Article

A Novel Approach in Solving Improper Integrals

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Abstract: To resolve several challenging applications in many scientific domains, general formulas of improper integrals are provided and established for use in this article. The suggested theorems can be considered generators for new improper integrals with precise solutions, without requiring complex computations. New criteria for handling improper integrals are illustrated in tables to simplify the usage and the applications of the obtained outcomes. The results of this research are compared with those obtained by I.S. Gradshteyn and I.M. Ryzhik in the classical table of integrations. Some well-known theorems on improper integrals are considered to be simple cases in the context of our work. Some applications related to finding Green's function, one-dimensional vibrating string problems, wave motion in elastic solids, and computing Fourier transforms are presented.

Keywords: improper integrals; power series; analytic function; Cauchy residue theorem; Ramanujan's master theorem

MSC: 30E20; 33E20; 44A99

1. Introduction

Numerous studies on the topic of improper integrals have been published in recent years in a variety of scientific disciplines, including physics and engineering [1–7]. Due to this, mathematicians have been particularly interested in finding new theorems and methods to solve these integrals. Particularly in engineering, applied mathematical physics, electrical engineering, and other fields, it is sometimes necessary to handle erroneous integrals in computations or when describing models [8–16]. While some of these integrations can be handled easily, others require complex calculations. Many of these integrals require computer software to be solved as they cannot be calculated so manually. Additionally, numerical techniques may be employed to resolve some incorrect integrals that the aforementioned techniques are unable to resolve [17–23].

The process of evaluating improper integrals is not usually based on certain rules or techniques that can be applied directly. Many methods and techniques were established and introduced by mathematicians and physicists to present a closed form for indefinite integrals, the technique of double integrals, series methods, residue theorems, calculus under the integral sign, and other methods that are used to solve improper complex integrals exactly or approximately [24–31].

The residue theorem was first established by A.L. Cauchy in 1826, which is considered a powerful theorem in complex analysis. However, the applications that can be calculated using the residue theorem to compute integrals on real numbers require many precise constraints that should be satisfied in order to solve the integrals, including finding appropriate closed contours and also determining the poles. Another challenge in the process of applying the residue theorem is the difficulty and efforts in finding solutions for some integrations.

According to his published memoirs, Cauchy developed powerful formulas in mathematics using the residue theorem [4]. Researchers consider these formulas essential in treating and solving improper integrals. However, these results are considered simple cases

when compared to the results that we present in this article. In addition, we show that the proposed theorems and results in this research are not based on the residue theorem.

One significant accomplishment in the sphere of definite and indefinite integrals is found in the master theorem of Ramanujan, which presents new expressions concerning the Milline transform of any continuous function in terms of the analytic Taylor series, and others [32–39]. It was implemented by Ramanujan and other researchers as a powerful tool in calculating definite and indefinite integrals and also in computing infinite series. The obtained results are as applicable and effective as Ramanujan's master theorem in handling and generating new formulas of integrals with direct solutions.

In this study, we introduce new theorems to simplify the procedure of computing improper integrals by presenting new theorems with proofs. Each theorem can generate many improper integral formulas that cannot be solved by usual techniques or would need a large amount of effort and time spent in order to be solved. The motivation of this work is to generate as many improper integrals and their values as possible to be used in different problems. The obtained results can be implemented to construct new tables of integrations so that researchers can use them in calculations and to check the accuracy of their answers while discovering new methods.

The main purpose of this work is to introduce simple new techniques to help researchers, mathematicians, engineers, physicists, etc., to solve some difficult improper integrals that cannot be treated or solved easily (and which require several theorems and a large amount of effort to solve). This goal is achieved by introducing some master theorems that can be implemented in order to solve difficult applications. The outcomes can be generalized and introduced in tables to obtain and to use the results of some improper integrals directly.

We organize this article as follows: In Section 2, we introduce some illustrative preliminaries; then, facts concerning analytic functions, master theorems, and results are presented in Section 3. Mathematical remarks and several applications are presented in Section 4. Finally, the conclusion of our research is presented in Section 5.

2. Preliminaries

In this section, some basic definitions and theorems related to our work are presented and illustrated for later use.

2.1. Basic Definitions and Lemmas

Definition 1 ([7]). *Suppose that a function f is analytic in a domain $\Omega \subseteq \mathbb{C}$, where $\mathbb{C}$ is the complex plane. Consider a disc $D \subseteq \Omega$ centered at z_0; then, the function f can be expressed in the following series expansion:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where a_n is the coefficients of the series.

Definition 2 ([8]). *Assume that f is an analytic function; then, Taylor series expansion at any point x_0 of f in its domain is given by*

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n,$$

which converges to f in a neighborhood of x_0 point wisely.

2.2. Basic Formulas of Series and Improper Integrals

In this section, we introduce some series and improper integrals that are needed in our work.

Lemma 1. *The following factorization formula holds for* $n \in \mathbb{N}$, *as follows*

$$
\frac{1}{(x^2+1^2)(x^2+3^2)\ldots(x^2+(2n+1)^2)}
$$
$$
= \frac{(-1)^n}{4^n(2n+1)!} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \frac{2n+1-2k}{(2n+1-2k)^2+x^2} \tag{1}
$$

Proof. To prove Equation (1), we define an integral whose solution can be expressed by two different forms: the left side of Equation (1) and the right side of the equation.

Let

$$
I = \int_{0}^{\infty} e^{-px}(sinx)^{2a+1}dx, \tag{2}
$$

where $p > 0$, $a \in \mathbb{N}$.

Taking the indefinite integral:

$$
J = p^2 \int e^{-px}(sinx)^{2a+1}dx \tag{3}
$$

Applying integration by parts on Equation (3) twice, we obtain a reduction formula as follows:

$$
J = -pe^{-px}(sinx)^{2a+1} - (2a+1)e^{-px}sin^{2a}xcosx
$$
$$
+(2a+1)\int e^{-px}[2a((sinx)^{2a-1}-(sinx)^{2a+1})-(sinx)^{2a+1}]\,dx. \tag{4}
$$

Taking the limit of the integrals in Equation (4) from 0 to ∞, we obtain:

$$
\int_{0}^{\infty} e^{-px}(sinx)^{2a+1}dx = \frac{(2a+1)(2a)}{p^2+(2a+1)^2} \int_{0}^{\infty} e^{-px}(sinx)^{2a-1}dx. \tag{5}
$$

Applying Equation (5) $(a-1)$ times to the integral $\int_{0}^{\infty} e^{-px}(sinx)^{2a-1}dx$, we obtain:

$$
\int_{0}^{\infty} e^{-px}(sinx)^{2a+1}dx
$$
$$
= \frac{(2a+1)(2a)(2a-1)(2a-2)\ldots(3)(2)}{((2a+1)^2+p^2)\,((2a-1)^2+p^2)\ldots(3+p^2)} \int_{0}^{\infty} e^{-px}sinxdx. \tag{6}
$$

The integral $\int_{0}^{\infty} e^{-px}sinxdx$ can be calculated easily using twice integration by parts to obtain:

$$
\int_{0}^{\infty} e^{-px}sinxdx = \frac{1}{1+p^2}. \tag{7}
$$

Substituting the fact in Equation (7) into Equation (6), we obtain:

$$
\int_{0}^{\infty} e^{-px}(sinx)^{2a+1}dx = \frac{(2a+1)!}{((2a+1)^2+p^2)\,((2a-1)^2+p^2)\ldots(3+p^2)\,(1+p^2)}. \tag{8}
$$

Therefore, the left side of Equation (1) is obtained.

Now, we express the solution of Equation (2) in another form, that is, to obtain the right side of Equation (1), as follows:

Using the power trigonometric formula deduced using De Moivre's formula, Euler's formula, and the binomial theorem [10] (p. 31)

$$
(sin(x))^{2a+1} = \frac{(-1)^a}{4^a} \sum_{k=0}^{a} (-1)^k \binom{2a+1}{k} sin[(2a+1-2k)x]. \tag{9}
$$

Substituting Equation (9) into Equation (2), we obtain:

$$\int_0^\infty e^{-px}(sinx)^{2a+1}dx = \int_0^\infty e^{-px}\frac{(-1)^a}{4^a}\sum_{k=0}^a (-1)^k \binom{2a+1}{k} sin[(2a+1-2k)x]dx \qquad (10)$$

Therefore, by changing the order of the integral and the sum in Equation (10), we obtain:

$$\int_0^\infty e^{-px}(sinx)^{2a+1}dx = \frac{(-1)^a}{4^a}\sum_{k=0}^a (-1)^k \binom{2a+1}{k} \int_0^\infty e^{-px}sin[(2a+1-2k)x]dx \qquad (11)$$

To evaluate the integral $\int_0^\infty e^{-px}sin[(2a+1-2k)x]dx$, we apply twice integration by parts to obtain:

$$\int_0^\infty e^{-px}sin[(2a+1-2k)x]dx = \frac{2a+1-2k}{(2a+1-2k)^2+p^2}. \qquad (12)$$

Substituting the result in Equation (12) into Equation (11), we obtain:

$$\int_0^\infty e^{-px}(sinx)^{2a+1}dx= \frac{(-1)^a}{4^a}\sum_{k=0}^a (-1)^k \binom{2a+1}{k} \frac{2a+1-2k}{(2a+1-2k)^2+p^2}. \qquad (13)$$

Therefore, the right side of Equation (1) is obtained.

Then, equating Equation (13) with Equation (8); this, thus, completes the proof of Equation (21). □

Lemma 2. *The following factorization holds for $n \in \mathbb{N}$ as,*

$$\frac{1}{x(x^2+2^2)(x^2+4^2)...(x^2+4n^2)} =$$
$$\frac{1}{2^{2n}\,(2n)!} \left(\frac{1}{x}\binom{2n}{n} + 2\sum_{k=0}^{n-1}(-1)^{n+k}\binom{2n}{k} \frac{x}{(2n-2k)^2+x^2} \right). \qquad (14)$$

Proof. The proof is obtained by repeating the same process in proving Lemma (1), but by using the integral $\int_0^\infty e^{-px}(sinx)^{2a}dx$, where $p > 0$ *and* $a \in \mathbb{N}$. □

Lemma 3. *The following factorization formula holds for* $n = 0, 1, \cdots,$ *and* $m = 1, 2, \cdots,$ *as follows:*

$$\frac{1}{\left[(x^2+1^2)(x^2+3^2)...\left(x^2+(2n+1)^2\right)\right]\,[x(x^2+2^2)(x^2+4^2)...(x^2+4m^2)]}$$
$$= \frac{(-1)^n}{(2^{2m+2n})(m!)^2(2n+1)!}\sum_{s=0}^n (-1)^s\binom{2n+1}{s} \frac{2n+1-2s}{x((2n+1-2s)^2+x^2)}$$
$$+ \frac{(-1)^n}{2^{2m+2n-1}\,(2m)!(2n+1)!} \left(\sum_{k=0}^{m-1}\sum_{s=0}^n (-1)^{m+k+s}\binom{2m}{k}\binom{2n+1}{s} \frac{x(2n+1-2s)}{((2m-2k)^2+x^2)\,((2n+1-2s)^2+x^2)} \right). \qquad (15)$$

Proof. This is a direct result obtained by multiplying Equation (1) by Equation (14). □

Lemma 4. *The following formulas of improper integrals are created using Lemmas (1–3):*

$$\int_0^\infty \frac{cos(\theta x)}{(x^2+1)(x^2+9)...(x^2+(2n+1)^2)}dx$$
$$= \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n+1}}\sum_{k=0}^n (-1)^k\binom{2n+1}{k}e^{\theta(2k-2n-1)}, \qquad (16)$$
$$\text{for } \theta \geq 0 \text{ , } n = 0, 1, \cdots$$

Proof. The formula is obtained by multiplying both sides of Equation (1) by $cos(\theta x)$, then integrating both sides from 0 to ∞, and using the well-known fact:

$$\int_0^{\infty} \frac{cos(\theta x)}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-a\theta},$$

where a and $\theta > 0$. $\square$

$$\int_0^{\infty} \frac{x sin(\theta x)}{(x^2+1)(x^2+9)...(x^2+(2n+1)^2)} dx$$
$$= \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n+1}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} (2n-2k+1) e^{\theta(2k-2n-1)}, \tag{17}$$
$$\text{for } \theta > 0, n = 0, 1, \cdots.$$

Proof. The formula is obtained by differentiating both sides of Equation (16) with respect to θ. $\square$

$$\int_0^{\infty} \frac{sin(\theta x)}{x(x^2+4)(x^2+16)...(x^2+(2n)^2)} dx$$
$$= \frac{(-1)^n}{(2n)!} \frac{\pi}{2^{2n+1}} \left((-1)^n \binom{2n}{n} + 2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} e^{2\theta(k-n)} \right), \tag{18}$$
$$\text{for } \theta > 0, n = 1, 2, \cdots.$$

Proof. The formula is obtained by multiplying both sides of Equation (14) by $sin(\theta x)$, then integrating both sides from 0 to ∞, and using the well-known fact:

$$\int_0^{\infty} \frac{sin(\theta x)}{x(a^2 + x^2)} dx = \frac{\pi}{2a^2} \left(1 - e^{-a\theta} \right),$$

where θ and $a > 0$

$$\int_0^{\infty} \frac{cos(\theta x)}{(x^2+4)(x^2+16)...(x^2+(2n)^2)} dx = \frac{(-1)^n \pi \, 2^{1-2n}}{(2n)!} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (k-n) e^{2\theta(k-n)},$$
$$\text{for } \theta \geq 0, n = 1, 2, \cdots. \tag{19}$$

$\square$

Proof. The formula is obtained by differentiating both sides of Equation (18) with respect to θ. $\square$

Lemma 5. *Let $\theta > 0$ and $n = 0, 1, \cdots, m = 1, 2, \cdots$. Then, we have the following improper integrals:*

$$\int_0^{\infty} \frac{sin(\theta x)}{\left((x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)\right) \, \left(x(x^2+4)(x^2+16)\cdots(x^2+4m^2)\right)} dx$$
$$= \frac{(-1)^n \pi}{(2^{2m+2n+1})(m!)^2(2n+1)!} \sum_{s=0}^{n} (-1)^s \binom{2n+1}{s} \frac{1-e^{-\theta(2n+1-2s)}}{(2n+1-2s)}$$
$$+ \frac{(-1)^n \pi}{2^{2m+2n} (2m)!(2n+1)!} \sum_{k=0}^{m-1} \sum_{s=0}^{n} (-1)^{m+k+s} \binom{2m}{k} \binom{2n+1}{s} \tag{20}$$
$$\frac{(2n+1-2s)\left(e^{-\theta(2n+1-2s)} - e^{-\theta(2m-2k)}\right)}{(2m-2k)^2 - (2n+1-2s)^2}.$$

Proof. The formula is obtained by multiplying both sides of Equation (15) by $sin(\theta x)$, then integrating both sides from 0 to ∞, and using the well-known facts:

$$\int_0^\infty \frac{sin(\theta x)}{x(a^2 + x^2)} dx = \frac{\pi}{2a^2}\left(1 - e^{-a\theta}\right),$$

and

$$\int_0^\infty \frac{xsin(\theta x)}{a^2 + x^2} dx = \frac{\pi}{2}e^{-a\theta},$$

where θ and $a > 0$

$$\int_0^\infty \frac{cos(\theta x)}{\left((x^2+1)(x^2+9)...(x^2+(2n+1)^2)\right)\left((x^2+4)(x^2+16)...(x^2+4m^2)\right)} dx$$

$$= \frac{(-1)^n \pi}{\left(2^{2m+2n+1}\right)(m!)^2(2n+1)!}\sum_{s=0}^{n}(-1)^s\binom{2n+1}{s}e^{-\theta(2n+1-2s)} + \frac{(-1)^n \pi}{2^{2m+2n}\,(2m)!(2n+1)!}$$

$$\sum_{k=0}^{m-1}\sum_{s=0}^{n}(-1)^{m+k+s}\binom{2m}{k}\binom{2n+1}{s}\frac{(2n+1-2s)\left((2m-2k)e^{-\theta(2m-2k)}-(2n+1-2s)e^{-\theta(2n+1-2s)}\right)}{(2m-2k)^2-(2n+1-2s)^2}$$

□

Proof. The formula can be obtained by differentiating both sides of Equation (20) with respect to θ. □

3. New Master Theorems

In this part, we present new theorems to help mathematicians, engineers, and physicists solve complicated improper integrals. To obtain our objective, we introduce some facts concerning analytic functions [7,9,12].

Assuming that f is an analytic function in a disc D centered at α, then using Taylor's expansion, where α, β and θ are real constants, we have

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k, \tag{21}$$

substituting $z = \alpha + \beta e^{i\theta x}$ into $f(z)$, where β is not completely arbitrary, since it must be smaller than the radius of D, we obtain

$$f\left(\alpha + \beta e^{i\theta x}\right) = \sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\beta^k e^{i\theta kx}, \ x \in \mathbb{R}. \tag{22}$$

Using the formulas

$$e^{i\theta x} + e^{-i\theta x} = 2cos(\theta x), \ e^{i\theta x} - e^{-i\theta x} = 2isin(\theta x),$$

one can obtain

$$\begin{aligned}\frac{1}{2}\left(f\left(\alpha + \beta e^{i\theta x}\right) + f\left(\alpha + \beta e^{-i\theta x}\right)\right) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\beta^k\left(e^{i\theta kx} + e^{-i\theta kx}\right)\\ &= \sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\beta^k cos(k\theta x)\\ &= f(\alpha) + f'(\alpha)\beta cos(\theta x) + \frac{f''(\alpha)}{2!}\beta^2 cos(2\theta x) + \dots.\end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned}\frac{1}{2i}\left(f\left(\alpha + \beta e^{i\theta x}\right) - f\left(\alpha + \beta e^{-i\theta x}\right)\right) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\beta^k\left(e^{i\theta kx} - e^{-i\theta kx}\right)\\ &= f'(\alpha)\beta sin(\theta x) + \frac{f''(\alpha)}{2!}\beta^2 sin(2\theta x) + \dots\\ &= \sum_{k=1}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\beta^k sin(k\theta x).\end{aligned} \tag{24}$$

Next, the parameters in Equations (23) and (24) can be modified in the following lemma.

Lemma 3. *Assume that $g(\alpha + z)$ is an analytic function that has the following series expansion:*

$$g(\alpha + z) = \sum_{k=0}^{\infty} M_k e^{-kz},\tag{25}$$

whether z be real or imaginary, and $\sum_{k=0}^{\infty} M_k$ is absolutely convergent. Then

$$\frac{1}{2}\big(g(\alpha - i\theta x) + g(\alpha + i\theta x)\big) = \frac{1}{2}\sum_{k=0}^{\infty} M_k\Big(e^{ik\theta x} + e^{-ik\theta x}\Big) = \sum_{k=0}^{\infty} M_k \cos(k\,\theta x),\tag{26}$$

and,

$$\frac{1}{2i}\big(g(\alpha - i\theta x) - g(\alpha + i\theta x)\big) = \frac{1}{2i}\sum_{k=1}^{\infty} M_k\Big(e^{ik\theta x} - e^{-ik\theta x}\Big) = \sum_{k=1}^{\infty} M_k \sin(k\,\theta x),\tag{27}$$

where $\theta > 0$, $\alpha \in \mathbb{R}$, and x is any real number.

The next part of this section includes the new master theorems that we establish. Moreover, we mention here that Cauchy's results in [3] are identical to our results with special choices of the parameters, as will be discussed later.

Theorem 1. *Let f be an analytic function in a disc D centered at α, where $\alpha \in \mathbb{R}$. Then, we have the following improper integral formula:*

$$\begin{aligned}\int_0^{\infty} &\frac{f(\alpha+\beta e^{i\theta x}) + f(\alpha + \beta e^{-i\theta x})}{(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)}\,dx \\ &= \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n}}\sum_{s=0}^{n}(-1)^s\binom{2n+1}{s}f\Big(\alpha + \beta e^{\theta(2s-2n-1)}\Big),\end{aligned}\tag{28}$$

where $\theta \geq 0$ and $n = 0, 1, 2, \cdots$.

Proof. Let

$$I = \int_0^{\infty} \frac{f\big(\alpha + \beta e^{i\theta x}\big) + f\big(\alpha + \beta e^{-i\theta x}\big)}{(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)}\,dx.\tag{29}$$

Now, since f is an analytic function around α, substituting the fact in Equation (23) into Equation (29), we obtain

$$I = \int_0^{\infty} \frac{2\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)\beta^k}{k!}\cos(k\theta x)}{(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)}\,dx.\tag{30}$$

Fubini's theorem implies changing the order of the summation and the improper integral to obtain

$$I = 2\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)\beta^k}{k!}\int_0^{\infty}\frac{\cos(k\theta x)}{(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)}\,dx.\tag{31}$$

The fact in Equation (1) implies that Equation (31) becomes

$$I = 2\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)\beta^k}{k!}\frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n+1}}\sum_{s=0}^{n}(-1)^s\binom{2n+1}{s}e^{k\theta(2s-2n-1)}.\tag{32}$$

The result comes directly, by comparing the definition of the function g in Equation (25) with the definition of the function f in Equation (22), to obtain

$$I = \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n}} \sum_{s=0}^{n} (-1)^s \binom{2n+1}{s} f\left(\alpha + \beta e^{\theta(2s-2n-1)}\right).$$

□

Theorem 2. *Let f be an analytic function in a disc D centered at α, where $\alpha \in \mathbb{R}$. Then, we have the following improper integral formula:*

$$
\begin{aligned}
\int_0^{\infty} \frac{x\left(f(\alpha+\beta e^{i\theta x}) - f(\alpha+\beta e^{-i\theta x})\right)}{i(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)} dx \\
= \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n}} \sum_{s=0}^{n} (-1)^s \binom{2n+1}{s} (2n-2s \\
+1)\left(f\left(\alpha + \beta e^{\theta(2s-2n-1)}\right) - f(\alpha)\right),
\end{aligned}
\tag{33}
$$

where $\theta > 0$ and $n = 0, 1, 2, \cdots$.

Proof. Let

$$I = \int_0^{\infty} \frac{x\left(f\left(\alpha + \beta e^{i\theta x}\right) - f\left(\alpha + \beta e^{-i\theta x}\right)\right)}{i(x^2+1)(x^2+9)\ldots(x^2+(2n+1)^2)} dx. \tag{34}$$

Now, since f is an analytic function around α and substituting the fact in Equation (24) into Equation (34), we obtain

$$I = 2\sum_{k=1}^{\infty} \frac{f^{(k)}(\alpha)\beta^k}{k!} \int_0^{\infty} \frac{x(\sin(k\theta x))}{(x^2+1)(x^2+9)\ldots(x^2+(2n+1)^2)} dx. \tag{35}$$

Substituting the fact in Equation (2) into Equation (35), we obtain

$$I = 2\sum_{k=1}^{\infty} \frac{f^{(k)}(\alpha)\beta^k}{k!} \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n+1}} \sum_{s=0}^{n} (-1)^s \binom{2n+1}{s} (2n-2s+1) e^{\theta k(2s-2n-1)}. \tag{36}$$

The fact in Equation (22) implies that Equation (36) becomes

$$I = \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n}} \sum_{s=0}^{n} (-1)^s \binom{2n+1}{s} (2n-2s+1)\left(f\left(\alpha + \beta e^{\theta(2s-2n-1)}\right) - f(\alpha)\right).$$

Hence, this completes the proof. □

We should point out that $f(\alpha)$ appears in Equation (33) because the lower index of the infinite summation started from $k = 1$ and not from $k = 0$, as is the case in Equation (29). Thus, when we want to express the answer in terms of the original function f, we add and subtract $f(\alpha)$ to obtain our result.

Theorem 3. *Let f be an analytic function in a disc D centered at α, where $\alpha \in \mathbb{R}$. Then, we have the following improper integral formula:*

$$\int_0^{\infty} \frac{f\left(\alpha + \beta e^{i\theta x}\right) - f\left(\alpha + \beta e^{-i\theta x}\right)}{i\, x\, (x^2+4)(x^2+16)\ldots(x^2+(2n)^2)} dx = \frac{(-1)^n}{(2n)!} \frac{\pi}{2^{2n}} \left((-1)^n \binom{2n}{n}\psi + 2\sum_{s=0}^{n-1} (-1)^s \binom{2n}{s}\phi(s)\right), \tag{37}$$

where $\theta > 0$, $n = 1, 2, \cdots$, $\psi = f(\alpha+\beta) - f(\alpha)$ and $\phi(s) = f\left(\alpha + \beta e^{2\theta(s-n)}\right) - f(\alpha)$.

Proof. The proof of Theorem 3 can be obtained by similar arguments to Theorem 2 and using the fact (3) in Lemma 1. □

Theorem 4. *Let f be an analytic function in a disc D centered at $\alpha \in \mathbb{R}$. Then, we have the following improper integral formula:*

$$\int_0^\infty \frac{f\left(\alpha + \beta e^{i\theta x}\right) + f\left(\alpha + \beta e^{-i\theta x}\right)}{(x^2 + 4)(x^2 + 16)\ldots(x^2 + (2n)^2)}dx = \frac{(-1)^n \pi \, 2^{2-2n}}{(2n)!}\sum_{s=0}^{n-1}(-1)^s \binom{2n}{s}(s-n)f\left(\alpha + \beta e^{2\theta(s-n)}\right), \quad (38)$$

where $\theta \geq 0$, $n = 1, 2, \cdots$.

Proof. The proof of Theorem 4 can be obtained by similar arguments to Theorem 1 and using the fact (4) in Lemma 1. □

Theorem 5. *Let f be an analytic function in a disc D centered at α, where $\alpha \in \mathbb{R}$. Then, we have the following improper integral formula:*

$$\int_0^\infty \frac{f\left(\alpha + \beta e^{i\theta x}\right) - f\left(\alpha + \beta e^{-i\theta x}\right)}{i\left((x^2+1)(x^2+9)\ldots(x^2+(2n+1)^2)\right)\left(x(x^2+4)(x^2+16)\cdots(x^2+4m^2)\right)}dx$$

$$= \frac{(-1)^n \pi}{(2^{2m+2n})(m!)^2(2n+1)!}\sum_{s=0}^n (-1)^s \binom{2n+1}{s}\frac{(\varphi - \psi(s))}{(2n+1-2s)}$$

$$+ \frac{(-1)^n \pi}{2^{2m+2n-1}(2m)!(2n+1)!}\left(\sum_{k=0}^{m-1}\sum_{s=0}^n (-1)^{m+k+s}\binom{2m}{k}\binom{2n+1}{s}(2n+1\right. \tag{39}$$

$$\left. -2s\right)\frac{(\psi(s) - \phi(k))}{\left((2m-2k)^2 - (2n+1-2s)^2\right)}\right),$$

where $\theta > 0$, $n = 0, 1, 2, \cdots$, $m = 1, 2, \cdots$, $\psi(s) = f\left(\alpha + \beta e^{-\theta(2n+1-2s)}\right)$, $\phi(k) = f\left(\alpha + \beta e^{-\theta(2m-2k)}\right)$, and $\varphi = f(\alpha + \beta)$.

Proof. Let

$$I = \int_0^\infty \frac{f\left(\alpha + \beta e^{i\theta x}\right) - f\left(\alpha + \beta e^{-i\theta x}\right)}{i\left((x^2+1)(x^2+9)\ldots(x^2+(2n+1)^2)\right)\left(x(x^2+4)(x^2+16)\cdots(x^2+4m^2)\right)}dx. \tag{40}$$

Now, since f is an analytic function around α and substituting the fact in Equation (24) into Equation (40), we obtain

$$I = 2\sum_{k=1}^\infty \frac{f^{(k)}(\alpha)\beta^k}{k!}\int_0^\infty \frac{\sin(\theta k x)}{\left((x^2+1)(x^2+9)\ldots(x^2+(2n+1)^2)\right)\left(x(x^2+4)(x^2+16)\cdots(x^2+4m^2)\right)}dx. \tag{41}$$

Substituting the fact in Equation (9) into Equation (41), we obtain

$$I = 2\sum_{k=1}^\infty \frac{f^{(k)}(\alpha)\beta^k}{k!}(A + B), \tag{42}$$

where

$$A = \frac{(-1)^n \pi}{(2^{2m+2n+1})(m!)^2(2n+1)!}\sum_{s=0}^n (-1)^s \binom{2n+1}{s}\frac{1 - e^{-\theta k(2n+1-2s)}}{(2n+1-2s)},$$

$$B = \frac{(-1)^n \pi}{2^{2m+2n} (2m)!(2n+1)!} \left(\sum_{k=0}^{m-1} \sum_{s=0}^{n} (-1)^{m+k+s} \binom{2m}{k} \binom{2n+1}{s} (2n+1-2s) \frac{e^{-\theta k(2n+1-2s)} - e^{-\theta k(2m-2k)}}{(2m-2k)^2 - (2n+1-2s)^2} \right).$$

The fact in Equation (22) implies that Equation (42) becomes

$$I$$
$$= \frac{(-1)^n \pi}{(2^{2m+2n})(m!)^2 (2n+1)!} \sum_{s=0}^{n} (-1)^s \binom{2n+1}{s} \frac{\varphi - \psi(s)}{2n+1-2s}$$
$$+ \frac{(-1)^n \pi}{2^{2m+2n-1} (2m)!(2n+1)!} \left(\sum_{k=0}^{m-1} \sum_{s=0}^{n} (-1)^{m+k+s} \binom{2m}{k} \binom{2n+1}{s} (2n \right.$$
$$\left. +1-2s) \frac{(\psi(s) - \phi(k))}{\left((2m-2k)^2 - (2n+1-2s)^2\right)} \right),$$

where $\psi(s) = f\left(\alpha + \beta e^{-\theta(2n+1-2s)}\right)$, $\phi(k) = f\left(\alpha + \beta e^{-\theta(2m-2k)}\right)$, and $\varphi = f(\alpha + \beta)$.

Hence, this completes the proof of Theorem 5. $\square$

Theorem 6. *Let f be an analytic function in a disc D centered at α, where $\alpha \in \mathbb{R}$. Then, we have the following improper integral formula:*

$$\int_0^{\infty} \frac{f\left(\alpha + \beta e^{i\theta x}\right) + f\left(\alpha + \beta e^{-i\theta x}\right)}{\left((x^2+1)(x^2+9)...(x^2+(2n+1)^2)\right) \left((x^2+4)(x^2+16)...(x^2+4m^2)\right)} dx$$
$$= \frac{(-1)^n \pi}{(2^{2m+2n})(m!)^2 (2n+1)!} \sum_{s=0}^{n} (-1)^s \binom{2n+1}{s} \psi(s)$$
$$+ \frac{(-1)^n \pi}{2^{2m+2n-1} (2m)!(2n+1)!} \left(\sum_{k=0}^{m-1} \sum_{s=0}^{n} (-1)^{m+k+s} \binom{2m}{k} \binom{2n+1}{s} \right.$$
$$\left. \frac{(2n+1-2s)\left((2m-2k)\ \phi(k) - (2n+1-2s)\psi(s)\right)}{\left((2m-2k)^2 - (2n+1-2s)^2\right)} \right), \tag{43}$$

where $\theta \geq 0$, $n = 0,1,2,\cdots$, $m = 1,2,\cdots$, $\psi(s) = f\left(\alpha + \beta e^{-\theta(2n+1-2s)}\right)$, and $\phi(k) = f\left(\alpha + \beta e^{-\theta(2m-2k)}\right)$.

Proof The proof of Theorem 6 can be obtained by similar arguments to Theorem 5 and using the fact (6) in Lemma 2. $\square$

The following table, Table 1 illustrates some corollaries of the theorems with special cases and presents some values of improper integrals under certain conditions.

Table 1. Improper integral formulas with the series representation as detailed in Equation (25).

	$f(x)$	$\int\limits_{0}^{\infty} f(x)dx$	Conditions	No. of Theorem
1.	$\dfrac{g(\alpha-i\theta x)+g(\alpha+i\theta x)}{(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)}$	$\dfrac{(-1)^n}{(2n+1)!}\dfrac{\pi}{2^{2n}}\sum\limits_{s=0}^{n}(-1)^s\dbinom{2n+1}{s}g(\alpha-\theta(2s-2n-1)),$	$\theta \geq 0,$ $n=1,2,\cdots$	Theorem 1
2.	$\dfrac{x(g(\alpha-i\theta x)-g(\alpha+i\theta x))}{i(x^2+1)(x^2+9)...(x^2+(2n+1)^2)}$	$\dfrac{(-1)^n}{(2n+1)!}\dfrac{\pi}{2^{2n}}\sum\limits_{s=0}^{n}(-1)^s\dbinom{2n+1}{s}(2n-2s+1)(g(\alpha-\theta(2s-2n-1))-g(\alpha)),$	$\theta > 0,$ $n=1,2,\cdots$	Theorem 2
3.	$\dfrac{g(\alpha-i\theta x)+f(\alpha+i\theta x)}{(x^2+4)(x^2+16)\cdots(x^2+(2n)^2)}$	$\dfrac{(-1)^n\,2^{2-2n}}{(2n)!}\left(\sum\limits_{s=0}^{n-1}(-1)^s\dbinom{2n}{s}(s-n)g(\alpha-2\theta(s-n))\right),$	$\theta \geq 0,$ $n=1,2,\cdots$	Theorem 4
4.	$\dfrac{g(\alpha-i\theta x)+g(\alpha+i\theta x)}{\left((x^2+1)(x^2+9)...(x^2+(2n+1)^2)\right)}$ $\dfrac{1}{\left((x^2+4)(x^2+16)...(x^2+4m^2)\right)}$	$\dfrac{(-1)^n\pi}{(2^{2m+2n})(m!)^2(2n+1)!}\sum\limits_{s=0}^{n}(-1)^s\dbinom{2n+1}{s}\psi(s)+$ $\dfrac{(-1)^n\pi}{2^{2m+2n-1}(2m)!(2n+1)!}\left(\sum\limits_{k=0}^{m-1}\sum\limits_{s=0}^{n}(-1)^{m+k+s}\dbinom{2m}{k}\dbinom{2n+1}{s}\right.$ $\left.\dfrac{(2n+1-2s)((2m-2k)\,\phi(k)-(2n+1-2s)\psi(s))}{\left((2m-2k)^2-(2n+1-2s)^2\right)}\right),$ where $\psi(s)=g(\alpha+\theta(2n+1-2s))$ and $\phi(k)=g(\alpha+\theta(2m-2k))$	$\theta \geq 0,$ $n=0,1,2,\cdots$ $m=1,2,\cdots$	Theorem 6

4. Applications and Examples

In this section, we present the results, applications, and observations of the proposed theorems. We also show that the simple cases of the master theorems are identical to the results obtained by Cauchy, as detailed in his memoirs, using Residue Theorem 4. Additionally, some examples on difficult integrals that cannot be treated directly by usual methods are addressed. In this section, we show the applicability of our results in handling such problems.

4.1. Some Remarks on the Theorems

Remark 1. *Letting $\alpha = 0$ and $n = 1$ in Theorem 3, we obtain*

$$\int_0^\infty \frac{f\left(\beta e^{i\theta x}\right) - f\left(\beta e^{-i\theta x}\right)}{i\, x(x^2 + 4)}\, dx = \frac{\pi}{4}\left(f(\beta) - f\left(\beta e^{-2\theta}\right)\right), \tag{44}$$

where $\theta > 0$.

By letting $\frac{x}{2} = y$,

$$\frac{1}{4}\int_0^\infty \frac{f\left(\beta e^{2i\theta y}\right) - f\left(\beta e^{-2i\theta y}\right)}{i\, y(y^2 + 1)}\, dy = \frac{\pi}{4}\left(f(\beta) - f\left(\beta e^{-2\theta}\right)\right).$$

Letting $2\theta = \varphi$

$$\int_0^\infty \frac{f\left(\beta e^{i\varphi y}\right) - f\left(\beta e^{-i\varphi y}\right)}{i\, y(y^2 + 1)}\, dy = \pi\left(f(\beta) - f\left(\beta e^{-\varphi}\right)\right).$$

This result appears in [10] (Theorem 4). Further, we show that Cauchy made a mistake in this result (see [4]) (P. 62 formula (10)).

The following table, Table 2 presents some remarks on improper integrals.

Table 2. Remarks on improper integrals, where $\theta > 0$.

	Conditions	Theorem	$g(x)$	$\int\limits_0^\infty g(x)\,dx$	Remarks
1	$\alpha = 0, \beta = 1$ and $n = 0$	1	$\frac{f(e^{i\theta x})+f(e^{-i\theta x})}{(1+x^2)}$	$\pi f\left(e^{-\theta}\right)$	Cauchy's theorem [4] (p. 62 Formula (8)) and in [10] (3.037 Theorem 1) is identical.
2	$\alpha = 0, \beta = 1$ and $n = 0$	2	$\frac{x\left(f(e^{i\theta x})-f(e^{-i\theta x})\right)}{i(1+x^2)}$	$\pi\left(f\left(e^{-\theta}\right)-f(0)\right)$	Cauchy made a mistake in this result see [4] (p. 62 Formula (8)). He corrected his result in his next memoir see [5,6].
3	$n = 1$	1	$\frac{f(\alpha+\beta e^{i\theta x})+f(\alpha+\beta e^{-i\theta x})}{(x^2+1^2)(x^2+3^2)}$	$\frac{\pi}{4!}\left(3f\left(\alpha+\beta e^{-\theta}\right)-f\left(\alpha+\beta e^{-3\theta}\right)\right)$	This result does not appear in [4,5,10].
4	$n = 1$	1	$\frac{x\left(f(\alpha+\beta e^{i\theta x})-f(\alpha+\beta e^{-i\theta x})\right)}{i\,(x^2+1^2)(x^2+3^2)}$	$\frac{\pi}{8}\left(f\left(\alpha+\beta e^{-\theta}\right)-f\left(\alpha+\beta e^{-3\theta}\right)\right)$	This result does not appear in [4,5,10].
5	$n = 2$	3	$\frac{f(\alpha+\beta e^{i\theta x})-f(\alpha+\beta e^{-i\theta x})}{i\,x\,(x^2+2^2)(x^2+4^2)}$	$\frac{\pi}{192}\left(3(f(\alpha+\beta))+f\left(\alpha+\beta e^{-4\theta}\right)-4f\left(\alpha+\beta e^{-2\theta}\right)\right)$	This result does not appear in [4,5,10].
6	$n = 1$	4	$\frac{f(\alpha+\beta e^{i\theta x})+f(\alpha+\beta e^{-i\theta x})}{(x^2+2^2)(x^2+4^2)}$	$\frac{\pi}{48}\left(2f\left(\alpha+\beta e^{-2\theta}\right)-f\left(\alpha+\beta e^{-4\theta}\right)\right)$	This result does not appear in [4,5,10].

4.2. Generating Improper Integrals

In this section, we show the mechanism of generating an infinite number of integrals by choosing the function $f(z)$ and finding the real or imaginary part. It is worth noting that some of these integrals with special cases appear in [40–43] when solving some applications related to finding Green's function, one-dimensional vibrating string problems, wave motion in elastic solids, and when using Fourier cosine and Fourier Sine transforms.

To illustrate the idea, we show some general examples that are applied on Theorems 1, 2, and 3, as follows:

1. Setting $f(z) = z^m$, $m \in \mathbb{R}^+$:

    *   Using Theorem (1) and setting $\alpha = 0$ and $\beta = 1$ we have:

$$f\left(e^{i\theta x}\right) + f\left(e^{-i\theta x}\right) = e^{i\theta mx} + e^{-i\theta mx} = 2\cos(\theta \, mx).$$

Thus,

$$\int_0^\infty \frac{2\cos(\theta \, mx)}{(x^2+1)(x^2+9)\cdots\left(x^2+(2n+1)^2\right)} dx = \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n}}\sum_{s=0}^n (-1)^s \binom{2n+1}{s} e^{m\theta(2s-2n-1)}.$$

where $\theta \geq 0$ and $n = 0, 1, 2, \cdots$

Setting $m = 1$, the obtained integral is a Fourier cosine transform [40,41] of the function $f(t) = \frac{1}{(t^2+1)(t^2+9)\cdots\left(t^2+(2n+1)^2\right)}$.

    *   Using Theorem (3), and setting $\alpha = 0$, $\beta = 1$ we have:

$$\frac{1}{i}\left(f\left(e^{i\theta x}\right) - f\left(e^{-i\theta x}\right)\right) = \frac{1}{i}\left(e^{i\theta mx} + e^{-i\theta mx}\right) = 2\sin(\theta \, mx).$$

Thus,

$$\int_0^\infty \frac{2\sin(\theta \, mx)}{x\left(x^2+4\right)(x^2+16)\dots\left(x^2+(2n)^2\right)} dx = \frac{(-1)^n}{(2n)!}\frac{\pi}{2^{2n}}\left((-1)^n\binom{2n}{n} + 2\sum_{s=0}^{n-1}(-1)^s\binom{2n}{s}e^{2\theta m(s-n)}\right).$$

Setting $m = 1$, the obtained integral is a Fourier sine transform [40,41] of the function $f(t) = \frac{1}{t\left(t^2+4\right)(t^2+16)\dots\left(t^2+(2n)^2\right)}$.

2. Setting $f(z) = e^z$.

    *   Using Theorem (1), we have:

$$f\left(\alpha + \beta e^{i\theta x}\right) + f\left(\alpha + \beta e^{-i\theta x}\right) = e^{\alpha+\beta e^{i\theta x}} + e^{\alpha+\beta e^{-i\theta x}} = 2e^{\alpha+\beta\cos(\theta x)}\cos(\beta\sin(\theta x)).$$

$$\int_0^\infty \frac{2e^{\alpha+\beta\cos(\theta x)}\cos(\beta\sin(\theta x))}{(x^2+1)(x^2+9)\cdots\left(x^2+(2n+1)^2\right)} dx = \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n}}\sum_{s=0}^n (-1)^s\binom{2n+1}{s}e^{\alpha+\beta e^{\theta(2s-2n-1)}},$$

where $\theta \geq 0$ and $n = 0, 1, 2, \cdots$.

    *   Using Theorem (2), we have:

$$\frac{1}{i}\left(f\left(\alpha + \beta e^{i\theta x}\right) - f\left(\alpha + \beta e^{-i\theta x}\right)\right) = \frac{1}{i}\left(e^{\alpha+\beta e^{i\theta x}} - e^{\alpha+\beta e^{-i\theta x}}\right) = 2e^{\alpha+\beta\cos(\theta x)}\sin(\beta\sin(\theta x)).$$

Thus,

$$\int_0^\infty \frac{2xe^{\alpha+\beta\cos(\theta x)}\sin(\beta\sin(\theta x))}{i(x^2+1)(x^2+9)\cdots\left(x^2+(2n+1)^2\right)}dx$$
$$= \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n}}\sum_{s=0}^n (-1)^s\binom{2n+1}{s}(2n-2s+1)\left(e^{\alpha+\beta e^{\theta(2s-2n-1)}}-e^\alpha\right).$$

3. Setting $f(z)=\sinh z$.
    - Using Theorem (1), we have:

    $$f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)=\sinh\left(\alpha+\beta e^{i\theta x}\right)+\sinh\left(\alpha+\beta e^{-i\theta x}\right)$$
    $$= 2\cos(\beta\sin(\theta x))\sinh(\alpha+\beta\cos(\theta x))$$

    Thus,

    $$\int_0^\infty \frac{2\cos(\beta\sin(\theta x))\sinh(\alpha+\beta\cos(\theta x))}{(x^2+1)(x^2+9)\cdots\left(x^2+(2n+1)^2\right)}dx$$
    $$= \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n}}\sum_{s=0}^n (-1)^s\binom{2n+1}{s}\sinh\left(\alpha+\beta e^{\theta(2s-2n-1)}\right)$$

    - Using Theorem (3), we have:

    $$\frac{1}{i}\left(f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)\right)=\frac{1}{i}\left(\sinh\left(\alpha+\beta e^{i\theta x}\right)-\sinh\left(\alpha+\beta e^{-i\theta x}\right)\right)$$
    $$= 2\sin(\beta\sin(\theta x))\cosh(\alpha+\beta\cos(\theta x)).$$

    Thus,

    $$\int_0^\infty \frac{2\sin(\beta\sin(\theta x))\cosh(\alpha+\beta\cos(\theta x))}{x\,(x^2+4)(x^2+16)\dots\left(x^2+(2n)^2\right)}dx$$
    $$= \frac{(-1)^n}{(2n)!}\frac{\pi}{2^{2n}}\left((-1)^n\binom{2n}{n}(\sinh(\alpha+\beta)-\sinh(\alpha))\right.$$
    $$\left.+2\sum_{s=0}^{n-1}(-1)^s\binom{2n}{s}\left(\sinh\left(\alpha+\beta e^{2\theta(s-n)}\right)-\sinh(\alpha)\right)\right),$$

where $\theta>0$, $n=1,2,\cdots$.

4. Setting $f(z)=\cos(e^z)$
    - Using Theorem (1), we have:

    $$f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{-i\theta x}\right)=\cos\left(e^{\alpha+\beta e^{i\theta x}}\right)+\cos\left(e^{\alpha+\beta e^{-i\theta x}}\right)$$
    $$= 2\cos\left(e^{\alpha+\beta\cos(\theta x)}\cos(\beta\sin(\theta x))\right)\cosh\left(\sin(\beta\sin(\theta x))e^{\alpha+\beta\cos(\theta x)}\right).$$

    Thus,

    $$\int_0^\infty \frac{2\cos\left(e^{\alpha+\beta\cos(\theta x)}\cos(\beta\sin(\theta x))\right)\cosh\left(\sin(\beta\sin(\theta x))e^{\alpha+\beta\cos(\theta x)}\right)}{(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)}dx$$
    $$= \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n}}\sum_{s=0}^n (-1)^s\binom{2n+1}{s}\cos\left(\alpha+\beta e^{\theta(2s-2n-1)}\right).$$

5. Setting $f(z)=\ln(1+z)$,
    - Using Theorem (1), we have:

    $$f\left(1+\alpha+\beta e^{i\theta x}\right)+f\left(1+\alpha+\beta e^{-i\theta x}\right)=\ln\left(1+\alpha+\beta e^{i\theta x}\right)+\ln\left(1+\alpha+\beta e^{-i\theta x}\right)$$
    $$= \ln((\alpha+1)^2+\beta^2+2(\alpha+1)\beta\cos(\theta x)).$$

Thus,

$$\int\limits_0^\infty \frac{\ln\left((\alpha+1)^2+\beta^2+2(\alpha+1)\beta\cos(\theta x)\right)}{(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)}\,dx$$
$$=\frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n}}\sum_{s=0}^n (-1)^s \binom{2n+1}{s}\ln\left(1+\alpha+\beta e^{\theta(2s-2n-1)}\right).$$

- Setting $\alpha = 0$ and $\beta = 1$, we have:

$$f\left(e^{i\theta x}\right)+f\left(e^{-i\theta x}\right)=\ln\left(1+e^{i\theta x}\right)+\ln\left(1+e^{-i\theta x}\right)$$
$$=2\ln\left|2\cos\left(\tfrac{\theta x}{2}\right)\right|.$$

Thus,

$$\int\limits_0^\infty \frac{2\ln\left|2\cos\left(\tfrac{\theta x}{2}\right)\right|}{(x^2+1)(x^2+9)\cdots(x^2+(2n+1)^2)}\,dx$$
$$=\frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n}}\sum_{s=0}^n (-1)^s \binom{2n+1}{s}\ln\left(1+e^{\theta(2s-2n-1)}\right).$$

4.3. Solving Improper Integrals

In this section, some applications on complicated problems are introduced and solved directly depending on our new theorems. We note that the Mathematica and Maple software cannot solve such examples.

Example 1. Evaluate the following integral:

$$\int\limits_0^\infty \frac{\ln^2\left|\tan\left(\tfrac{\theta x}{2}-\tfrac{\pi}{4}\right)\right|}{(x^2+4)(x^2+16)}\,dx,$$

where $\theta > 0$.

Solution: Using Theorem 1 and setting $\alpha = 0$, $\beta = 1$, and $n = 1$ or using Remark 6 Table 2 and setting $\alpha = 0$ and $\beta = 1$, we set

$$f(z)=\left(\tan^{-1}z\right)^2=\frac{-1}{4}\ln^2\left(\frac{1-iz}{1+iz}\right).$$

Therefore, we have $f\left(e^{i\theta x}\right) = -\frac{1}{4}\ln^2\left(\frac{1-ie^{i\theta x}}{1+ie^{i\theta x}}\right)$, and $f\left(e^{i\theta x}\right)+f\left(e^{-i\theta x}\right)=2\,Ref\left(e^{i\theta x}\right)$. Thus, we obtain

$$\int\limits_0^\infty \frac{f\left(e^{i\theta x}\right)+f\left(e^{-i\theta x}\right)}{(x^2+4)(x^2+16)}\,dx=\frac{-1}{4}\int\limits_0^\infty \frac{\ln^2\left(\frac{1-ie^{i\theta x}}{1+ie^{i\theta x}}\right)+\ln^2\left(\frac{1-ie^{-i\theta x}}{1+ie^{-i\theta x}}\right)}{(x^2+4)(x^2+16)}\,dx$$

$$=\frac{-1}{4}\int\limits_0^\infty \frac{2\,Re\left(\ln^2\left(\frac{1+\sin(\theta x)-i\cos(\theta x)}{1-\sin(\theta x)+i\cos(\theta x)}\right)\right)}{(x^2+4)(x^2+16)}\,dx$$

$$=\frac{-1}{2}\int\limits_0^\infty \frac{Re(\ln(1+\sin(\theta x)-i\cos(\theta x))-\ln(1-\sin(\theta x)+i\cos(\theta x)))^2}{(x^2+4)(x^2+16)}\,dx$$

$$=\frac{-1}{2}\int\limits_0^\infty \frac{Re\left(\pm i\tfrac{\pi}{2}-\ln\left|\tan\left(\tfrac{\theta x}{2}-\tfrac{\pi}{4}\right)\right|\right)^2}{(x^2+4)(x^2+16)}\,dx$$

$$=\frac{-1}{2}\int\limits_0^\infty \frac{Re\left(-\tfrac{\pi^2}{4}+\ln^2\left|\tan\left(\tfrac{\theta x}{2}-\tfrac{\pi}{4}\right)\right|\pm i\pi\ln\left|\tan\left(\tfrac{\theta x}{2}-\tfrac{\pi}{4}\right)\right|\right)}{(x^2+4)(x^2+16)}\,dx$$

$$=\frac{-1}{2}\int\limits_0^\infty \frac{-\tfrac{\pi^2}{4}+\ln^2\left|\tan\left(\tfrac{\theta x}{2}-\tfrac{\pi}{4}\right)\right|}{(x^2+4)(x^2+16)}\,dx=\frac{\pi}{24}\left(3\left(\tan^{-1}e^{-\theta}\right)^2-\left(\tan^{-1}e^{-3\theta}\right)^2\right).$$

$$\therefore \int\limits_0^\infty \frac{\ln^2\left|\tan\left(\tfrac{\theta x}{2}-\tfrac{\pi}{4}\right)\right|}{(x^2+4)(x^2+16)}\,dx=\frac{\pi^3}{384}-\frac{\pi}{48}\left(3\left(\tan^{-1}e^{-\theta}\right)^2-\left(\tan^{-1}e^{-3\theta}\right)^2\right).$$

Example 2. Evaluate the following integral:

$$PV \int_0^\infty \frac{x \, tan(\pi x)}{(x^2 + 2^2)(x^2 + 4^2) \dots \left(x^2 + (2n)^2\right)} dx, \tag{45}$$

where $n = 1, 2, \cdots$.

Solution. Using Theorem 4, let $\alpha = 0$, $\beta = 1$ and $f(z) = ln(1 + z)$.

Therefore, we have

$$f\left(e^{i\theta x}\right) + f\left(e^{-i\theta x}\right) = ln\left(1 + e^{i\theta x}\right) + ln\left(1 + e^{-i\theta x}\right) = ln(2cos(\theta x) + 2)$$
$$= 2ln\left|2cos\left(\frac{\theta x}{2}\right)\right|.$$

Therefore, we have

$$I(\theta) = PV \int_0^\infty \frac{2ln\left|2cos\left(\frac{\theta x}{2}\right)\right|}{(x^2 + 2^2)(x^2 + 4^2) \dots \left(x^2 + (2n)^2\right)} dx = \frac{(-1)^n \pi \, 2^{2-2n}}{(2n)!} \left(\sum_{s=0}^{n-1} (-1)^s \binom{2n}{s} (s - n) ln\left(1 + e^{2\theta(s-n)}\right)\right).$$

Now, taking the derivative of $I(\theta)$ with respect to θ, we obtain

$$\frac{\partial I}{\partial \theta} = PV \int_0^\infty \frac{-x \, tan\left(\frac{\theta x}{2}\right)}{(x^2 + 2^2)(x^2 + 4^2) \cdots (x^2 + (2n)^2)} dx = \frac{(-1)^n \pi \, 2^{2-2n}}{(2n)!} \left(\sum_{s=0}^{n-1} (-1)^s \binom{2n}{s} (s - n) \frac{2(s - n)e^{2\theta(s-n)}}{e^{2\theta(s-n)} + 1}\right).$$

Therefore,

$$PV \int_0^\infty \frac{x \, tan(\pi x)}{(x^2 + 2^2)(x^2 + 4^2) \cdots (x^2 + (2n)^2)} dx = \frac{(-1)^{n+1} \pi \, 2^{2-2n}}{(2n)!} \left(\sum_{s=0}^{n-1} (-1)^s \binom{2n}{s} (s - n) \frac{2(s - n)e^{4\pi(s-n)}}{e^{4\pi(s-n)} + 1}\right)$$

Putting $n = 1$ in Equation (45), we obtain the following integral:

$$PV \int_0^\infty \frac{x \, tan(\pi x)}{x^2 + 4} dx = \pi \frac{e^{-4\pi}}{e^{-4\pi} + 1} = \frac{\pi}{(e^{4\pi} + 1)}.$$

Example 3. Evaluate the following integral:

$$\int_0^\infty \frac{1 + 2cos(\theta x)}{(x^2 + 1)(x^2 + 4) \, (1 + 4cos(\theta x) + 4)} dx,$$

where $\theta \geq 0$.

Solution. Using Theorem 5 and taking $\alpha = 0$ and $\beta = 1$, let $f(z) = \frac{1}{1 + 2e^z}$.

Thus, we have

$$f\left(e^{i\theta x}\right) + f\left(e^{-i\theta x}\right) = \left(\frac{1}{1 + 2e^{i\theta x}} + \frac{1}{1 + 2e^{-i\theta x}}\right) = \frac{2(1 + 2 \, cos(\theta x))}{1 + 4cos(\theta x) + 4}.$$

Therefore, setting $n = 0$ and $m = 1$, in Theorem 5, we obtain

$$\int_0^\infty \frac{1 + 2cos(\theta x)}{(x^2 + 1)(x^2 + 4) \, (1 + 4cos(\theta x) + 4)} dx = \frac{\pi}{12} \left(\frac{2}{1 + 2e^{-\theta}} - \frac{1}{1 + 2e^{-\theta}}\right).$$

5. Conclusions

In this research, we introduce new theorems that simplify calculating improper integrals. These results can establish many instances of formulas of improper integrals and solve them directly without complicated calculations or computer software. We illustrate some remarks that analyze our work.

- The proposed theorems are considered powerful techniques for generating improper integrals and testing the results when using other methods to solve similar examples.
- These theorems can be illustrated in tables of integrations, with different values of functions and generate more results.
- The obtained improper integrals cannot be solved manually (simply) or by computer software such as Mathematica and Maple.

We intend to generalize the proposed theorems and make tables and algorithms to simplify their use during the applications. Additionally, these results can be used to solve differential equations by inverting the integrals into differential equations.

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