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Article

Fourier Transform of the Orthogonal Polynomials on the Unit Ball and Continuous Hahn Polynomials

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Abstract: Some systems of univariate orthogonal polynomials can be mapped into other families by the Fourier transform. The most-studied example is related to the Hermite functions, which are eigenfunctions of the Fourier transform. For the multivariate case, by using the Fourier transform and Parseval's identity, very recently, some examples of orthogonal systems of this type have been introduced and orthogonality relations have been discussed. In the present paper, this method is applied for multivariate orthogonal polynomials on the unit ball. The Fourier transform of these orthogonal polynomials on the unit ball is obtained. By Parseval's identity, a new family of multivariate orthogonal functions is introduced. The results are expressed in terms of the continuous Hahn polynomials.

Keywords: Gegenbauer polynomials; multivariate orthogonal polynomials; Hahn polynomials; Fourier transform; Parseval's identity; hypergeometric function

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1. Introduction

From a historical point of view, mathematical transforms started with some works of L. Euler within the context of second-order differential equation problems [1]. Since then, due to their interesting mathematical properties, as well as their applications, integral transforms have attracted research interests in many areas of engineering, mathematics, physics, as well as several other scientific branches. Just to give an idea, without the intention of completeness, integral transforms such as the Fourier, Laplace, Beta, Hankel, Mellin, and Whittaker transforms with various special functions as kernels play an important role in various problems of physics [2,3], mathematics [4–13], and in vibration analysis [14], sound engineering [15,16], communication [17], data processing [18], automatization [18], etc.

As for the relation between orthogonal polynomials and integral transforms, by the Fourier transform or other integral transforms, it is shown that some systems of univariate orthogonal polynomials are mapped into other families [7]. For example, Hermite functions, which are Hermite polynomials $H_n(x)$ multiplied by $\exp(-x^2/2)$, are eigenfunctions of the Fourier transform [9–11,19]. Some other interesting works are related to families of classical discrete orthogonal polynomials [20]. In [11], by the Fourier–Jacobi transform, it was investigated that classical Jacobi polynomials can be mapped onto Wilson polynomials. Furthermore, the Fourier transform of Jacobi polynomials and their close relation with continuous Hahn polynomials were discussed by Koelink [9].

Recently, in the univariate case, the Fourier transforms of finite classical orthogonal polynomials by Koepf and Masjed-Jamei [10], generalized ultraspherical and generalized Hermite polynomials, and symmetric sequences of finite orthogonal polynomials [12,21,22] have been studied. As for the multivariate case, Tratnik [23,24] presented a multivariable generalization both of all continuous and discrete families of the Askey tableau, providing hypergeometric representation, the orthogonality weight function, which applies with respect to subspaces of lower degree, and biorthogonality within a given subspace. A non-trivial interaction for multivariable continuous Hahn polynomials was presented by Koelink et al. [25]. Moreover, in [26–28], Fourier transforms of multivariate orthogonal polynomials and their applications were investigated, obtaining some families of orthogonal functions in terms of continuous Hahn polynomials. In particular, in [26], a new family of orthogonal functions was derived by using Fourier transforms of bivariate orthogonal polynomials on the unit disc and Parseval's identity.

The main aims of this investigation are to find the Fourier transformation of the classical orthogonal polynomials on the unit ball $\mathbb{B}^r$ and to obtain a new family of multivariate orthogonal functions in terms of multivariable Hahn polynomials. We first state the results for $r = 1$, $r = 2$ and $r = 3$ to illustrate the results and illuminate how the results on $\mathbb{B}^r$ are obtained, then we give the results on the unit ball $\mathbb{B}^r$ by induction.

The work is organized as follows. In Section 2, the basic definitions and notations are introduced. The main results are stated and proven in Section 3. Finally, the discussion and conclusions are given.

2. Preliminaries

In this section, we state background materials on orthogonal polynomials that we shall need. The first subsection recalls the properties of two families of (univariate) orthogonal polynomials, namely the Gegenbauer polynomials and the continuous Hahn polynomials, as well as some definitions. In the second subsection, we recall the basic results on the (multivariate) classical orthogonal polynomials on the unit ball. The notations and nomenclature followed are that of the the book of Koekoek, Lesky, and Swarttouw [29] for the univariate case and of the book of Dunkl and Xu [30] for the multivariate case.

2.1. The Classical Univariate Gegenbauer Polynomials

Let

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x+1)^k (x-1)^{n-k} \tag{1}$$

be the univariate Jacobi polynomial of degree n, orthogonal with respect to the weight function [31] (p. 68, Equation (4.3.2)):

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \;\; \alpha, \beta > -1, \;\; x \in [-1, 1]. \tag{2}$$

The univariate Gegenbauer polynomials are a special case of the Jacobi polynomial, defined by [32] (p. 277, Equation (4))

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{\left(\lambda + \frac{1}{2}\right)_n} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x), \tag{3}$$

where for $n \geq 1$, $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$, which denotes the Pochhammer symbol with the convention $(\alpha)_0 = 1$. These polynomials can also be written in terms of hypergeometric series as

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+2\lambda \\ \lambda + \frac{1}{2} \end{matrix} \;\middle|\; \frac{1-x}{2} \right), \tag{4}$$

where [32] (p. 73, Equation (2))

$${}_pF_q\left(\begin{matrix} a_1,\, a_2, \ldots, a_p \\ b_1,\, b_2, \ldots, b_q \end{matrix} \;\middle|\; x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!}. \tag{5}$$

The Gegenbauer polynomials satisfy the orthogonality relation [32] (p. 281, Equation (28)):

$$\int_{-1}^{1}\left(1-x^2\right)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)dx = h_n^{\lambda}\, \delta_{n,m}, \qquad (m,n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}), \tag{6}$$

where h_n^{λ} is given by

$$h_n^{\lambda} = \frac{(2\lambda)_n \Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{n!(n+\lambda)\Gamma(\lambda)}, \tag{7}$$

$\delta_{n,m}$ is the Kronecker delta, and the Gamma function $\Gamma(x)$ is defined by [33] (p. 254, (6.1.1))

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t}dt, \qquad \Re(x)>0. \tag{8}$$

The beta function is given by [33] (p. 258, (6.2.1))

$$B(a,b) = \int_{0}^{1} x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad \Re(a),\Re(b)>0. \tag{9}$$

For our purposes, we also need to introduce the continuous Hahn polynomials [34]:

$$p_n(x;a,b,c,d) = i^n\frac{(a+c)_n(a+d)_n}{n!}\, {}_3F_2\left(\begin{matrix}-n,\ n+a+b+c+d-1,\ a+ix\\ a+c,\ a+d\end{matrix}\;\middle|\; 1\right). \tag{10}$$

which can also be written as a limiting case of the Wilson polynomials [34].

2.2. Orthogonal Polynomials on the Unit Ball

Let $\|\boldsymbol{x}\| := \left(x_1^2+\cdots+x_r^2\right)^{1/2}$ for $\boldsymbol{x}=(x_1,\ldots,x_r)\in\mathbb{R}^r$. The unit ball in $\mathbb{R}^r$ is denoted by $\mathbb{B}^r := \{\boldsymbol{x}\in\mathbb{R}^r : \|\boldsymbol{x}\|\le 1\}$. Let W_μ be the weight function defined by

$$W_\mu(\boldsymbol{x}) = \left(1-\|\boldsymbol{x}\|^2\right)^{\mu-1/2},\ \ \mu>-1/2. \tag{11}$$

We shall consider orthogonal polynomials on the unit ball, by considering the inner product:

$$\langle f,g\rangle_\mu = \int_{\mathbb{B}^r} W_\mu(\boldsymbol{x}) f(\boldsymbol{x}) g(\boldsymbol{x}) d\mathbf{x}, \tag{12}$$

where $d\mathbf{x} = dx_1\cdots dx_r$.

Let Π^r denote the space of polynomials in r real variables. Let Π_n^r denote the linear space of polynomials in several variables of (total) degree at most n for $n=0,1,2,\ldots$. Let $\mathcal{V}_n^r(W_\mu)$ be the space of orthogonal polynomials of total degree n with respect to $W_\mu(\boldsymbol{x})$. Then, $\dim \mathcal{V}_n^r(W_\mu) = \binom{n+r-1}{n}$. The elements of the space $\mathcal{V}_n^r(W_\mu)$ are eigenfunctions of a second-order partial differential equation [30] (p. 141, Equation (5.2.3)):

$$\sum_{i=1}^{r}\frac{\partial^2 P}{\partial x_i^2} - \sum_{j=1}^{r}\frac{\partial}{\partial x_j}x_j\left[2\mu-1+\sum_{i=1}^{r}x_i\frac{\partial}{\partial x_i}\right]P = -(n+r)(n+2\mu-1)P. \tag{13}$$

The space $\mathcal{V}_n^r$ has several different bases. One orthogonal basis of the space $\mathcal{V}_n^r$ can be expressed in terms of the Gegenbauer polynomials (4) as [30] (p. 143)

$$P_{\mathbf{n}}^{\mu}(\boldsymbol{x}) = \prod_{j=1}^{r}\left(1-\|\boldsymbol{x}_{j-1}\|^2\right)^{\frac{n_j}{2}} C_{n_j}^{(\lambda_j)}\left(\frac{x_j}{\sqrt{1-\|\boldsymbol{x}_{j-1}\|^2}}\right), \tag{14}$$

where $\lambda_j = \mu + \left|\mathbf{n}^{j+1}\right| + \frac{r-j}{2}$,

$$\begin{cases} \boldsymbol{x}_0 = 0, \ \ \boldsymbol{x}_j = (x_1, \dots, x_j), \\ \mathbf{n} = (n_1, \dots, n_r), \ \ |\mathbf{n}| = n_1 + \cdots + n_r = n, \\ \mathbf{n}^j = (n_j, \dots, n_r), \ \ \left|\mathbf{n}^j\right| = n_j + \cdots + n_r, \ \ 1 \le j \le r, \end{cases} \tag{15}$$

and $\mathbf{n}^{r+1} := 0$. More precisely,

$$\int_{\mathbb{B}^r} W_{\mu}(\boldsymbol{x}) P_{\mathbf{n}}^{\mu}(\boldsymbol{x}) P_{\mathbf{m}}^{\mu}(\boldsymbol{x}) d\mathbf{x} = h_{\mathbf{n}}^{\mu}\delta_{\mathbf{n},\mathbf{m}}, \tag{16}$$

where $\delta_{\mathbf{n},\mathbf{m}} = \delta_{n_1,m_1}\dots\delta_{n_r,m_r}$ and $h_{\mathbf{n}}^{\mu}$ is given by [30]

$$h_{\mathbf{n}}^{\mu} = \frac{\pi^{r/2}\Gamma\left(\mu+\frac{1}{2}\right)\left(\mu+\frac{r}{2}\right)_{|\mathbf{n}|}}{\Gamma\left(\mu+\frac{r+1}{2}+|\mathbf{n}|\right)} \prod_{j=1}^{r} \frac{\left(\mu+\frac{r-j}{2}\right)_{\left|\mathbf{n}^j\right|}\left(2\mu+2\left|\mathbf{n}^{j+1}\right|+r-j\right)_{n_j}}{n_j!\left(\mu+\frac{r-j+1}{2}\right)_{\left|\mathbf{n}^j\right|}}. \tag{17}$$

3. Main Results

In this section, we define Fourier transforms of functions in terms of orthogonal polynomials on the unit ball and obtain a new family of multivariate orthogonal functions by a similar method applied in [26] for bivariate Koornwinder polynomials. While doing these, firstly, we define specific special functions so that they are determined with the motivation to use the orthogonality relation of orthogonal polynomials on the ball in Parseval's identity created with the help of the Fourier transform.

Let us introduce

$$f_r(\mathbf{x};\mathbf{n},a,\mu) := f_r(x_1,\dots,x_r;n_1,\dots,n_r,a,\mu) = \prod_{j=1}^{r}\left(1-\tanh^2 x_j\right)^{a+\frac{r-j}{4}} P_{\mathbf{n}}^{\mu}(v_1,\dots,v_r), \tag{18}$$

namely

$$f_r(x_1,\dots,x_r;n_1,\dots,n_r,a,\mu) = \prod_{j=1}^{r}\left(1-\tanh^2 x_j\right)^{a+\frac{r-j}{4}} \prod_{j=1}^{r-1}\left(1-\tanh^2 x_j\right)^{\frac{n_{j+1}+\dots+n_r}{2}} \prod_{j=1}^{r} C_{n_j}^{(\lambda_j)}(\tanh x_j)$$

for $r \ge 1$, where a, μ are real parameters and

$$v_1(x_1) = v_1 = \tanh x_1, \tag{19}$$

$$v_j(x_1,\dots,x_j) = v_j = \tanh x_j\sqrt{\left(1-\tanh^2 x_1\right)\left(1-\tanh^2 x_2\right)\cdots\left(1-\tanh^2 x_{j-1}\right)}, \tag{20}$$

for $j = 2,\dots,r$. Note that $\sqrt{1-\tanh^2 x} = \frac{1}{coshx}$ for every real number x . From the latter expression, we can write f_r defined in (18) in terms of f_{r-1} in the following forms:

$$f_r(x_1,\ldots,x_r;n_1,\ldots,n_r,a,\mu)=\left(1-\tanh^2 x_1\right)^{a+\frac{n_2+\cdots+n_r}{2}+\frac{r-1}{4}} C_{n_1}^{\left(n_2+\cdots+n_r+\mu+\frac{r-1}{2}\right)}(\tanh x_1) \times f_{r-1}(x_2,\ldots,x_r;n_2,\ldots,n_r,a,\mu), \quad (21)$$

and

$$f_r(x_1,\ldots,x_r;n_1,\ldots,n_r,a,\mu)=\left(1-\tanh^2 x_r\right)^{a} C_{n_r}^{(\mu)}(\tanh x_r) \times f_{r-1}\left(x_1,\ldots,x_{r-1};n_1,\ldots,n_{r-1},a+\frac{n_r}{2}+\frac{1}{4},\mu+n_r+\frac{1}{2}\right), \quad (22)$$

for $r\geq 1$, where the univariate Gegenbauer polynomials $C_n^{(\lambda)}(x)$ are defined in (3). For $r=1$,

$$f_1(x_1;n_1,a,\mu)=\left(1-\tanh^2 x_1\right)^{a} C_{n_1}^{(\mu)}(\tanh x_1). \quad (23)$$

3.1. The Fourier Transform of Orthogonal Polynomials on the Unit Ball

The Fourier transform of a given univariate function $f(x)$ is defined by [6] (p. 111, Equation (7.1))

$$\mathcal{F}(f(x))=\int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx. \quad (24)$$

In the r-variable case, the Fourier transform of a given multivariate function $f(x_1,\ldots,x_r)$ is defined by ([6], p. 182, Equation (11.1a))

$$\mathcal{F}(f(x_1,\ldots,x_r))=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i(\xi_1 x_1+\cdots+\xi_r x_r)} f(x_1,\ldots,x_r)dx_1\cdots dx_r. \quad (25)$$

Next, we calculate the Fourier transform of the function $f_r(\mathbf{x};\mathbf{n},a,\mu)$ defined in (18) by using the induction method. In doing so, we first start with the following theorem.

Theorem 1. *Let $f_r(\mathbf{x};\mathbf{n},a,\mu)$ be defined in (18). The following result holds true:*

$$\mathcal{F}(f_r(\mathbf{x};\mathbf{n},a,\mu))=\mathcal{F}(f_r(x_1,\ldots,x_r;n_1,\ldots,n_r,a,\mu))=\frac{2^{|\mathbf{n}^2|+2a+\frac{r-3}{2}}\left(2\left(|\mathbf{n}^2|+\mu+\frac{r-1}{2}\right)\right)_{n_1}}{n_1!} \times B\left(a+\frac{|\mathbf{n}^2|+i\xi_1}{2}+\frac{r-1}{4},a+\frac{|\mathbf{n}^2|-i\xi_1}{2}+\frac{r-1}{4}\right) {}_3F_2\left(\begin{matrix}-n_1,\ n_1+2\left(|\mathbf{n}^2|+\mu+\frac{r-1}{2}\right),\ a+\frac{|\mathbf{n}^2|+i\xi_1}{2}+\frac{r-1}{4}\\ |\mathbf{n}^2|+2a+\frac{r-1}{2},\ |\mathbf{n}^2|+\mu+\frac{r}{2}\end{matrix}\ \Big|\ 1\right) \times \mathcal{F}(f_{r-1}(x_2,\ldots,x_r;n_2,\ldots,n_r,a,\mu)), \quad (26)$$

and

$$\mathcal{F}(f_r(\mathbf{x};\mathbf{n},a,\mu))=\mathcal{F}(f_r(x_1,\ldots,x_r;n_1,\ldots,n_r,a,\mu))=\frac{2^{2a-1}(2\mu)_{n_r}}{n_r!}B\left(a+\frac{i\xi_r}{2},a-\frac{i\xi_r}{2}\right) {}_3F_2\left(\begin{matrix}-n_r,\ n_r+2\mu,\ a+\frac{i\xi_r}{2}\\ 2a,\ \mu+\frac{1}{2}\end{matrix}\ \Big|\ 1\right) \times \mathcal{F}\left(f_{r-1}\left(x_1,\ldots,x_{r-1};n_1,\ldots,n_{r-1},a+\frac{n_r}{2}+\frac{1}{4},\mu+n_r+\frac{1}{2}\right)\right). \quad (27)$$

Proof. By using (21), the Fourier transform of the function f_r defined in (18) can be calculated as follows by using Relation (4):

$$\mathcal{F}(f_r(x_1,\ldots,x_r;n_1,\ldots,n_r,a,\mu)) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i(\xi_1x_1+\cdots+\xi_rx_r)}\left(1-\tanh^2 x_1\right)^{a+\frac{n_2+\cdots+n_r}{2}+\frac{r-1}{4}}$$
$$\times C_{n_1}^{\left(\mu+n_2+\cdots+n_r+\frac{r-1}{2}\right)}(\tanh x_1)f_{r-1}(x_2,\ldots,x_r;n_2,\ldots,n_r,a,\mu)dx_r\cdots dx_1$$
$$= \int_{-\infty}^{\infty} e^{-i\xi_1x_1}\left(1-\tanh^2 x_1\right)^{a+\frac{n_2+\cdots+n_r}{2}+\frac{r-1}{4}} C_{n_1}^{\left(\mu+n_2+\cdots+n_r+\frac{r-1}{2}\right)}(\tanh x_1)dx_1$$
$$\times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-i(\xi_2x_2+\cdots+\xi_rx_r)}f_{r-1}(x_2,\ldots,x_r;n_2,\ldots,n_r,a,\mu)dx_r\cdots dx_2$$
$$= \mathcal{F}(f_{r-1}(x_2,\ldots,x_r;n_2,\ldots,n_r,a,\mu))$$
$$\times \int_{-1}^{1}(1+u)^{a+\frac{n_2+\cdots+n_r-i\xi_1}{2}+\frac{r-5}{4}}(1-u)^{a+\frac{n_2+\cdots+n_r+i\xi_1}{2}+\frac{r-5}{4}}C_{n_1}^{\left(\mu+n_2+\cdots+n_r+\frac{r-1}{2}\right)}(u)du$$
$$= \frac{2^{n_2+\cdots+n_r+2a+\frac{r-3}{2}}\left(2\left(\mu+n_2+\cdots+n_r+\frac{r-1}{2}\right)\right)_{n_1}}{n_1!}\mathcal{F}(f_{r-1}(x_2,\ldots,x_r;n_2,\ldots,n_r,a,\mu))$$
$$\times\sum_{l=0}^{n_1}\frac{(-n_1)_l(n_1+2(\mu+n_2+\cdots+n_r)+r-1)_l}{l!\left(\mu+n_2+\cdots+n_r+\frac{r}{2}\right)_l}$$

$$\times\int_0^1(1-t)^{a+\frac{n_2+\cdots+n_r-i\xi_1}{2}+\frac{r-5}{4}}t^{a+\frac{n_2+\cdots+n_r+i\xi_1}{2}+\frac{r-5}{4}+l}dt = \frac{2^{n_2+\cdots+n_r+2a+\frac{r-3}{2}}\left(2\left(\mu+n_2+\cdots+n_r+\frac{r-1}{2}\right)\right)_{n_1}}{n_1!}$$
$$\times\mathcal{F}(f_{r-1}(x_2,\ldots,x_r;n_2,\ldots,n_r,a,\mu))B\left(a+\frac{n_2+\cdots+n_r+i\xi_1}{2}+\frac{r-1}{4},a+\frac{n_2+\cdots+n_r-i\xi_1}{2}+\frac{r-1}{4}\right)$$
$$\times\ {}_3F_2\left(\begin{matrix}-n_1,n_1+2(\mu+n_2+\cdots+n_r)+r-1,a+\frac{n_2+\cdots+n_r+i\xi_1}{2}+\frac{r-1}{4}\\ \mu+n_2+\cdots+n_r+\frac{r}{2},2a+n_2+\cdots+n_r+\frac{r-1}{2}\end{matrix}\ \middle|\ 1\right), \quad (28)$$

which proves (26). Similarly, when we repeat this process by using the Equation (22), it follows that

$$\mathcal{F}(f_r(x_1,\ldots,x_r;n_1,\ldots,n_r,a,\mu)) = \int_{-1}^{1}(1-u)^{a+\frac{i\xi_r}{2}-1}(1+u)^{a-\frac{i\xi_r}{2}-1}C_{n_r}^{(\mu)}(u)du$$
$$\times\mathcal{F}\left(f_{r-1}\left(x_1,\ldots,x_{r-1};n_1,\ldots,n_{r-1},a+\frac{n_r}{2}+\frac{1}{4},\mu+n_r+\frac{1}{2}\right)\right)$$
$$= \frac{2^{2a-1}(2\mu)_{n_r}}{n_r!}B\left(a+\frac{i\xi_r}{2},a-\frac{i\xi_r}{2}\right)\ {}_3F_2\left(\begin{matrix}-n_r,\ n_r+2\mu,\ a+\frac{i\xi_r}{2}\\ \mu+1/2,\ 2a\end{matrix}\ \middle|\ 1\right)$$
$$\times\mathcal{F}\left(f_{r-1}\left(x_1,\ldots,x_{r-1};n_1,\ldots,n_{r-1},a+\frac{n_r}{2}+\frac{1}{4},\mu+n_r+\frac{1}{2}\right)\right). \quad (29)$$

□

By applying Theorem 1 consecutively, we can give the next theorem.

Theorem 2. *The Fourier transform of the function $f_r(\mathbf{x};\mathbf{n},a,\mu)$ defined in (18) is explicitly given as follows:*

$$\mathcal{F}(f_r(\mathbf{x};\mathbf{n},a,\mu)) = \mathcal{F}(f_r(x_1,\ldots,x_r;n_1,\ldots,n_r,a,\mu))$$
$$= 2^{2ra+\frac{r(r-5)}{4}+\sum\limits_{j=1}^{r-1} jn_{j+1}} \prod_{j=1}^{r}\left\{\frac{\left(2\left(\left|\mathbf{n}^{j+1}\right|+\mu+\frac{r-j}{2}\right)\right)_{n_j}}{n_j!}\Theta_j^r(a,\mu,\mathbf{n};\xi_j)\right\}, \quad (30)$$

where

$$\Theta_j^r(a,\mu,\mathbf{n};\xi_j) = B\left(a+\frac{\left|\mathbf{n}^{j+1}\right|+i\xi_j}{2}+\frac{r-j}{4}, a+\frac{\left|\mathbf{n}^{j+1}\right|-i\xi_j}{2}+\frac{r-j}{4}\right)$$
$$\times\ {}_3F_2\left(\begin{matrix}-n_j,\ n_j+2\left(\left|\mathbf{n}^{j+1}\right|+\mu+\frac{r-j}{2}\right),\ a+\frac{\left|\mathbf{n}^{j+1}\right|+i\xi_j}{2}+\frac{r-j}{4}\\ \left|\mathbf{n}^{j+1}\right|+\mu+\frac{r-j+1}{2},\ \left|\mathbf{n}^{j+1}\right|+2a+\frac{r-j}{2}\end{matrix}\ \Big|\ 1\right), \quad (31)$$

which can be also expressed in terms of the continuous Hahn polynomials defined in (10):

$$\Theta_j^r(a,\mu,\mathbf{n};\xi_j) = \frac{n_j!}{i^{n_j}\left(\left|\mathbf{n}^{j+1}\right|+\mu+\frac{r-j+1}{2}\right)_{n_j}\left(\left|\mathbf{n}^{j+1}\right|+2a+\frac{r-j}{2}\right)_{n_j}}$$
$$\times B\left(a+\frac{\left|\mathbf{n}^{j+1}\right|+i\xi_j}{2}+\frac{r-j}{4}, a+\frac{\left|\mathbf{n}^{j+1}\right|-i\xi_j}{2}+\frac{r-j}{4}\right)$$
$$\times p_{n_j}\left(\frac{\xi_j}{2}; a+\frac{\left|\mathbf{n}^{j+1}\right|}{2}+\frac{r-j}{4}, \mu-a+\frac{\left|\mathbf{n}^{j+1}\right|+1}{2}+\frac{r-j}{4}\right.$$
$$\left., \mu-a+\frac{\left|\mathbf{n}^{j+1}\right|+1}{2}+\frac{r-j}{4}, a+\frac{\left|\mathbf{n}^{j+1}\right|}{2}+\frac{r-j}{4}\right). \quad (32)$$

Proof. The proof follows by induction on r by applying Theorem 1 successively. In order to give the results on $\mathbb{B}^r$, we first discuss the results for $r=1$, $r=2$, and $r=3$.

When $r=1$, the unit ball $\mathbb{B}^r$ becomes the interval $[-1,1]$, and the corresponding orthogonal polynomials are Gegenbauer polynomials $C_n^{(\lambda)}(x)$ on the interval $[-1,1]$, which are the special case of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. The Fourier transform of the specific function in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ was obtained in terms of continuous Hahn polynomials in [9]. In fact, for $r=1$, the Fourier transform of

$$f_1(x_1;n_1,a,\mu) = \left(1-\tanh^2 x_1\right)^a C_{n_1}^{(\mu)}(\tanh x_1) \quad (33)$$

follows from (4) (see [9])

$$\mathcal{F}(f_1(x_1;n_1,a,\mu)) = \int_{-\infty}^{\infty} e^{-i\xi_1 x_1}\left(1-\tanh^2 x_1\right)^a C_{n_1}^{(\mu)}(\tanh x_1)dx_1$$
$$= \frac{2^{2a-1}(2\mu)_{n_1}}{n_1!}\Theta_1^1(a,\mu,n_1;\xi_1), \quad (34)$$

where

$$\Theta_1^1(a,\mu,n_1;\xi_1) = {}_3F_2\left(\begin{matrix}-n_1,\ n_1+2\mu,\ a+\frac{i\xi_1}{2}\\ 2a,\ \mu+1/2\end{matrix}\ \Big|\ 1\right)B\left(a+\frac{i\xi_1}{2}, a-\frac{i\xi_1}{2}\right). \quad (35)$$

It can be rewritten [9] in terms of the continuous Hahn polynomials $p_n(x;a,b,c,d)$ from (10) as

$$\mathcal{F}(f_1(x_1;n_1,a,\mu)) = \frac{2^{2a-1}(2\mu)_{n_1}}{i^{n_1}(2a)_{n_1}(\mu+1/2)_{n_1}} B\left(a+\frac{i\xi_1}{2},\, a-\frac{i\xi_1}{2}\right) \times p_{n_1}\left(\frac{\xi_1}{2};a,\mu-a+1/2,\mu-a+1/2,a\right). \quad (36)$$

For the case $r=2$, in view of (21), we can write

$$f_2(x_1,x_2;n_1,n_2,a,\mu) = \left(1-\tanh^2 x_1\right)^{a+\frac{n_2}{2}+\frac{1}{4}} C_{n_1}^{\left(n_2+\mu+\frac{1}{2}\right)}(\tanh x_1) f_1(x_2;n_2,a,\mu). \quad (37)$$

By using now (26), it yields

$$\mathcal{F}(f_2(x_1,x_2;n_1,n_2,a,\mu)) = \frac{2^{n_2+2a-\frac{1}{2}}\left(2\left(n_2+\mu+\frac{1}{2}\right)\right)_{n_1}}{n_1!} \mathcal{F}(f_1(x_2;n_2,a,\mu)) \times B\left(a+\frac{n_2+i\xi_1}{2}+\frac{1}{4}, a+\frac{n_2-i\xi_1}{2}+\frac{1}{4}\right) \times {}_3F_2\left(\begin{matrix} -n_1,\ n_1+2\left(n_2+\mu+\frac{1}{2}\right),\ a+\frac{n_2+i\xi_1}{2}+\frac{1}{4} \\ n_2+2a+\frac{1}{2},\ n_2+\mu+1 \end{matrix} \,\Big|\, 1\right). \quad (38)$$

From (34), we can write

$$\begin{aligned} &\mathcal{F}(f_2(x_1,x_2;n_1,n_2,a,\mu)) \\ &= \frac{2^{n_2+4a-\frac{3}{2}}(2\mu)_{n_2}\left(2\left(n_2+\mu+\frac{1}{2}\right)\right)_{n_1}}{n_1!n_2!} B\left(a+\frac{n_2+i\xi_1}{2}+\frac{1}{4}, a+\frac{n_2-i\xi_1}{2}+\frac{1}{4}\right) \\ &\quad \times {}_3F_2\left(\begin{matrix} -n_1,\ n_1+2\left(n_2+\mu+\frac{1}{2}\right),\ a+\frac{n_2+i\xi_1}{2}+\frac{1}{4} \\ n_2+2a+\frac{1}{2},\ n_2+\mu+1 \end{matrix} \,\Big|\, 1\right) \\ &\quad \times B\left(a+\frac{i\xi_2}{2},\, a-\frac{i\xi_2}{2}\right) {}_3F_2\left(\begin{matrix} -n_2,\ n_2+2\mu,\ a+\frac{i\xi_2}{2} \\ 2a,\ \mu+1/2 \end{matrix} \,\Big|\, 1\right) \\ &= \frac{2^{n_2+4a-\frac{3}{2}}(2\mu)_{n_2}\left(2\left(n_2+\mu+\frac{1}{2}\right)\right)_{n_1}}{n_1!n_2!} \Theta_1^2(a,\mu,n_1,n_2;\xi_1)\Theta_2^2(a,\mu,n_1,n_2;\xi_2), \end{aligned} \quad (39)$$

where

$$\Theta_1^2(a,\mu,n_1,n_2;\xi_1) = B\left(a+\frac{n_2+i\xi_1}{2}+\frac{1}{4}, a+\frac{n_2-i\xi_1}{2}+\frac{1}{4}\right) \times_3 F_2\left(\begin{matrix} -n_1,\ n_1+2\left(n_2+\mu+\frac{1}{2}\right),\ a+\frac{n_2+i\xi_1}{2}+\frac{1}{4} \\ n_2+2a+\frac{1}{2},\ n_2+\mu+1 \end{matrix} \,\Big|\, 1\right), \quad (40)$$

and

$$\Theta_2^2(a,\mu,n_1,n_2;\xi_2) = B\left(a+\frac{i\xi_2}{2},\, a-\frac{i\xi_2}{2}\right) {}_3F_2\left(\begin{matrix} -n_2,\ n_2+2\mu,\ a+\frac{i\xi_2}{2} \\ 2a,\ \mu+1/2 \end{matrix} \,\Big|\, 1\right). \quad (41)$$

Since the latter ${}_3F_2$ hypergeometric function can be expressed in terms of the continuous Hahn polynomials given by (10), both expressions above can be written again in terms of the continuous Hahn polynomials as

$$\begin{aligned}\Theta_1^2(a,\mu,n_1,n_2;\xi_1) = {} & \frac{n_1!}{i^{n_1}(n_2+\mu+1)_{n_1}\left(n_2+2a+\frac{1}{2}\right)_{n_1}} \\ & \times B\left(a+\frac{n_2+i\xi_1}{2}+\frac{1}{4}, a+\frac{n_2-i\xi_1}{2}+\frac{1}{4}\right) \\ & \times p_{n_1}\left(\frac{\xi_1}{2}; a+\frac{n_2}{2}+\frac{1}{4}, \mu-a+\frac{2n_2+3}{4}, \mu-a+\frac{2n_2+3}{4}, a+\frac{n_2}{2}+\frac{1}{4}\right), \quad (42)\end{aligned}$$

and

$$\begin{aligned}\Theta_2^2(a,\mu,n_1,n_2;\xi_2) = {} & \frac{n_2!}{i^{n_2}\left(\mu+\frac{1}{2}\right)_{n_2}(2a)_{n_2}} B\left(a+\frac{i\xi_2}{2}, a-\frac{i\xi_2}{2}\right) \\ & \times p_{n_2}\left(\frac{\xi_2}{2}; a, \mu-a+\frac{1}{2}, \mu-a+\frac{1}{2}, a\right). \quad (43)\end{aligned}$$

For the case $r=3$, in view of (21), we can write

$$\begin{aligned} f_3(x_1,x_2,x_3;n_1,n_2,n_3,a,\mu) = {} & \left(1-\tanh^2 x_1\right)^{a+\frac{n_2+n_3}{2}+\frac{1}{2}} C_{n_1}^{(n_2+n_3+\mu+1)}(\tanh x_1) \\ & \times f_2(x_2,x_3;n_2,n_3,a,\mu),\end{aligned}$$

from which it follows from (26):

$$\begin{aligned}\mathcal{F}(f_3(x_1,x_2,x_3;n_1,n_2,n_3,a,\mu)) = {} & \frac{2^{n_2+n_3+2a}(2(n_2+n_3+\mu+1))_{n_1}}{n_1!} \\ & \times B\left(a+\frac{n_2+n_3+i\xi_1}{2}+\frac{1}{2}, a+\frac{n_2+n_3-i\xi_1}{2}+\frac{1}{2}\right) \\ & \times {}_3F_2\left(\begin{matrix}-n_1,\ n_1+2(n_2+n_3+\mu+1),\ a+\frac{n_2+n_3+i\xi_1}{2}+\frac{1}{2} \\ n_2+n_3+2a+1,\ n_2+n_3+\mu+\frac{3}{2}\end{matrix}\ \Big|\ 1\right) \\ & \times \mathcal{F}(f_2(x_2,x_3;n_2,n_3,a,\mu)). \quad (44)\end{aligned}$$

From (39), we can write

$$\begin{aligned}&\mathcal{F}(f_3(x_1,x_2,x_3;n_1,n_2,n_3,a,\mu)) \\ &= 2^{6a-\frac{3}{2}+n_2+2n_3}\frac{(2(n_2+n_3+\mu+1))_{n_1}\left(2\left(n_3+\mu+\frac{1}{2}\right)\right)_{n_2}(2\mu)_{n_3}}{n_1!n_2!n_3!} \\ &\times \Theta_1^3(a,\mu,n_1,n_2,n_3;\xi_1)\Theta_2^3(a,\mu,n_1,n_2,n_3;\xi_2)\Theta_3^3(a,\mu,n_1,n_2,n_3;\xi_3), \quad (45)\end{aligned}$$

where

$$\begin{aligned}\Theta_1^3(a,\mu,n_1,n_2,n_3;\xi_1) = {} & B\left(a+\frac{n_2+n_3+i\xi_1}{2}+\frac{1}{2}, a+\frac{n_2+n_3-i\xi_1}{2}+\frac{1}{2}\right) \\ & \times {}_3F_2\left(\begin{matrix}-n_1,\ n_1+2(n_2+n_3+\mu+1),\ a+\frac{n_2+n_3+i\xi_1}{2}+\frac{1}{2} \\ n_2+n_3+\mu+\frac{3}{2},\ n_2+n_3+2a+1\end{matrix}\ \Big|\ 1\right), \quad (46)\end{aligned}$$

$$\Theta_2^3(a,\mu,n_1,n_2,n_3;\xi_2) = B\left(a+\frac{n_3+i\xi_2}{2}+\frac{1}{4}, a+\frac{n_3-i\xi_2}{2}+\frac{1}{4}\right) \times {}_3F_2\left(\begin{matrix} -n_2,\ n_2+2\left(n_3+\mu+\frac{1}{2}\right),\ a+\frac{n_3+i\xi_2}{2}+\frac{1}{4} \\ n_3+\mu+1,\ n_3+2a+\frac{1}{2} \end{matrix} \,\Big|\, 1\right), \quad (47)$$

$$\Theta_3^3(a,\mu,n_1,n_2,n_3;\xi_3) = B\left(a+\frac{i\xi_3}{2}, a-\frac{i\xi_3}{2}\right) {}_3F_2\left(\begin{matrix} -n_3,\ n_3+2\mu,\ a+\frac{i\xi_3}{2} \\ \mu+\frac{1}{2},\ 2a \end{matrix} \,\Big|\, 1\right). \quad (48)$$

If we write the ${}_3F_2$ hypergeometric function in terms of the continuous Hahn polynomials $p_n(x;a,b,c,d)$ from (10), the expressions above can be written as in (32) for $r=3$.

The proof follows now by induction on r. □

3.2. The Class of Special Functions Using the Fourier Transform of the Orthogonal Polynomials on the Unit Ball

The Parseval identity corresponding to (24) is given by [6] (p. 118, Equation (7.17))

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(f(x))\overline{\mathcal{F}(g(x))}d\xi, \quad (49)$$

and in the r-variable case, Parseval's identity corresponding to (25) is [6] (p. 183, (iv))

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\ldots,x_r)\overline{g(x_1,\ldots,x_r)}dx_1\cdots dx_r = \frac{1}{(2\pi)^r}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathcal{F}(f(x_1,\ldots,x_r))\overline{\mathcal{F}(g(x_1,\ldots,x_r))}d\xi_1\cdots d\xi_r. \quad (50)$$

By substituting the results in Theorem 2 in Parseval's identity, we have the next theorem. The proof is included in Appendix A.

Theorem 3. *Let* $\mathbf{n}$ *and* $\mathbf{n}^j$ *be defined as in* (15)*, and let* $\mathbf{a} = (a_1,a_2)$ *and* $|\mathbf{a}| = a_1+a_2$*. Then, the following equality is satisfied:*

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} {}_rD_{\mathbf{n}}(i\boldsymbol{x};a_1,a_2)\, {}_rD_{\mathbf{m}}(-i\boldsymbol{x};a_2,a_1)\mathbf{dx} = (2\pi)^r 2^{-2r|\mathbf{a}|+r+1} h_{\mathbf{n}}^{(a_1+a_2-\frac{1}{2})} \times \prod_{j=1}^{r} \frac{(n_j!)^2\Gamma\left(|\mathbf{n}^{j+1}|+2a_1+\frac{r-j}{2}\right)\Gamma\left(|\mathbf{n}^{j+1}|+2a_2+\frac{r-j}{2}\right)}{2^{2|\mathbf{n}^{j+1}|}\left(\left(2|\mathbf{n}^{j+1}|+2|\mathbf{a}|+r-j-1\right)_{n_j}\right)^2}\delta_{n_j,m_j}, \quad (51)$$

for $a_1, a_2 > 0$*, where* $h_{\mathbf{n}}^{(a_1+a_2-\frac{1}{2})}$ *is given in* (17) *and*

$${}_rD_{\mathbf{n}}(\boldsymbol{x};a_1,a_2) = \prod_{j=1}^{r}\left\{\Gamma\left(a_1+\frac{|\mathbf{n}^{j+1}|-x_j}{2}+\frac{r-j}{4}\right)\Gamma\left(a_1+\frac{|\mathbf{n}^{j+1}|+x_j}{2}+\frac{r-j}{4}\right) \times {}_3F_2\left(\begin{matrix} -n_j,\ n_j+2\left(|\mathbf{n}^{j+1}|+|\mathbf{a}|+\frac{r-j-1}{2}\right),\ a_1+\frac{|\mathbf{n}^{j+1}|+x_j}{2}+\frac{r-j}{4} \\ |\mathbf{n}^{j+1}|+|\mathbf{a}|+\frac{r-j}{2},\ |\mathbf{n}^{j+1}|+2a_1+\frac{r-j}{2} \end{matrix}\,\Big|\, 1\right)\right\}, \quad (52)$$

which can be expressed in terms of the continuous Hahn polynomials (10) by

$$
{}_rD_{\mathbf{n}}(\boldsymbol{x};a_1,a_2)=\prod_{j=1}^{r}\left\{\frac{n_j!\,i^{-n_j}}{\left(\left|\mathbf{n}^{j+1}\right|+2a_1+\frac{r-j}{2}\right)_{n_j}\left(\left|\mathbf{n}^{j+1}\right|+|\mathbf{a}|+\frac{r-j}{2}\right)_{n_j}}\right.
\times\Gamma\left(a_1+\frac{\left|\mathbf{n}^{j+1}\right|-x_j}{2}+\frac{r-j}{4}\right)\Gamma\left(a_1+\frac{\left|\mathbf{n}^{j+1}\right|+x_j}{2}+\frac{r-j}{4}\right)
\times p_{n_j}\left(-\frac{ix_j}{2};a_1+\frac{\left|\mathbf{n}^{j+1}\right|}{2}+\frac{r-j}{4},a_2+\frac{\left|\mathbf{n}^{j+1}\right|}{2}+\frac{r-j}{4}\right.
\left.\left.,a_2+\frac{\left|\mathbf{n}^{j+1}\right|}{2}+\frac{r-j}{4},a_1+\frac{\left|\mathbf{n}^{j+1}\right|}{2}+\frac{r-j}{4}\right)\right\}, \tag{53}
$$

for $r\geq 1$.

4. Discussion and Conclusions

In [9], the author derived the Fourier transform of Jacobi polynomials on the interval $[-1,1]$ in terms of continuous Hahn polynomials and discussed some applications. Motivated by this investigation, the Fourier transforms of bivariate orthogonal polynomials were studied in [26]. In our recent study, we introduced and dealt with the Fourier transform of a family of multivariate orthogonal polynomials. In such a framework, we defined specific functions in terms of the orthogonal polynomials on the unit ball. We applied the Fourier transform to the corresponding functions. By Parseval's identity, the class of the resulting orthogonal functions in terms of continuous Hahn polynomials was discussed.

In the future, integral transforms of other families of multivariate orthogonal polynomials could be obtained by similar methods used in this paper, and further relationships with some other well-known orthogonal polynomials such as Wilson polynomials could be investigated.

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Appendix A

In this Appendix, we give a proof of Theorem 3.

The proof follows by using induction on r. For $r=1$, we obtain the specific functions from (33):

$$\begin{cases} f_1(x_1;n_1,a_1,\mu_1) = \left(1-\tanh^2 x_1\right)^{a_1} P_{n_1}^{\mu_1}(v_1) = \left(1-\tanh^2 x_1\right)^{a_1} C_{n_1}^{(\mu_1)}(\tanh x_1), \\ g_1(x_1;m_1,a_2,\mu_2) = \left(1-\tanh^2 x_1\right)^{a_2} P_{m_1}^{\mu_2}(v_1) = \left(1-\tanh^2 x_1\right)^{a_2} C_{m_1}^{(\mu_2)}(\tanh x_1), \end{cases} \tag{A1}$$

where $v_1 = \tanh x_1$. According to (A1) and (34), we use Parseval's identity to obtain

$$\begin{aligned} &2\pi \int_{-\infty}^{\infty} \left(1-\tanh^2 x_1\right)^{a_1+a_2} C_{n_1}^{(\mu_1)}(\tanh x_1) C_{m_1}^{(\mu_2)}(\tanh x_1) dx_1 \\ &\quad = 2\pi \int_{-1}^{1} \left(1-u^2\right)^{a_1+a_2-1} C_{n_1}^{(\mu_1)}(u) C_{m_1}^{(\mu_2)}(u) du = \frac{2^{2(a_1+a_2-1)}(2\mu_1)_{n_1}(2\mu_2)_{m_1}}{n_1! m_1! \Gamma(2a_1)\Gamma(2a_2)} \\ &\quad \times \int_{-\infty}^{\infty} \Gamma\left(a_1+\frac{i\xi_1}{2}\right)\Gamma\left(a_1-\frac{i\xi_1}{2}\right) \overline{\Gamma\left(a_2+\frac{i\xi_1}{2}\right)\Gamma\left(a_2-\frac{i\xi_1}{2}\right)} \\ &\times {}_3F_2\left(\begin{matrix} -n_1,\ n_1+2\mu_1,\ a_1+\frac{i\xi_1}{2} \\ 2a_1,\ \mu_1+1/2 \end{matrix} \,\Big|\, 1\right) \overline{{}_3F_2\left(\begin{matrix} -m_1,\ m_1+2\mu_2,\ a_2+\frac{i\xi_1}{2} \\ 2a_2,\ \mu_2+1/2 \end{matrix} \,\Big|\, 1\right)} d\xi_1. \end{aligned} \tag{A2}$$

By assuming

$$\mu_1 = \mu_2 = a_1 + a_2 - \frac{1}{2}, \tag{A3}$$

and considering the orthogonality relation (6), we obtain that the special function:

$$\begin{aligned} {}_1D_{n_1}(x_1;a_1,a_2) &= \Gamma\left(a_1-\frac{x_1}{2}\right)\Gamma\left(a_1+\frac{x_1}{2}\right) {}_3F_2\left(\begin{matrix} -n_1,\ n_1+2(a_1+a_2)-1,\ a_1+\frac{x_1}{2} \\ a_1+a_2,\ 2a_1 \end{matrix} \,\Big|\, 1\right) \\ &= \frac{n_1! i^{-n_1}}{(2a_1)_{n_1}(a_1+a_2)_{n_1}} \Gamma\left(a_1-\frac{x_1}{2}\right)\Gamma\left(a_1+\frac{x_1}{2}\right) p_{n_1}\left(\frac{-ix_1}{2};a_1,a_2,a_2,a_1\right) \end{aligned} \tag{A4}$$

has the orthogonality relation:

$$\begin{aligned} &\int_{-\infty}^{\infty} {}_1D_{n_1}(ix_1;a_1,a_2)\, {}_1D_{m_1}(-ix_1;a_2,a_1) dx_1 \\ &\quad = \frac{2\pi n_1! \Gamma(2a_1)\Gamma(2a_2)\Gamma^2(a_1+a_2)}{\left(n_1+a_1+a_2-\frac{1}{2}\right)\Gamma(2a_1+2a_2+n_1-1)} \delta_{n_1,m_1} \\ &\quad = \frac{2\pi (n_1!)^2 \Gamma(2a_1)\Gamma(2a_2)}{2^{2(a_1+a_2-1)}\left((2a_1+2a_2-1)_{n_1}\right)^2} h_{n_1}^{(a_1+a_2-\frac{1}{2})} \delta_{n_1,m_1}, \end{aligned} \tag{A5}$$

where $h_{n_1}^{(a_1+a_2-\frac{1}{2})}$ is given in (7). As a consequence, it follows that

$$\begin{aligned} &\int_{-\infty}^{\infty} \Gamma(a_1+ix_1)\Gamma(a_1-ix_1)\Gamma(a_2-ix_1)\Gamma(a_2+ix_1) \\ &\qquad \times p_{n_1}(x_1;a_1,a_2,a_2,a_1) p_{m_1}(x_1;a_1,a_2,a_2,a_1) dx_1 \\ &\qquad = \frac{\pi \Gamma(2a_1+n_1)\Gamma(2a_2+n_1)\Gamma^2(a_1+a_2+n_1)}{n_1!\left(n_1+a_1+a_2-\frac{1}{2}\right)\Gamma(2a_1+2a_2+n_1-1)} \delta_{n_1,m_1}, \end{aligned} \tag{A6}$$

for $a_1, a_2 > 0$, which gives the orthogonality relation for continuous Hahn polynomials $p_{n_1}(x_1;a_1,a_2,a_2,a_1)$, which was proven by Koelink [9].

For $r = 2$, we consider the specific functions from (18):

$$\begin{cases} f_2(x_1, x_2; n_1, n_2, a_1, \mu_1) &= \left(1-\tanh^2 x_1\right)^{a_1+\frac{1}{4}} \left(1-\tanh^2 x_2\right)^{a_1} P^{\mu_1}_{n_1,n_2}(v_1, v_2), \\ g_2(x_1, x_2; m_1, m_2, a_2, \mu_2) &= \left(1-\tanh^2 x_1\right)^{a_2+\frac{1}{4}} \left(1-\tanh^2 x_2\right)^{a_2} P^{\mu_2}_{m_1,m_2}(v_1, v_2), \end{cases} \tag{A7}$$

where $v_1 = \tanh x_1$ and $v_2 = \tanh x_2\sqrt{1-\tanh^2 x_1}$. According to (A7) and (39), if we use Parseval's identity again and apply the transforms $\tanh x_1 = u$, $\tanh x_2 = \frac{v}{\sqrt{1-u^2}}$, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(1-\tanh^2 x_1\right)^{a_1+a_2+\frac{1}{2}} \left(1-\tanh^2 x_2\right)^{a_1+a_2} P^{\mu_1}_{n_1,n_2}(v_1,v_2)P^{\mu_2}_{m_1,m_2}(v_1,v_2)dx_1dx_2 \\ &= \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(1-u^2-v^2\right)^{a_1+a_2-1} P^{\mu_1}_{n_1,n_2}(u,v)P^{\mu_2}_{m_1,m_2}(u,v)dvdu \\ &= \frac{2^{n_2+m_2+4a_1+4a_2-3}(2\mu_1)_{n_2}(2\mu_2)_{m_2}\left(2\left(n_2+\mu_1+\frac{1}{2}\right)\right)_{n_1}\left(2\left(m_2+\mu_2+\frac{1}{2}\right)\right)_{m_1}}{4\pi^2 n_1!n_2!m_1!m_2!} \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B\left(a_1+\frac{n_2+i\xi_1}{2}+\frac{1}{4}, a_1+\frac{n_2-i\xi_1}{2}+\frac{1}{4}\right)B\left(a_1+\frac{i\xi_2}{2},\ a_1-\frac{i\xi_2}{2}\right) \\ &\times {}_3F_2\left(\begin{matrix}-n_1,\ n_1+2\left(n_2+\mu_1+\frac{1}{2}\right),\ a_1+\frac{n_2+i\xi_1}{2}+\frac{1}{4} \\ n_2+2a_1+\frac{1}{2},\ n_2+\mu_1+1\end{matrix} \,\Big|\, 1\right) \\ &\times {}_3F_2\left(\begin{matrix}-n_2,\ n_2+2\mu_1,\ a_1+\frac{i\xi_2}{2} \\ 2a_1,\ \mu_1+1/2\end{matrix} \,\Big|\, 1\right) \\ &\times \overline{B\left(a_2+\frac{m_2+i\xi_1}{2}+\frac{1}{4}, a_2+\frac{m_2-i\xi_1}{2}+\frac{1}{4}\right)B\left(a_2+\frac{i\xi_2}{2},\ a_2-\frac{i\xi_2}{2}\right)} \\ &\times \overline{{}_3F_2\left(\begin{matrix}-m_1,\ m_1+2\left(m_2+\mu_2+\frac{1}{2}\right),\ a_2+\frac{m_2+i\xi_1}{2}+\frac{1}{4} \\ m_2+2a_2+\frac{1}{2},\ m_2+\mu_2+1\end{matrix} \,\Big|\, 1\right)} \\ &\times \overline{{}_3F_2\left(\begin{matrix}-m_2,\ m_2+2\mu_2,\ a_2+\frac{i\xi_2}{2} \\ 2a_2,\ \mu_2+1/2\end{matrix} \,\Big|\, 1\right)}d\xi_1 d\xi_2. \end{aligned} \tag{A8}$$

If we fix

$$\mu_1 = \mu_2 = a_1 + a_2 - \frac{1}{2} \tag{A9}$$

and use the orthogonality relation (6), it yields

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} {}_2D_{n_1,n_2}(ix_1, ix_2; a_1, a_2)\ {}_2D_{m_1,m_2}(-ix_1, -ix_2; a_2, a_1)dx_1dx_2 \\ &= \frac{\Gamma(2a_1)\Gamma(2a_2)\Gamma\left(2a_1+n_2+\frac{1}{2}\right)\Gamma\left(2a_2+n_2+\frac{1}{2}\right)4\pi^2(n_1!)^2(n_2!)^2}{2^{2n_2+4a_1+4a_2-3}(2(n_2+a_1+a_2))^2_{n_1}(2a_1+2a_2-1)^2_{n_2}} \\ &\times h^{\left(a_1+a_2-\frac{1}{2}\right)}_{n_1,n_2}\delta_{n_1,m_1}\delta_{n_2,m_2}, \end{aligned} \tag{A10}$$

where $h_{n_1,n_2}^{(a_1+a_2-\frac{1}{2})}$ is given in (17) and

$$
\begin{aligned}
{}_2D_{n_1,n_2}(x_1,x_2;a_1,a_2) = {}&{}_3F_2\left(\begin{matrix}-n_1,\ n_1+2(n_2+a_1+a_2),\ a_1+\frac{x_1}{2}+\frac{n_2}{2}+\frac{1}{4}\\ n_2+2a_1+\frac{1}{2},\ n_2+a_1+a_2+\frac{1}{2}\end{matrix}\,\Big|\,1\right)\\
&\times\,{}_3F_2\left(\begin{matrix}-n_2,\ n_2+2a_1+2a_2-1,\ a_1+\frac{x_2}{2}\\ 2a_1,\ a_1+a_2\end{matrix}\,\Big|\,1\right)\\
&\times\Gamma\left(a_1+\frac{n_2+x_1}{2}+\frac{1}{4}\right)\Gamma\left(a_1+\frac{n_2-x_1}{2}+\frac{1}{4}\right)\Gamma\left(a_1-\frac{x_2}{2}\right)\Gamma\left(a_1+\frac{x_2}{2}\right), \quad (A11)
\end{aligned}
$$

which can be expressed in terms of the continuous Hahn polynomials (10) as

$$
\begin{aligned}
{}_2D_{n_1,n_2}(x_1,x_2;a_1,a_2) = {}&\frac{n_1!n_2!i^{-n_1-n_2}}{(2a_1)_{n_2}(a_1+a_2)_{n_2}\left(n_2+2a_1+\frac{1}{2}\right)_{n_1}\left(n_2+a_1+a_2+\frac{1}{2}\right)_{n_1}}\\
&\times p_{n_1}\left(\frac{-ix_1}{2};a_1+\frac{n_2}{2}+\frac{1}{4},a_2+\frac{n_2}{2}+\frac{1}{4},a_1+\frac{n_2}{2}+\frac{1}{4},a_2+\frac{n_2}{2}+\frac{1}{4}\right)p_{n_2}\left(\frac{-ix_2}{2};a_1,a_2,a_2,a_1\right)\\
&\times\Gamma\left(a_1+\frac{n_2+x_1}{2}+\frac{1}{4}\right)\Gamma\left(a_1+\frac{n_2-x_1}{2}+\frac{1}{4}\right)\Gamma\left(a_1-\frac{x_2}{2}\right)\Gamma\left(a_1+\frac{x_2}{2}\right). \quad (A12)
\end{aligned}
$$

Similar to the cases $r = 1$ and $r = 2$, if we substitute (18) and (30) in the Parseval identity (50), the necessary calculations give the desired result.

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