cdot \mathbf{a}_{(i_2,i_1)}$ (18)*:*

$$\begin{aligned}
u_{(0,0)}(x) &= w_\mu(x) + w_\mu(S_1x) + w_\mu(S_2x) + w_\mu(S_2S_1x) + w_\mu(S_2^2x) + w_\mu(S_2^2S_1x), \\
u_{(0,1)}(x) &= w_\mu(x) - w_\mu(S_1x) + w_\mu(S_2x) - w_\mu(S_2S_1x) + w_\mu(S_2^2x) - w_\mu(S_2^2S_1x), \\
u_{(1,0)}(x) &= w_\mu(x) + w_\mu(S_1x) + \lambda w_\mu(S_2x) + \lambda w_\mu(S_2S_1x) + \bar{\lambda} w_\mu(S_2^2x) + \bar{\lambda} w_\mu(S_2^2S_1x), \\
u_{(1,1)}(x) &= w_\mu(x) - w_\mu(S_1x) + \lambda w_\mu(S_2x) - \lambda w_\mu(S_2S_1x) + \bar{\lambda} w_\mu(S_2^2x) - \bar{\lambda} w_\mu(S_2^2S_1x), \\
u_{(2,0)}(x) &= w_\mu(x) + w_\mu(S_1x) + \bar{\lambda} w_\mu(S_2x) + \bar{\lambda} w_\mu(S_2S_1x) + \lambda w_\mu(S_2^2x) + \lambda w_\mu(S_2^2S_1x), \\
u_{(2,0)}(x) &= w_\mu(x) - w_\mu(S_1x) + \bar{\lambda} w_\mu(S_2x) - \bar{\lambda} w_\mu(S_2S_1x) + \lambda w_\mu(S_2^2x) - \lambda w_\mu(S_2^2S_1x).
\end{aligned}$$

If we use the eigenvalues of the matrix $A_{(2)}(\mathbf{a})$ *from* (13)*, then the eigenvalues of the problem* (1) *and* (2) *corresponding to the eigenfunctions written above look like* $\mu_{(i_2,i_1)} = \mu\bar{\lambda}_{(i_2,i_1)} = \mu\lambda_{(-i_2,-i_1)}$*:*

$$\begin{aligned}
&\mu_{(0,0)} = \mu\lambda_{(0,0)}, \quad \mu_{(0,1)} = \mu\lambda_{(0,1)}, \quad \mu_{(1,0)} = \mu\lambda_{(2,0)}, \\
&\mu_{(1,1)} = \mu\lambda_{(2,1)}, \quad \mu_{(2,0)} = \mu\lambda_{(1,0)}, \quad \mu_{(2,1)} = \mu\lambda_{(1,1)}.
\end{aligned}$$

Next, we need to expand a given polynomial into a sum of "generalized parity" polynomials. Let $H(x)$ be some function defined on Ω. Let us denote

$$F_{(i_2,i_1)}[H](x) = \frac{1}{l_2 l_1} \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} H(S_2^{j_2} S_1^{j_1} x), \quad x \in \Omega. \tag{19}$$

Lemma 2. *The function* $F_{(i_2,i_1)}[H](x)$ *has the "generalized parity" property*

$$F_{(i_2,i_1)}[H](S_2^{k_2} S_1^{k_1} x) = \bar{\lambda}_{i_2}^{k_2} \bar{\lambda}_{i_1}^{k_1} F_{(i_2,i_1)}[H](x) \tag{20}$$

and the following equality

$$F_{(k_2,k_1)}\left[F_{(i_2,i_1)}[H]\right](x) = \begin{cases} F_{(i_2,i_1)}[H](x) & (i_2,i_1) = (k_2,k_1), \\ 0 & (i_2,i_1) \neq (k_2,k_1). \end{cases} \tag{21}$$

holds true. In addition, the function $H(x)$ *can be expanded in the form*

$$H(x) = \sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} F_{(i_2,i_1)}[H](x), \quad x \in \Omega. \tag{22}$$

Proof. It is not hard to see that

$$F_{(i_2,i_1)}[H]\left(S_2^{k_2}S_1^{k_1}x\right)=\frac{1}{l_2l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}H\left(S_2^{k_2+j_2}S_1^{k_1+j_1}x\right)$$

$$=\frac{1}{l_2l_1}\sum_{(m_2,m_1)=0}^{(l_2-1,l_1-1)}\lambda_{i_2}^{m_2-k_2}\lambda_{i_1}^{m_1-k_1}H\left(S_2^{m_2}S_1^{m_1}x\right)=\lambda_{i_2}^{-k_2}\lambda_{i_1}^{-k_1}F_{(i_2,i_1)}[H](x)=\bar{\lambda}_{i_2}^{k_2}\bar{\lambda}_{i_1}^{k_1}F_{(i_2,i_1)}[H](x),$$

where, as in Theorem 2, the replacement of the index $k\oplus j=m$ is done. Therefore, equality (20) holds true.

Consider now the equality (22). It is easy to see that

$$\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)}F_{(i_2,i_1)}[H](x)=\frac{1}{l_2l_1}\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}H\left(S_2^{j_2}S_1^{j_1}x\right)$$
$$=\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}H\left(S_2^{j_2}S_1^{j_1}x\right)\frac{1}{l_2l_1}\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)}\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}. \tag{23}$$

Let us transform the inner sum from the right side of (23). Let $(j_2,j_1)\neq 0$, then, for example, $j_2\neq 0$ which means $\lambda_{j_2}\neq 1$. Taking into account that $\lambda_{j_2}^{l_2}=1$, by a simple combinatorial identity, we find

$$\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)}\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}=\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)}\lambda_{j_2}^{i_2}\lambda_{j_1}^{i_1}=\sum_{i_2=0}^{l_2-1}\lambda_{j_2}^{i_2}\sum_{i_1=0}^{l_1-1}\lambda_{j_1}^{i_1}=\frac{\lambda_{j_2}^{l_2}-1}{\lambda_{j_2}-1}\sum_{i_1=0}^{l_1-1}\lambda_{j_1}^{i_1}=0. \tag{24}$$

If $(j_2,j_1)=0$, then $\lambda_{j_2}=1$, $\lambda_{j_1}=1$ and so

$$\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)}\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}=l_2l_1.$$

Therefore, the expression on the right side of (23) is equal to $H(S_2^0S_1^0x)=H(x)$. This proves the equality (22).

Now, let us prove (21). It is not hard to see that, using (20) and (22), we can write

$$F_{(k_2,k_1)}\left[F_{(i_2,i_1)}[H]\right](x)$$
$$=\frac{1}{l_2l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}\lambda_{k_2}^{j_2}\lambda_{k_1}^{j_1}F_{(i_2,i_1)}[H]\left(S_2^{j_2}S_1^{j_1}x\right)=\frac{1}{l_2l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}\lambda_{k_2}^{j_2}\lambda_{k_1}^{j_1}\bar{\lambda}_{i_2}^{j_2}\bar{\lambda}_{i_1}^{j_1}F_{(i_2,i_1)}[H](x)$$
$$=F_{(i_2,i_1)}[H](x)\frac{1}{l_2l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}(\lambda_{k_2}\bar{\lambda}_{i_2})^{j_2}(\lambda_{k_1}\bar{\lambda}_{i_1})^{j_1}=F_{(i_2,i_1)}[H](x)\frac{1}{l_2l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}\lambda_{k_2-i_2}^{j_2}\lambda_{k_1-j_1}^{j_1}.$$

Since, by virtue of (24), the formula

$$\frac{1}{l_2l_1}\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)}\lambda_{k_2-i_2}^{j_2}\lambda_{k_1-j_1}^{j_1}=\begin{cases}1 & (k_2,k_1)=(i_2,i_1),\\ 0 & (k_2,k_1)\neq(i_2,i_1)\end{cases}$$

holds true, then (21) follows from the last equality. Here, the equalities $\lambda_{k_2}\bar{\lambda}_{i_2}=\lambda_{k_2}\lambda_{-i_2}=\lambda_{k_2-i_2}$ are taken into account. The lemma is proved. □

Example 6. *Let $l_2=l_1=2$, $S_1x=(-x_1,x_2)$, $S_2x=(x_1,-x_2)$. Taking into account that $\lambda_{i_2}=(-1)^{i_2}$, $\lambda_{i_1}=(-1)^{i_1}$, we obtain*

$$F_{(0,0)}[H](x) = \frac{1}{4}(H(x_1,x_2) + H(-x_1,x_2) + H(x_1,-x_2) + H(-x_1,-x_2)),$$
$$F_{(0,1)}[H](x) = \frac{1}{4}(H(x_1,x_2) - H(-x_1,x_2) + H(x_1,-x_2) - H(-x_1,-x_2)),$$
$$F_{(1,0)}[H](x) = \frac{1}{4}(H(x_1,x_2) + H(-x_1,x_2) - H(x_1,-x_2) - H(-x_1,-x_2)),$$
$$F_{(1,1)}[H](x) = \frac{1}{4}(H(x_1,x_2) - H(-x_1,x_2) - H(x_1,-x_2) + H(-x_1,-x_2)).$$

Let the function $H(x)$ be even in x_1. Then, its components $F_{(i_2,i_1)}[H]$ of generalized parity $(0,1)$ and $(1,1)$ are zero.

Consider homogeneous harmonic polynomial $H_m(x_1,x_2)$ of degree m and let (r,φ) be the polar coordinates of $x=(x_1,x_2)$. Then, there exist $\alpha,\beta \in \mathbb{R}$ such that

$$H_m(x) = \alpha\,\mathrm{Re}(x_1+ix_2)^m + \beta\,\mathrm{Im}(x_1+ix_2)^m = r^m(\alpha\cos m\varphi + \beta\sin m\varphi)$$

and hence

$$H_m(-x_1,x_2) = \alpha\,\mathrm{Re}(-x_1+ix_2)^m + \beta\,\mathrm{Im}(-x_1+ix_2)^m = (-r)^m(\alpha\cos m\varphi - \beta\sin m\varphi),$$
$$H_m(x_1,-x_2) = \alpha\,\mathrm{Re}(x_1-ix_2)^m + \beta\,\mathrm{Im}(x_1-ix_2)^m = r^m(\alpha\cos m\varphi - \beta\sin m\varphi),$$
$$H_m(-x_1,-x_2) = (-r)^m(\alpha\cos m\varphi + \beta\sin m\varphi).$$

The operator $F_{(i_2,i_1)}[\cdot]$ extracts the following components of the harmonic polynomial $H_m(x)$:

$$F_{(0,0)}[H_m](x) = \alpha r^m\frac{1+(-1)^m}{2}\cos m\varphi, \quad F_{(0,1)}[H_m](x) = \alpha r^m\frac{1-(-1)^m}{2}\cos m\varphi,$$
$$F_{(1,0)}[H_m](x) = \beta r^m\frac{1-(-1)^m}{2}\sin m\varphi, \quad F_{(1,1)}[H_m](x) = \beta r^m\frac{1+(-1)^m}{2}\sin m\varphi.$$

Thus, for $m \in \mathbb{N}_0$,

$$\begin{aligned} F_{(0,0)}[H_{2m}](x) &= \alpha r^{2m}\cos 2m\varphi, & F_{(1,1)}[H_{2m}](x) &= \beta r^{2m}\sin 2m\varphi, \\ F_{(0,1)}[H_{2m+1}](x) &= \alpha r^{2m+1}\cos(2m+1)\varphi, & F_{(1,0)}[H_{2m+1}](x) &= \beta r^{2m+1}\sin(2m+1)\varphi, \end{aligned} \tag{25}$$

and the rest of the components vanish

$$F_{(0,1)}[H_{2m}](x) = F_{(1,0)}[H_{2m}](x) = 0, \quad F_{(0,0)}[H_{2m+1}](x) = F_{(1,1)}[H_{2m+1}](x) = 0.$$

4. Finding Solutions to Problem S_2

Let us rewrite the result of Theorem 5 in a more convenient form.

Theorem 6. *Solutions to the boundary value problem (1) and (2) can be represented as*

$$\hat{u}_{(i_2,i_1)}(x) = F_{(i_2,i_1)}[w_\mu](x), \quad \lambda_{\mu,(i_2,i_1)} = \mu\sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)}\bar{\lambda}_{i_2}^{j_2}\bar{\lambda}_{i_1}^{j_1}, \tag{26}$$

where the operator $F_{(i_2,i_1)}[\cdot]$ is defined in (19), the function $w_\mu(x)$ is a solution to the boundary value problem (16) and (17)

$$\Delta w(x) + \mu w(x) = 0,\ x \in \Omega; \quad w(x) = 0,\ x \in \partial\Omega$$

for some $\mu \in \mathbb{R}_+$. Eigenfunctions $\hat{u}_{(i_2,i_1)}(x)$ for $(i_2,i_1) = 0,\ldots,(l_2-1,l_1-1)$ and fixed μ are orthogonal in $L_2(\Omega)$.

The functions $\hat{u}_{(i_2,i_1)}(x)$ are a part of the function $w_\mu(x)$ in the sense that

$$\sum_{(i_2,i_1)=0}^{(l_2-1,l_1-1)} \hat{u}_{(i_2,i_1)}(x) = w_\mu(x). \tag{27}$$

Proof. Denote $\hat{u}_{(i_2,i_1)}(x) = \frac{1}{l_2 l_1} u_{(i_2,i_1)}(x)$. It is clear that $\hat{u}_{(i_2,i_1)}(x)$ is also an eigenfunction of the problem (1) and (2). It is not hard to see that (18) implies

$$\hat{u}_{(i_2,i_1)}(x) = \frac{1}{l_2 l_1} u_{(i_2,i_1)}(x) = \frac{1}{l_2 l_1} W_\mu(x) \cdot \mathbf{a}_{(i_2,i_1)}$$
$$= \frac{1}{l_2 l_1} \Big(w_\mu(S_2^{j_2} S_1^{j_1} x) \Big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)} \cdot \Big(\lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} \Big)^T_{(j_2,j_1)=0,\dots,(l_2-1,l_1-1)}$$
$$= \frac{1}{l_2 l_1} \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} w_\mu(S_2^{j_2} S_1^{j_1} x) \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} = F_{(i_2,i_1)}[w_\mu](x),$$

which proves the first formula from (26).

The eigenvalues of the problem (1) and (2) corresponding to eigenfunction $\hat{u}_{(i_2,i_1)}(x)$, by Theorem 5 and (14) from Corollary 4 can be taken in the form

$$\lambda_{\mu,(i_2,i_1)} = \mu \bar{\lambda}_{(i_2,i_1)} = \mu \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)} \bar{\lambda}_{i_2}^{j_2} \bar{\lambda}_{i_1}^{j_1}.$$

We now prove that the functions $\hat{u}_{(i_2,i_1)}(x)$ and $\hat{u}_{(j_2,j_1)}(x)$ for $(i_2, i_1) \neq (j_2, j_1)$ are orthogonal in $L_2(\Omega)$. Indeed, if $(i_2 - j_2, i_1 - j_1) \neq 0$, then either $i_2 - j_2 \neq 0 \bmod l_2$ or $i_1 - j_1 \neq 0 \bmod l_1$. Let, for example, $i_2 - j_2 \neq 0$ and hence $\lambda_{i_2-j_2} \neq 1$. Then, using Lemma 4.1 from [10], we obtain the following equality for $g \in C(\Omega)$

$$\int_\Omega g(S_2\xi)\, d\xi = \int_\Omega g(\xi)\, d\xi.$$

Therefore, by the Equality (20) from Lemma 2, we have

$$\int_\Omega \hat{u}_{(i_2,i_1)}(x) \bar{\hat{u}}_{(j_2,j_1)}(x)\, dx = \int_\Omega F_{(i_2,i_1)}[w_\mu](x) \bar{F}_{(j_2,j_1)}[w_\mu](x)\, dx$$
$$= \int_\Omega F_{(i_2,i_1)}[w_\mu](S_2 x) \bar{F}_{(j_2,j_1)}[w_\mu](S_2 x)\, dx$$
$$= \bar{\lambda}_{i_2} \lambda_{j_2} \int_\Omega F_{(i_2,i_1)}[w_\mu](x) \bar{F}_{(j_2,j_1)}[w_\mu](x)\, dx = \bar{\lambda}_{i_2-j_2} \int_\Omega \hat{u}_{(i_2,i_1)}(x) \bar{\hat{u}}_{(j_2,j_1)}(x)\, dx. \tag{28}$$

Since $\lambda_{i_2-j_2} \neq 1$, then this immediately implies the orthogonality

$$\int_\Omega \hat{u}_{(i_2,i_1)}(x) \bar{\hat{u}}_{(j_2,j_1)}(x)\, dx = 0.$$

Finally, the equality (27) is a consequence of the equality (22) from Lemma 2 for $H = w_\mu$. The theorem is proved. □

Corollary 6. *If $H(x)$ is a harmonic polynomial with real coefficients, then the harmonic polynomials $F_{(i_2,i_1)}[H](x)$, $(i_2, i_1) = 0, \dots, (l_2 - 1, l_1 - 1)$ are orthogonal and linearly independent.*

Proof. Indeed, let $(i_2, i_1) \neq (j_2, j_1)$, which is possible, for example, for $i_1 \neq j_1$ whence $\lambda_{i_1 - j_1} \neq 1$. By analogy with (28) and according to Lemma 4.1 from [10], we obtain

$$\int_{\partial\Omega} F_{(i_2,i_1)}[H](x)\bar{F}_{(j_2,j_1)}[H](x)\,ds = \int_{\partial\Omega} F_{(i_2,i_1)}[H](S_2 x)\bar{F}_{(j_2,j_1)}[H](S_2 x)\,ds$$
$$= \bar{\lambda}_{i_1}\lambda_{j_1}\int_{\partial\Omega} F_{(i_2,i_1)}[H](x)\bar{F}_{(j_2,j_1)}[H](x)\,ds = \bar{\lambda}_{i_1-j_1}\int_{\partial\Omega} F_{(i_2,i_1)}[H](x)\bar{F}_{(j_2,j_1)}[H](x)\,ds,$$

where, because $\lambda_{i_1-j_1} \neq 1$, we obtain the orthogonality of $F_{(i_2,i_1)}[H](x)$ and $F_{(j_2,j_1)}[H](x)$ on $\partial\Omega$, and hence their linear independence. The corollary is proved. □

Corollary 7. *The matrix $E(\mathbf{a}) = \left(\mathbf{a}_{(j_2,j_1)}\right)_{(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}$ consisting of the eigenvectors of the matrix $A_{(2)}(\mathbf{a})$ is orthogonal and symmetric.*

Proof. Let $\mathbf{a}_{(j_2,j_1)}$ and $\mathbf{a}_{(i_2,i_1)}$ be two different columns of the matrix $E(\mathbf{a})$. Then, using the equality $\lambda_{j_2}\bar{\lambda}_{i_2} = \lambda_{j_2-i_2}$, we write

$$\mathbf{a}_{(j_2,j_1)} \cdot \bar{\mathbf{a}}_{(i_2,i_1)} = \left(\lambda_{j_2}^{k_2}\lambda_{j_1}^{k_1}\right)^T_{(k_2,k_1)=0,\ldots,(l_2-1,l_1-1)} \cdot \left(\bar{\lambda}_{i_2}^{k_2}\bar{\lambda}_{i_1}^{k_1}\right)^T_{(k_2,k_1)=0,\ldots,(l_2-1,l_1-1)}$$
$$= \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} \lambda_{j_2}^{k_2}\lambda_{j_1}^{k_1}\bar{\lambda}_{i_2}^{k_2}\bar{\lambda}_{i_1}^{k_1} = \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} (\lambda_{j_2}\bar{\lambda}_{i_2})^{k_2}(\lambda_{j_1}\bar{\lambda}_{i_1})^{k_1} = \sum_{(k_2,k_1)=0}^{(l_2-1,l_1-1)} \lambda_{j_2-i_2}^{k_2}\lambda_{j_1-i_1}^{k_1}$$
$$= \sum_{k_2=0}^{l_2-1} \lambda_{j_2-i_2}^{k_2} \sum_{k_1=0}^{l_1-1} \lambda_{j_1-i_1}^{k_1}.$$

If $(j_2, j_1) \neq (i_2, i_1)$, then $(j_2 - i_2, j_1 - i_1) \neq 0$, which means that one of the equalities $j_2 - i_2 \neq 0$ or $j_1 - i_1 \neq 0$ holds true. Therefore, either $\lambda_{j_2-i_2} \neq 1$ or $\lambda_{j_1-i_1} \neq 1$, which means that, similarly to (24), we obtain $\mathbf{a}_{(j_2,j_1)} \cdot \bar{\mathbf{a}}_{(i_2,i_1)} = 0$.

The symmetry of the matrix $E(\mathbf{a})$ follows from the equalities

$$E^T(\mathbf{a}) = \left(\lambda_{j_2}^{i_2}\lambda_{j_1}^{i_1}\right)^T_{\substack{(i_2,i_1)=0,\ldots,(l_2-1,l_1-1)\\(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}}$$
$$= \left(\lambda_{i_2}^{j_2}\lambda_{i_1}^{j_1}\right)_{\substack{(i_2,i_1)=0,\ldots,(l_2-1,l_1-1)\\(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}} = \left(\lambda_{j_2}^{i_2}\lambda_{j_1}^{i_1}\right)_{\substack{(i_2,i_1)=0,\ldots,(l_2-1,l_1-1)\\(j_2,j_1)=0,\ldots,(l_2-1,l_1-1)}} = E(\mathbf{a}).$$

The corollary is proved. □

Note that the matrix of eigenvectors in the case of multiple involution and for $l_1 = \ldots = l_n = 2$ has a similar property [24].

Now, we can explore the completeness of the eigenfunctions of Problem $\mathbf{S}_2$.

Let $\mathcal{H}_m$ be the space of homogeneous harmonic polynomials of degree m. By Lemma 2, it can be split into a sum of $l_2 l_1$ orthogonal on $\partial\Omega$ subspaces $F_{(i_2,i_1)}[\mathcal{H}_m]$, $(i_2, i_1) = 0, \ldots, (l_2 - 1, l_1 - 1)$ of homogeneous harmonic polynomials of parity (i_2, i_1). Let $\{H_m^{(i_2,i_1),k} : k = 1, 2, \ldots, m_{(i_2,i_1)}\}$ be a complete in $F_{(i_2,i_1)}[\mathcal{H}_m]$ system of orthogonal on $\partial\Omega$ polynomials.

Theorem 7. *Let the numbers $\lambda_{(i_2,i_1)}$ defined in (14) be all not zero. Then, the system of eigenfunctions of the Dirichlet problem (1) and (2) is complete in $L_2(\Omega)$ and has the form*

$$u_{\mu,m,(i_2,i_1),k}(x) = \frac{1}{|x|^{m+n/2-1}} J_{m+n/2-1}(\sqrt{\mu}|x|) H_m^{(i_2,i_1),k}(x), \quad (29)$$

where $m \in \mathbb{N}_0$, $(i_2, i_1) = 0, \dots, (l_2 - 1, l_1 - 1)$, $k = 1, \dots, m_{(i_2,i_1)}$, $J_\nu(t)$ is the Bessel function of the first kind, and $\sqrt{\mu}$ is a root of the Bessel function $J_{m+n/2-1}(t)$. The eigenvalues of Problem $\mathbf{S}_2$ are numbers $\lambda_{\mu,(i_2,i_1)}$, defined in (26)

$$\lambda_{\mu,(i_2,i_1)} = \mu \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} a_{(j_2,j_1)} \bar{\lambda}_{i_2}^{j_2} \bar{\lambda}_{i_1}^{j_1}.$$

Proof. By Theorem 6, to find eigenfunctions of the problem (1) and (2), it is necessary to find the function $w_\mu(x)$—a solution to the problem (16) and (17) and then write out the function $\hat{u}_{(i_2,i_1)}(x) = F_{(i_2,i_1)}[w_\mu](x)$. A maximal system of eigenfunctions of the problem (16) and (17) have the form (see, for example, [27,28])

$$w_{\mu,m,j}(x) = \frac{1}{|x|^{m+n/2-1}} J_{m+n/2-1}(\sqrt{\mu}|x|) H_m^j(x), \tag{30}$$

where $\left\{H_m^j(x) : j = 1, \dots, h_m\right\}$, $h_m = \frac{2m+n-2}{n-2}\binom{m+n-3}{n-3}$ $(n > 2)$ is the maximal system of linearly independent homogeneous harmonic polynomials of degree m, and $\sqrt{\mu}$ is a root of the Bessel function $J_{m+n/2-1}(t)$. Then, since $|S_2 S_1 x| = |x|$, then

$$\begin{aligned}
\hat{u}_{(i_2,i_1)}(x) = F_{(i_2,i_1)}[w_{\mu,m,j}](x) &= \frac{1}{l_2 l_1} \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} w_{\mu,m,j}(S_2^{j_2} S_1^{j_1} x) \\
= \frac{1}{l_2 l_1} \sum_{(j_2,j_1)=0}^{(l_2-1,l_1-1)} \lambda_{i_2}^{j_2} \lambda_{i_1}^{j_1} &\frac{1}{|S_2^{j_2} S_1^{j_1} x|^{m+n/2-1}} J_{m+n/2-1}\left(\sqrt{\mu}|S_2^{j_2} S_1^{j_1} x|\right) H_m^j\left(S_2^{j_2} S_1^{j_1} x\right) \\
&= \frac{1}{|x|^{m+n/2-1}} J_{m+n/2-1}(\sqrt{\mu}|x|) F_{(i_2,i_1)}[H_m^j](x).
\end{aligned} \tag{31}$$

Since $F_{(i_2,i_1)}[H_m^j] \subset F_{(i_2,i_1)}[\mathcal{H}_m]$, then choose in the space $F_{(i_2,i_1)}[\mathcal{H}_m]$ a complete system of polynomials $\left\{H_m^{(i_2,i_1),k} : k = 1, 2, \dots, m_{(i_2,i_1)}\right\}$ orthogonal on $\partial\Omega$, to which correspond some polynomials from $\mathcal{H}_m$. Note that, for some value of m, it is possible $m_{(i_2,i_1)} = 0$, that is, for such m, the component $H_m^{(i_2,i_1)}(x)$ is missing (see Example 6) and therefore $\hat{u}_{(i_2,i_1)} = 0$. Choosing in the resulting expression for $\hat{u}_{(i_2,i_1)}(x)$ instead of $F_{(i_2,i_1)}[H_m^j](x)$ the harmonic polynomials $H_m^{(i_2,i_1),k}(x)$ and adding indices, indicating the dependence of the eigenfunction $\hat{u}_{(i_2,i_1)}(x)$ on μ, m and k, we have (29).

Since, in formula (29), $H_m^{(i_2,i_1),k}(x)$ are homogeneous harmonic polynomials of degree m, the functions $u_{\mu,m,(i_2,i_1),k}(x)$ have the form (30) and hence are eigenfunctions of problem (16) and (17). The reverse is also true. Each homogeneous harmonic polynomial $H_m^j(x)$ by the formula (22) can be represented as a linear combination of harmonic polynomials of the form $H_m^{(i_2,i_1)}(x)$, and those are linear combinations of the polynomials $H_m^{(i_2,i_1),k}(x)$ and hence any function from (30) is a linear combination of functions of the form (29). The eigenvalues of the problem (1) and (2), in accordance with Theorem (6), are found from (26).

Let us study the orthogonality of the functions $u_{\mu,m,(i_2,i_1),k}(x)$. The equality holds true

$$\begin{aligned}
&\int_\Omega u_{\mu_1,m_1,(i_2,i_1),k_1}(x) \bar{u}_{\mu_2,m_2,(j_2,j_1),k_2}(x)\, dx \\
&= \int_0^1 \rho J_{m_1+n/2-1}(\sqrt{\mu_1}\rho) J_{m_2+n/2-1}(\sqrt{\mu_2}\rho)\, d\rho \cdot \int_{\partial\Omega} H_{m_1}^{(i_2,i_1),k_1}(\xi) \bar{H}_{m_2}^{(j_2,j_1),k_2}(\xi)\, ds_\xi = 0.
\end{aligned}$$

Consider the right side of the obtained equality. For $\mu_1 \neq \mu_2$ and $m_1 = m_2$, due to the properties of the Bessel functions (orthogonality in $L_2((0,1);t)$), the first factor is zero. If $m_1 \neq m_2$, by the property of harmonic polynomials, the second factor from the right side is zero. If $\mu_1 = \mu_2$, $m_1 = m_2$, then, for $(i_2, i_1) \neq (j_2, j_1)$, the second factor from the right side is zero by Corollary 6. Finally, if $(i_2, i_1) = (j_2, j_1)$ and $k_1 \neq k_2$, then the second factor is zero in accordance with the scheme for constructing polynomials $H_m^{(i_2,i_1),k}(x)$.

By Lemma 2 from ([29], p. 33), the obtained system (29) of functions is complete in $L_2(\Omega) = L_2((0,1) \times \partial\Omega)$ because the system $\{J_{m+n/2-1}(\sqrt{\mu}\rho) : J_{m+n/2-1}(\sqrt{\mu}) = 0\}$ is orthogonal and complete in $L_2((0,1);t)$ for each m, and the system $\left\{H_m^{(i_2,i_1),k)}(\xi)\right\}$ is orthogonal and complete in $L_2(\partial\Omega)$ for different m, (i_2, i_1) and k. The theorem is proved. □

Example 7. *Let $n = 2$, $l_2 = l_1 = 2$, $S_1x = (-x_1, x_2)$, $S_2x = (x_1, -x_2)$ then Problem* $\mathbf{S}_2$ *has the form*

$$a_0\Delta u(x_1, x_2) + a_1\Delta u(-x_1, x_2) + a_2\Delta u(x_1, -x_2) + a_3\Delta u(-x_1, -x_2) + \lambda u(x) = 0,\ x \in \Omega,$$
$$u(x) = 0,\ x \in \partial\Omega.$$

Using Example 6, we find the eigenfunctions of the problem (1) and (2). In the polar coordinate system, the eigenfunctions of the problem (16) and (17) are determined according to the equality (30) in the form

$$w_{\mu,m,0}(x) = J_m(\sqrt{\mu}r)\cos m\varphi, \quad w_{\mu,m,1}(x) = J_m(\sqrt{\mu}r)\sin m\varphi, \quad m \in \mathbb{N}_0,$$

where $\sqrt{\mu}$ is a root of the Bessel function $J_m(t)$

$$J_m(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+m)!j!}\left(\frac{t}{2}\right)^{2j+m}.$$

Using (31), we have

$$u_{\mu,m,(i_2,i_1)}(x) = J_m(\sqrt{\mu}|x|)F_{(i_2,i_1)}[H_m](x/|x|).$$

In the written formula, the dependence of the eigenfunction $u_{\mu,m,(i_2,i_1)}(x)$ on the index k is not indicated because, in accordance with Example 6, the dimension of the space $F_{(i_2,i_1)}[\mathcal{H}_m]$ is equal to 1. According to (25) and taking into account (26), we write

$$\begin{aligned}
u_{\mu,2m,(0,0)}(x) &= J_{2m}(\sqrt{\mu}r)\cos(2m\varphi), & \lambda_{\mu,(0,0)} &= \mu(a_0 + a_1 + a_2 + a_3),\\
u_{\mu,2m+1,(0,1)}(x) &= J_{2m+1}(\sqrt{\mu}r)\cos((2m+1)\varphi), & \lambda_{\mu,(0,1)} &= \mu(a_0 - a_1 + a_2 - a_3),\\
u_{\mu,2m+1,(1,0)}(x) &= J_{2m+1}(\sqrt{\mu}r)\sin((2m+1)\varphi), & \lambda_{\mu,(1,0)} &= \mu(a_0 + a_1 - a_2 - a_3),\\
u_{\mu,2m,(1,1)}(x) &= J_{2m}(\sqrt{\mu}r)\sin(2m\varphi), & \lambda_{\mu,(1,1)} &= \mu(a_0 - a_1 - a_2 + a_3),
\end{aligned}$$

where $m \in \mathbb{N}_0$, $\sqrt{\mu}$ is a root of the Bessel function $J_m(t)$ and $\lambda_{\mu,(i_2,i_1)} \neq 0$. It is clear that the obtained system of eigenfunctions is complete in $L_2(\Omega)$.

Let $n = 3$. Then, the maximal system of homogeneous harmonic polynomials of degree m in (30) $\{H_m^{j,0}(x), H_m^{j,1}(x) : j = 0, \ldots, m\}$ has the form [27]

$$H_m^{j,0}(x_1, x_2, x_3) = G_{m-j}^j(x)\,\mathrm{Re}(x_1 + ix_2)^j, \quad H_m^{j,1}(x_1, x_2, x_3) = G_{m-j}^j(x)\,\mathrm{Im}(x_1 + ix_2)^j, \tag{32}$$

where $0 \leq j \leq m$ and

$$G_m^j(x) \equiv G_m^j(x_1, x_2, x_3) = \sum_{k=0}^{[m/2]} (-1)^k \frac{(x_1^2 + x_2^2)^k x_3^{m-2k}}{(2k)!!(2j+2k)!!(m-2k)!}.$$

The system (32) *has* $2m+1$ *members for every* m *because* $H_0^{j,1}(x) = 0$. *In the spherical coordinate system* (r, φ, θ), *we can write it in a more compact way*

$$H_m^{j,0}(x_1, x_2, x_3) = r^m G_{m-j}^j(r, \varphi, \theta) \cos j\varphi, \quad H_m^{j,1}(x_1, x_2, x_3) = r^m G_{m-j}^j(r, \varphi, \theta) \sin j\varphi,$$

where

$$G_m^j(r, \varphi, \theta) = \sum_{k=0}^{[m/2]} (-1)^k \frac{\sin^{2k+j}\theta \cos^{m-2k}\theta}{(2k)!!(2j+2k)!!(m-2k)!},$$

since $\cos\theta = x_3/|x|$, $\sin\theta = \sqrt{x_1^2 + x_2^2}/|x|$ *and* $\cos\varphi = x_1/\sqrt{x_1^2 + x_2^2}$. *In this case, the dimension of the space* $F_{(i_2,i_1)}[\mathcal{H}_m]$ *is greater than 1. Note that the operator* $F_{(i_2,i_1)}[\cdot]$ *acts only on the second multiplier of polynomials in* (32). *For example, in the space* $F_{(0,0)}[\mathcal{H}_m]$, *one can choose the following basic polynomials* $\{H_m^{2k,0}(r, \varphi, \theta) : k = 0, \dots, [m/2]\}$, *which means*

$$u_{\mu,m,(0,0),k}(x) = r^{-1/2} J_{m+1/2}(\sqrt{\mu} r) G_{m-2k}^{2k}(r, \varphi, \theta) \cos(2k\varphi), \quad k = 0, \dots, [m/2].$$

The remaining eigenfunctions are obtained similarly.

5. Conclusions

The results obtained allow one to find explicitly, using the formula (29), the eigenfunctions and eigenvalues of the boundary value problem (1) and (2) for the nonlocal differential equation with double involution. The completeness of the system of eigenfunctions make it possible to use the Fourier method to construct solutions of initial-boundary value problems for nonlocal parabolic and hyperbolic equations.

Possible applications of the obtained results can be found in the modeling of optical systems, since differential equations with involution are an important part of the general theory of functional differential equations, which has numerous applications in optics. Applications of equations with involution in modeling optical systems are given, for example, in [30,31]. In particular, in [30], mathematical models important for applications in nonlinear optics are considered in the form of nonlinear functional-differential equations of a parabolic type with feedback and transformation of spatial variables, which is specified by the involution operator. The following parabolic functional differential equation is considered

$$\frac{\partial u}{\partial t} + u = \mu \Delta u + K(1 + \gamma \cos Qu),\ r_1 \le r \le r_2,\ t \ge 0,\ \mu > 0,$$

which describes the dynamics of the phase modulation of a light wave, in an optical system with an involution operator Q such that $Q^l = I$.

With the above in mind, as further research steps on the topic of the presented article, we are going to investigate nonlocal initial-boundary value problems with involution for parabolic equations. In addition, we are going to study nonlocal boundary value problems in the case of multiple involution of arbitrary orders, generalizing the results obtained in [24], and also consider similar boundary value problems for a nonlocal biharmonic equation.

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