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Extended Gevrey Regularity via Weight Matrices

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Abstract: The main aim of this paper is to compare two recent approaches for investigating the interspace between the union of Gevrey spaces $\mathcal{G}_t(U)$ and the space of smooth functions $C^\infty(U)$. The first approach in the style of Komatsu is based on the properties of two parameter sequences $M_p = p^{\tau p^\sigma}$, $\tau > 0, \sigma > 1$. The other one uses weight matrices defined by certain weight functions. We prove the equivalence of the corresponding spaces in the Beurling case by taking projective limits with respect to matrix parameters, while in the Roumieu case we need to consider a larger space than the one obtained as the inductive limit of extended Gevrey classes.

Keywords: ultradifferentiable functions; weight sequences; weight matrices; Gevrey classes

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1. Introduction

Classes of ultradifferentable functions are usually studied in the framework of one of the two most widely used approaches. The first one is based on the properties of the defining sequences M_p, $p \in \mathbf{N}$, which control the derivatives of the functions, [1]. For the same purpose, the other approach uses weights with the certain asymptotical properties [2,3]. The relation between these weights and the so-called associated function (associated with a weight sequence M_p) provides a way to compare the theories of ultradifferentable functions and their dual spaces of ultradistributions. In many situations, these approaches are equivalent. For example, it is proved in [3] that the corresponding classes of functions are equal if the sequence M_p satisfies Komatsu's conditions $(M.1)$, $(M.2)$ and $(M.3)$, see Section 2. These conditions are relaxed in [4] where $(M.3)$ is replaced by

$$(\exists Q \in \mathbf{N}) \quad \liminf_{p\to\infty} \frac{m_{Qp}}{m_p} > 1, \tag{1}$$

with $m_p = M_p / M_{p-1}$.

In this paper, we study the equivalence of the approaches by considering specific sequences which do not satisfy $(M.2)$. To that end, we exploit the powerful technique based on weight matrices introduced in [5]. Broadly speaking, weight matrices are families of sequences. For instance, $\{p!^t\}_{t>0}$ is a weight matrix that consists of Gevrey sequences. More generally, for a given weight function ω (see Section 1.1 for the definition), one can observe matrices of the form $\mathcal{M} = \{M_p^H\}_{H>0}$ where

$$M_p^H = e^{\frac{1}{H}\varphi^*(Hp)}, \quad p \in \mathbf{N}, \tag{2}$$

and φ^* is the Young conjugate of $\varphi(t) = \omega(e^t)$; see (6). This approach is used to prove that the corresponding classes of functions are equivalent to weight matrix classes in certain situations even if $(M.2)$ is violated; see [5,6].

We consider $M_p^{\tau,\sigma}$ sequences with two parameters $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$, cf. [7]. Such sequences do not satisfy $(M.2)$ for any choice of parameters $\tau > 0$ and $\sigma > 1$, hence we cannot use standard arguments (see [1]) to prove that $\mathcal{E}_{\tau,\sigma}(U)$ are ultradifferentiable. However, we can use a modified construction to obtain related ultradifferentiable classes by taking their unions and intersections (inductive and projective limits) with respect to the parameter τ (this follows from Proposition 1 *(iv)*).

Extended Gevrey regularity turned out to be convenient when describing certain aspects of some hyperbolic PDE's. In particular, $\mathcal{E}_{1,2}(U)$ appears in the study of local solvability of strictly hyperbolic PDE's, for which the initial value problem is ill-posed in the Gevrey settings (see [8]). In addition, sequences $M_p^{\tau,\sigma}$ for $1 < \sigma \leq 2$ are recently used in [9] to study the surjectivity of Borel maps for ultraholomorphic classes. For more details, concerning $M_p^{\tau,\sigma}$ and $\mathcal{E}_{\tau,\sigma}(U)$, we refer to [7,10,11].

In this paper, we prove that the extended Gevrey classes are a special case of classes investigated in [5,6] only when considering projective and inductive limits with respect to the (matrix) parameter τ. More precisely, in the Beurling case, we prove the equality of the corresponding spaces, while, in the Roumieu case, the equivalence holds when the corresponding inductive limit is replaced by a larger space of test functions (see (22) and (31)).

We start by proving that the function $T_{\tau,\sigma,h}(k)$ associated with the sequence $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ is equivalent to a weight function in the sense of [4] (see Theorem 1). For that purpose, we need to estimate $T_{\tau,\sigma,h}(k)$. This is done in [10] (Theorem 2.1) by using the properties of the Lambert W function. In Proposition 2 (see also Lemma 2), we use another technique to obtain similar estimates. Consequently, we conclude that $\{M_p^{\tau,\sigma}\}_{\tau>0}$ and $\{e^{\frac{1}{H}\varphi_\sigma^*(Hp)}\}_{H>0}$ are equivalent matrices for a suitable function φ_σ, which implies that the classes of functions given by these matrices coincide.

Although Theorem 1, as the main result of the paper, connects the approach from [7,10,11] with the one given in [5,6], let us mention an important difference between them. In contrast to the usual Carleman classes and the corresponding part in [5,6], in the norm (17), we consider $h^{|\alpha|^\sigma}$, $\sigma > 1$, in the denominator. Thus, the parameter σ plays an important role in our construction which can not be revealed by using the techniques from [5,6]. For example, the spaces $\mathcal{E}_{\tau,\sigma}(U)$ are closed under finite order differentiation for any choice of parameters $\tau > 0$ and $\sigma > 1$. In addition, the parameters h and σ provide a "fine tuning" in the gap between the union of Gevrey spaces and C^∞ (see Proposition 1 *(i)*).

We end this introductory section with a review of some basic notions.

1.1. Basic Notions and Notation

We use the standard notation $\mathbf{N}$, $\mathbf{Z}_+$, $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{C}$, for the sets of nonnegative integers, positive integers, real numbers, positive real numbers, and complex numbers, respectively. The floor function of $x \in \mathbf{R}_+$ is denoted by $\lfloor x \rfloor := \max\{m \in \mathbf{N} : m \leq x\}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$, we write $\partial^\alpha = \partial^{\alpha_1} \dots \partial^{\alpha_d}$ and $|\alpha| = |\alpha_1| + \dots |\alpha_d|$. By $\#A$, we denote the number of elements of the finite set A. We write $\ln_+ x = \max\{0, \ln x\}$, $x > 0$.

An essential role in our analysis is played by the *Lambert W function*, which is defined as the inverse of ze^z, $z \in \mathbf{C}$. By $W(x)$, $x \geq 0$, we denote the restriction of its principal branch, and we review some of its basic properties as follows:

$(W1)$ $W(0) = 0$, $W(e) = 1$, $W(x)$ is continuous, increasing and concave on $[0, \infty)$,

$(W2)$ $W(xe^x) = x$ and $x = W(x)e^{W(x)}$, $x \geq 0$,

$(W3)$ $\ln x - \ln(\ln x) \leq W(x) \leq \ln x - \frac{1}{2}\ln(\ln x)$, $x \geq e$.

Note that $(W2)$ implies

$$W(x \ln x) = \ln x, \quad x > 1. \tag{3}$$

By using $(W3)$, we obtain

$$W(x) \sim \ln x, \quad x \to \infty, \tag{4}$$

and therefore

$$W(Cx) \sim W(x), \quad x \to \infty, \tag{5}$$

for any $C > 0$. We refer to [12] for more details concerning the Lambert function.

A non-negative, continuous, even and increasing function ω defined on $\mathbf{R}$, $\omega(0) = 0$, is called *weight function* (see [4]) if it satisfies the following conditions:

(α) $\omega(2t) = O(\omega(t)), \quad t \to \infty,$

(β) $\omega(t) = O(t), \quad t \to \infty$

(γ) $o(\omega(t)) = \log t, \quad t \to \infty,$

(δ) $\varphi(t) = \omega(e^t),$ is convex.

Young's conjugate of the function φ (defined as above) is given by

$$\varphi^*(k) = \sup_{t>0}(kt - \varphi(t)), \quad k \geq 0. \tag{6}$$

Some classical examples of weight functions are

$$\omega(t) = \ln_+^s |t|, \qquad \omega(t) = \frac{|t|}{\ln^{s-1}(e + |t|)}, \quad s > 1,\ t \in \mathbf{R}. \tag{7}$$

Moreover, $\omega(t) = |t|^s$ is a weight function if and only if $0 < s \leq 1$. Note that, by (4), it follows that $\omega(t) = W(|t|)$ is not a weight function since condition (γ) is not satisfied.

Functions f and g are called equivalent if $f = O(g)$ and $g = O(f)$, and we will write $f \asymp g$. In particular, if ω is a weight function and $\omega_1 \asymp \omega$, then

$$A\varphi^*(y/A) \leq \varphi_1^*(y) \leq B\varphi^*(y/B) \quad y > 0, \tag{8}$$

for some $A, B > 0$, where $\varphi(t) = \omega(e^t)$, $\varphi_1(t) = \omega_1(e^t)$ and φ^*, φ_1^* are their Young conjugates, respectively (see [3]).

Throughout the paper, we assume that $\tau > 0$ and $\sigma > 1$, unless stated otherwise.

2. Preliminaries

In this section, we recall the definitions of weight functions, weight sequences, their associated functions, and classes of ultradifferentiable functions related to the extended Gevrey regularity. We also list their main properties that will be used in Section 3. We proceed with weight sequences introduced in [7].

2.1. Weight Sequences

In the sequel, we consider sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $M_0^{\tau,\sigma} = 1$, $\tau > 0$, $\sigma > 1$. Since $(M_p^{\tau,\sigma})^{1/p} \to \infty$, when $p \to \infty$, such sequences are examples of weight sequences as considered in [6].

Note that

$$M_p^{\tau_1,\sigma_1} \leq M_p^{\tau_2,\sigma_2}, \quad 0 < \tau_1 \leq \tau_2, \quad 1 < \sigma_1 \leq \sigma_2, \quad p \in \mathbf{N}. \tag{9}$$

Moreover, $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$ ($M_0^{\tau,\sigma} = 1$), satisfies the following conditions (see [7] for the proof):

$(M.1)$ $(M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma} M_{p+1}^{\tau,\sigma}$, $p \in \mathbf{Z}_+$,

$\widetilde{(M.2)}'$ $(\exists C > 0)$ $\quad M_{p+1}^{\tau,\sigma} \leq C^{p^\sigma} M_p^{\tau,\sigma}$, $p \in \mathbf{N}$,

$\widetilde{(M.2)}$ $(\exists C>0)\quad M_{p+q}^{\tau,\sigma}\le C^{p^\sigma+q^\sigma}M_p^{2^{\sigma-1}\tau,\sigma}M_q^{2^{\sigma-1}\tau,\sigma},\quad p,q\in\mathbf{N},$

$(M.3)'\ \sum_{p=1}^{\infty}\frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}}<\infty,$

$\widetilde{(M.4)}$ $(\forall h>0)\,(\exists C>0)\quad M_p^{\tau_1,\sigma}\le C\,h^{p^\sigma}M_p^{\tau_2,\sigma},\quad 0<\tau_1<\tau_2,\quad \sigma>1,$

$\widetilde{(M.5)}$ $(\forall h>0)\,(\exists C>0)\quad M_p^{\tau_1,\sigma_1}\le Ch^{p^{\sigma_2}}M_p^{\tau_2,\sigma_2},\ \tau_1,\tau_2>0,\ 1<\sigma_1<\sigma_2.$

Note that $\widetilde{(M.4)}$ implies

$$Ch^{p^\sigma}M_p^{\tau,\sigma}\ge M_p^{\tau/2,\sigma}\ge 1,\quad C,h>0,\quad p\in\mathbf{N},$$

and hence we obtain the weaker inequality

$\widetilde{(M.4)}'$ $(\forall h>0)\,(\exists C>0)\quad h^{p^\sigma}M_p^{\tau,\sigma}\ge C,\quad p\in\mathbf{N}.$

Remark 1. *Let us briefly comment on the case $\sigma=1$. Then, the conditions $\widetilde{(M.2)}'$ and $\widetilde{(M.2)}$ are $(M.2)'$ and $(M.2)$ (respectively) of Komatsu (see [1]) for the Gevrey sequence $M_p=p!^\tau$. Moreover, $\widetilde{(M.4)}$ also holds. The theory of Gevrey functions is a classical one (see [13,14] and references therein), hence we are interested in the case $\sigma>1$.*

Note that $\widetilde{(M.5)}$ is also true for the case $\sigma_2>\sigma_1=1$ (see [7]).

A family of weight sequences $\mathcal{M}$ is called *weight matrix* (see [6]) if

$$(\forall M_p,N_p\in\mathcal{M})\quad M_p\le N_p\ \vee\ N_p\le M_p,\quad p\in\mathbf{N}. \tag{10}$$

Example 1. *For fixed $\sigma>1$ and $\sigma_2=\sigma_1=\sigma$, (9) implies that $\mathcal{M}_\sigma=\{M_p^{\tau,\sigma}\}_{\tau>0}$ is a weight matrix. Similarly, $\mathcal{M}_\tau=\{M_p^{\tau,\sigma}\}_{\sigma>1}$ is a weight matrix for any given $\tau>0$. Nevertheless, if we observe $\mathcal{M}=\{M_p^{\tau,\sigma}\}_{\tau>0,\sigma>1}$, then, for $\tau_1>\tau_2$ and $\sigma_1<\sigma_2$, we can only prove that*

$$M_p^{\tau_1,\sigma_1}\le CM_p^{\tau_2,\sigma_2},\quad p\in\mathbf{N},$$

for a large positive constant C (see $\widetilde{(M.5)}$). Thus, $\mathcal{M}=\{M_p^{\tau,\sigma}\}_{\tau>0,\sigma>1}$ does not satisfy (10).

For two weight matrices $\mathcal{M}$ and $\mathcal{N}$, we write $\mathcal{M}\precsim\mathcal{N}$ if

$$(\forall M_p\in\mathcal{M})\,(\exists N_p\in\mathcal{N})\,(\exists C>0)\quad M_p\le C^pN_p,\quad p\in\mathbf{N}.$$

We say that $\mathcal{M}$ and $\mathcal{N}$ are *equivalent* if $\mathcal{M}\precsim\mathcal{N}$ and $\mathcal{N}\precsim\mathcal{M}$ (see [6]).

Remark 2. *Let ω be a weight function and ω_1 equivalent to ω. Notice that $\{M_p^H=e^{\frac{1}{H}\varphi^*(Hp)}\}_{H>0}$ and $\{M_p^{H_1}=e^{\frac{1}{H_1}\varphi_1^*(H_1p)}\}_{H_1>0}$ are equivalent matrices due to (8).*

Put

$$m_p^{\tau,\sigma}=\frac{M_p^{\tau,\sigma}}{M_{p-1}^{\tau,\sigma}},\qquad p\in\mathbf{Z}_+. \tag{11}$$

By $(M.1)$, it follows that $m_p^{\tau,\sigma}$ is an increasing sequence. Moreover, the following Lemma holds.

Lemma 1. *Let $M_p^{\tau,\sigma}=p^{\tau p^\sigma}$, $M_0^{\tau,\sigma}=1$, $\tau>0$, $\sigma>1$, and let $m_p^{\tau,\sigma}$ be given by (11). Then, there exist constants $C_1,C_2>0$ depending on τ and σ such that*

$$C_2^{p^{\sigma-1}}p^{\frac{\tau\sigma p^{\sigma-1}}{2^{\sigma-1}}}\le m_p^{\tau,\sigma}\le C_1^{p^{\sigma-1}}p^{\tau\sigma p^{\sigma-1}},\quad p\in\mathbf{Z}_+. \tag{12}$$

Proof. In the sequel, we prove (12) for $p \geq 2$, since the case $p = 1$ is obvious (with $C_1 = C_2 = 1$).

Set $f_{\tau,\sigma}(x) = \tau x^\sigma \ln x$, $x > 0$. By the mean value theorem, for every $p \in \mathbf{Z}_+$, there exists θ_p such that

$$f_{\tau,\sigma}(p) - f_{\tau,\sigma}(p-1) = \tau\theta_p^{\sigma-1}\ln(e\theta_p^\sigma), \quad p-1 < \theta_p < p. \tag{13}$$

For $p \geq 2 \iff p/2 \leq p-1$, we obtain

$$\frac{\tau p^{\sigma-1}}{2^{\sigma-1}} \ln\frac{ep^\sigma}{2^\sigma} \leq \tau(p-1)^{\sigma-1}\ln(e(p-1)^\sigma) < \tau\theta_p^{\sigma-1}\ln(e\theta_p^\sigma) < \tau p^{\sigma-1}\ln(ep^\sigma),$$

and by (13) we conclude

$$\frac{\tau p^{\sigma-1}}{2^{\sigma-1}} \ln\frac{ep^\sigma}{2^\sigma} \leq \tau p^\sigma \ln p - \tau(p-1)^\sigma \ln(p-1) \leq \tau p^{\sigma-1}\ln(ep^\sigma), \quad p \geq 2.$$

After taking exponentials, we obtain

$$\left(\frac{e}{2^\sigma}\right)^{\frac{\tau p^{\sigma-1}}{2^{\sigma-1}}} p^{\frac{\tau\sigma p^{\sigma-1}}{2^{\sigma-1}}} \leq m_p^{\tau,\sigma} \leq e^{\tau p^{\sigma-1}} p^{\tau\sigma p^{\sigma-1}}, \quad p \geq 2, \tag{14}$$

which implies (12).
□

Remark 3. *Note that $\widetical{(M.2)}'$ follows from the right-hand side of (14). In particular,*

$$M_p^{\tau,\sigma} \leq e^{\tau p^{\sigma-1}} p^{\tau\sigma p^{\sigma-1}} M_{p-1}^{\tau,\sigma} \leq C^{p^\sigma} M_{p-1}^{\tau,\sigma}, \quad p \in \mathbf{Z}_+.$$

for a sufficiently large $C > 0$ (which depends on τ and σ).

2.2. Associated Function

In this subsection, we recall the definition and some elementary properties of $T_{\tau,\sigma,h}(k)$, $h > 0$, the associated function to the sequence $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ given by

$$T_{\tau,\sigma,h}(k) = \sup_{p\in\mathbf{N}} \ln_+ \frac{h^{p^\sigma} k^p}{M_p^{\tau,\sigma}}, \quad k > 0. \tag{15}$$

We refer to [10] for more details on $T_{\tau,\sigma,h}(k)$. One of the aims of this paper is to prove that $\omega(k) = T_{\tau,\sigma,h}(|k|)$ is equivalent to a weight function, see Theorem 1 *(i)*.

Remark 4. *Consider $1 < \sigma \leq 2$. Then, by $\widetilde{(M.4)}'$ and Example 21 from [4], we obtain*

$$T_{\tau,\sigma,h}(k) \leq A \sup_{p\in\mathbf{N}} \ln_+ \frac{k^p}{e^{p^\sigma}} + B \leq A_1 \ln_+^{\frac{\sigma}{\sigma-1}} k + B_1 \quad k > 0,$$

for suitable $A_1 > 0$ and $B_1 \in \mathbf{R}$ (depending on τ, σ, h). Hence, we conclude that $T_{\tau,\sigma,h}(k)$ is dominated by a weight function (see (7)). However, this fact does not imply that $T_{\tau,\sigma,h}(|k|)$ is equivalent to a weight function. We will provide additional arguments in the proof of Theorem 1.

Sharp estimates for $T_{\tau,\sigma,h}(k)$ are given in [10], where it is proved that, for some $A_1, A_2 > 0$ and $B_1, B_2 \in \mathbf{R}$ (depending on τ, σ, h), the following inequalities hold:

$$A_1 W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k)) \ln_+^{\frac{\sigma}{\sigma-1}} k + B_1 \leq T_{\tau,\sigma,h}(k) \leq A_2 W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k)) \ln_+^{\frac{\sigma}{\sigma-1}} k + B_2, \tag{16}$$

where

$$\mathfrak{R}(h,k) := h^{-\frac{\sigma-1}{\tau}} e^{\frac{\sigma-1}{\sigma}} \frac{\sigma-1}{\tau\sigma} \ln(e+k), \quad h,k>0,$$

and W is the principal branch of the Lambert function. Note that (16) holds for any choice of parameters $h>0$, $\tau>0$, and $\sigma>1$.

We write $T_{\tau,\sigma}(k) := T_{\tau,\sigma,1}(k)$ for the function, which is associated with $M_p^{\tau,\sigma}$, in the sense of [1]. We end this subsection with a simple result, which will be used in the sequel.

Lemma 2. *Let $T_{\tau,\sigma,h}(k)$ be given by (15), and let $T_{\tau,\sigma}(k) := T_{\tau,\sigma,1}(k)$. Then, for any given $h>0$ and $\tau_2>\tau>\tau_1>0$ there exist $A,B\in\mathbf{R}$ such that*

$$T_{\tau_2,\sigma}(k)+A \le T_{\tau,\sigma,h}(k) \le T_{\tau_1,\sigma}(k)+B, \quad k>0.$$

Proof. By $\widetilde{(M.4)}$ it follows that there exist $C_1,C_2>0$ such that

$$C_2\frac{k^p}{M_p^{\tau_2,\sigma}} \le \frac{h^{p^\sigma}k^p}{M_p^{\tau,\sigma}} \le C_1\frac{k^p}{M_p^{\tau_1,\sigma}}, \quad k>0, \quad p\in\mathbf{N},$$

and the conclusion follows after taking logarithms and the supremum with respect to $p\in\mathbf{N}$. □

2.3. Extended Gevrey Classes

In this subsection, we recall the definition of extended Gevrey classes and some of their basic properties.

Let U be an open set in $\mathbf{R}^d$ and $K \subset\subset U$ be a regular compact set. We denote by $\mathcal{E}_{\tau,\sigma,h}(K)$ the Banach space of functions $\phi\in C^\infty(K)$ such that

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma,h}(K)} = \sup_{\alpha\in\mathbf{N}^d}\sup_{x\in K}\frac{|\partial^\alpha\phi(x)|}{h^{|\alpha|^\sigma}M_{|\alpha|}^{\tau,\sigma}} < \infty. \tag{17}$$

Note that

$$\mathcal{E}_{\tau_1,\sigma_1,h_1}(K) \hookrightarrow \mathcal{E}_{\tau_2,\sigma_2,h_2}(K), \quad 0<h_1<h_2,\ 0<\tau_1<\tau_2,\ 1<\sigma_1<\sigma_2,$$

where $\hookrightarrow$ denotes a strict and dense inclusion. We define spaces of Roumieu and Beurling type by introducing the following inductive and projective limit topologies (respectively)

$$\mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{K\subset\subset U}\varinjlim_{h\to\infty}\mathcal{E}_{\tau,\sigma,h}(K), \tag{18}$$

$$\mathcal{E}_{(\tau,\sigma)}(U) = \varprojlim_{K\subset\subset U}\varprojlim_{h\to 0}\mathcal{E}_{\tau,\sigma,h}(K). \tag{19}$$

We omit the brackets if we consider either $\{\tau,\sigma\}$ or (τ,σ).

Remark 5. *The condition $(M.3)'$ implies that $\mathcal{E}_{\tau,\sigma}(U)$ contains compactly supported functions. The construction of smooth compactly supported functions, which are not in Gevrey classes but which belong to $\mathcal{E}_{\tau,\sigma}(U)$, can be found in [7].*

Extended Gevrey classes given by (18) and (19) are studied in [7,10,11]. For the convenience of the reader, we collect some of their basic properties in the following Proposition. Recall the Gevrey class of index $t>1$ is given by $\mathcal{G}_t(U)=\mathcal{E}_{\{t,1\}}(U)$, see (18).

Proposition 1. *Let U be an open set in $\mathbf{R}^d$. Let $\mathcal{E}_{\{\tau,\sigma\}}(U)$ and $\mathcal{E}_{(\tau,\sigma)}(U)$ be given by (18) and (19), respectively, and let $\varinjlim$ and $\varprojlim$ denote the corresponding inductive and projective limits, respectively. Then, the following is true:*

(*i*) *For* $\sigma_2 > \sigma_1 > 1$, *we have*

$$\varinjlim_{t\to\infty} \mathcal{G}_t(U) \hookrightarrow \varprojlim_{\tau\to 0} \mathcal{E}_{\{\tau,\sigma_1\}}(U) = \varprojlim_{\tau\to 0} \mathcal{E}_{(\tau,\sigma_1)}(U) \hookrightarrow \varinjlim_{\tau\to\infty} \mathcal{E}_{(\tau,\sigma_1)}(U)$$
$$= \varinjlim_{\tau\to\infty} \mathcal{E}_{\{\tau,\sigma_1\}}(U) \hookrightarrow \varprojlim_{\tau\to 0} \mathcal{E}_{\{\tau,\sigma_2\}}(U) \hookrightarrow C^\infty(U).$$

(*ii*) $\mathcal{E}_{\tau,\sigma}(U)$ *are closed under the pointwise multiplication;*
(*iii*) $\mathcal{E}_{\tau,\sigma}(U)$ *are closed under finite order derivation;*
(*iv*) *For* $a_\alpha \in \mathcal{E}_{(\tau,\sigma)}(U)$ *(resp.* $a_\alpha \in \mathcal{E}_{\{\tau,\sigma\}}(U)$*), define*

$$P(x,\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x)\partial^\alpha,$$

such that, for every $K \subset\subset U$*, there exists* $L > 0$ *and for every* $h > 0$ *there exists* $A > 0$ *(resp. for every* $K \subset\subset U$*, there exists* $h > 0$ *and, for every* $L > 0$*, there exists* $A > 0$*) so that*

$$\sup_{x\in K} |\partial^\beta a_\alpha(x)| \le A h^{|\beta|^\sigma} M^{\tau,\sigma}_{|\beta|} \frac{L^{|\alpha|^\sigma}}{M^{2^{\sigma-1}\tau,\sigma}_{|\alpha|}}, \quad \alpha,\beta \in \mathbf{N}^d.$$

Then, $P(x,\partial) : \mathcal{E}_{(\tau,\sigma)}(U) \to \mathcal{E}_{(2^{\sigma-1}\tau,\sigma)}(U)$ *(resp.* $P(x,\partial) : \mathcal{E}_{\{\tau,\sigma\}}(U) \to \mathcal{E}_{\{2^{\sigma-1}\tau,\sigma\}}(U)$*) is a continuous and linear mapping.*

Let $h = 1$ in (17). We introduce the following spaces:

$$\mathcal{E}_{\{\sigma\}}(U) = \varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma}(U), \quad \mathcal{E}_{(\sigma)}(U) = \varprojlim_{\tau\to 0} \mathcal{E}_{\tau,\sigma}(U), \tag{20}$$

$$\mathcal{E}^R_{\{\sigma\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma,1}(K), \quad \mathcal{E}^B_{(\sigma)}(U) = \varprojlim_{K\subset\subset U} \varprojlim_{\tau\to 0} \mathcal{E}_{\tau,\sigma,1}(K). \tag{21}$$

Remark 6. *By Proposition 1, (i), it follows that the definition of classes (20) does not depend on the choice of the classes (18) and (19). Similar holds for classes in (21).*

Proposition 1 *(i)*, and the order of quantifiers in the definition of spaces (20) and (21) imply the following embeddings

$$\varinjlim_{t\to\infty} \mathcal{G}_t(U) \hookrightarrow \mathcal{E}_{(\sigma)}(U) = \mathcal{E}^B_{(\sigma)}(U) \hookrightarrow \mathcal{E}_{\{\sigma\}}(U) \hookrightarrow \mathcal{E}^R_{\{\sigma\}}(U) \hookrightarrow C^\infty(U). \tag{22}$$

Notice that, unlike $\mathcal{E}_{\tau,\sigma}(U)$, $\mathcal{E}_{\{\sigma\}}(U)$, $\mathcal{E}_{(\sigma)}(U)$, are classes of ultradifferentiable functions. This follows from Proposition 1 *(iv)*. Moreover, the ultradifferentiability of $\mathcal{E}^R_{\{\sigma\}}(U)$ follows from the arguments given in [5].

3. Main Result

In this section, we first give an estimate for $T_{\tau,\sigma}(k) = T_{\tau,\sigma,1}(k)$ which is introduced in Section 2.2. $T_{\tau,\sigma}(k)$ obviously satisfies (16) with $h = 1$. Therefore, the next Proposition follows directly from [10] (Theorem 2.1). However, here we give an independent proof.

Proposition 2. *Let* $T_{\tau,\sigma}(k) = T_{\tau,\sigma,1}(k)$ *be given by (15) with* $h = 1$*, and let* $W(t)$*,* $t > 0$*, denote the restriction of the principal branch of the Lambert W function. If* $\varphi_\sigma(t) = \dfrac{t^{\frac{\sigma}{\sigma-1}}}{W^{\frac{1}{\sigma-1}}(t)}$*,* $t > 0$*, and* $\varphi_\sigma(0) = 0$*, then we have*

$$B_{\tau,\sigma}\varphi_\sigma(\ln_+ k) + \widetilde{B}_{\tau,\sigma} \le T_{\tau,\sigma}(k) \le A_{\tau,\sigma}\varphi_\sigma(\ln_+ k) + \widetilde{A}_{\tau,\sigma}, \quad k > 0, \tag{23}$$

for suitable constants $A_{\tau,\sigma}, B_{\tau,\sigma} > 0$ *and* $\widetilde{A}_{\tau,\sigma}, \widetilde{B}_{\tau,\sigma} \in \mathbf{R}$.

Proof. For $\lambda > 0$, we let

$$m_{\tau,\sigma}(\lambda) = \#\{p \in \mathbf{Z}_+ \mid m_p^{\tau,\sigma} \leq \lambda\},$$

and note that $m_{\tau,\sigma}(\lambda) = 0$ for all $0 < \lambda < 1$. This is due to the fact that $m_1^{\tau,\sigma} = 1$ and $m_p^{\tau,\sigma}$ is increasing.

Since $M_p^{\tau,\sigma}$ satisfies $(M.1)$, we can write (see [1,15])

$$T_{\tau,\sigma}(k) = \int_0^k \frac{m_{\tau,\sigma}(\lambda)}{\lambda} d\lambda = \int_1^k \frac{m_{\tau,\sigma}(\lambda)}{\lambda} d\lambda.$$

In the sequel, we estimate $m_{\tau,\sigma}(\lambda)$ when $\lambda \geq 1$.

Put

$$m^C_{\tau,\sigma}(\lambda) = \#\{p \in \mathbf{Z}_+ \mid C^{p^{\sigma-1}} p^{\tau p^{\sigma-1}} \leq \lambda\}, \quad C > 0.$$

Then, (14) implies that

$$m^{C_1}_{(\tau\sigma),\sigma}(\lambda) \leq m_{\tau,\sigma}(\lambda) \leq m^{C_2}_{(\tau\sigma)/2^{\sigma-1},\sigma}(\lambda), \quad \lambda \geq 1, \tag{24}$$

where C_1 and C_2 are as in (12). In particular,

$$\int_1^k \frac{m^{C_1}_{(\tau\sigma),\sigma}(\lambda)}{\lambda} d\lambda \leq T_{\tau,\sigma}(k) \leq \int_1^k \frac{m^{C_2}_{(\tau\sigma)/2^{\sigma-1},\sigma}(\lambda)}{\lambda} d\lambda. \tag{25}$$

Next, we note that

$$C^{p^{\sigma-1}} p^{\tau p^{\sigma-1}} \leq \lambda \iff C^{\frac{\sigma-1}{\tau}} p^{\sigma-1} \ln(C^{\frac{\sigma-1}{\tau}} p^{\sigma-1}) \leq C^{\frac{\sigma-1}{\tau}} \frac{\sigma-1}{\tau} \ln\lambda \iff$$
$$\ln(C^{\frac{\sigma-1}{\tau}} p^{\sigma-1}) \leq W(C^{\frac{\sigma-1}{\tau}} \frac{\sigma-1}{\tau} \ln\lambda) \iff p \leq C^{-\frac{1}{\tau}} e^{\frac{1}{\sigma-1} W(C^{\frac{\sigma-1}{\tau}} \frac{\sigma-1}{\tau} \ln\lambda)}, \quad C > 0,\ \lambda \geq 1,$$

where, for the second equivalence, we used property (3) of the Lambert function.

This calculation shows that

$$m^C_{\tau,\sigma}(\lambda) = \left\lfloor C^{-\frac{1}{\tau}} e^{\frac{1}{\sigma-1} W(C^{\frac{\sigma-1}{\tau}} \frac{\sigma-1}{\tau} \ln\lambda)} \right\rfloor, \quad \lambda \geq 1, \tag{26}$$

and therefore

$$\int_1^k \frac{m^C_{\tau,\sigma}(\lambda)}{\lambda} d\lambda \asymp C^{-\frac{1}{\tau}} \int_1^k \frac{e^{\frac{1}{\sigma-1} W(C^{\frac{\sigma-1}{\tau}} \frac{\sigma-1}{\tau} \ln\lambda)}}{\lambda} d\lambda.$$

It remains to compute

$$I^C_{\tau,\sigma}(k) := C^{-\frac{1}{\tau}} \int_1^k \frac{e^{\frac{1}{\sigma-1} W(C^{\frac{\sigma-1}{\tau}} \frac{\sigma-1}{\tau} \ln\lambda)}}{\lambda} d\lambda, \quad C > 0.$$

Set $C_{\tau,\sigma} = C^{\frac{\sigma-1}{\tau}} \frac{\sigma-1}{\tau}$. Note that, after the substitution $t = C_{\tau,\sigma} \ln\lambda$, we obtain

$$I^C_{\tau,\sigma}(k) = C^{-\frac{\sigma}{\tau}} \frac{\tau}{\sigma-1} \int_0^{C_{\tau,\sigma}\ln k} e^{\frac{1}{\sigma-1} W(t)} dt. \tag{27}$$

Another change of variables $W(t) = s$ ($t = se^s, dt = (s+1)e^s ds$), and integration by parts yields

$$\int e^{\frac{1}{\sigma-1} W(t)} dt = \int e^{\frac{\sigma s}{\sigma-1}} (s+1) ds = e^{\frac{\sigma s}{\sigma-1}} \frac{\sigma-1}{\sigma} \Big(s + \frac{1}{\sigma}\Big), \tag{28}$$

where we use indefinite integral just for the notational convenience.

Now, using property $(W2)$ of the Lambert function and (5), by (27) and (28), we have

$$I^C_{\tau,\sigma}(k) = \frac{\tau}{\sigma} C^{-\frac{\sigma}{\tau}} e^{\frac{\sigma s}{\sigma-1}} \Big(s + \frac{1}{\sigma}\Big)\Big|_{s=0}^{s=W(t)} = \frac{\tau}{\sigma} C^{-\frac{\sigma}{\tau}} \Big(\frac{t}{W(t)}\Big)^{\frac{\sigma}{\sigma-1}} (W(t) + \frac{1}{\sigma})\Big|_{t=0}^{t=C_{\tau,\sigma}\ln k} \asymp \tau^{-\frac{1}{\sigma-1}} \varphi_\sigma(\ln_+ k) + \widetilde{C}_{\tau,\sigma},\ k > 0, \quad (29)$$

for some $\widetilde{C}_{\tau,\sigma} \in \mathbf{R}$, where the hidden constants depend only on σ.

More precisely, using (25) and (29), we conclude that

$$B_\sigma \tau^{-\frac{1}{\sigma-1}} \varphi_\sigma(\ln_+ k) + \widetilde{B}_{\tau,\sigma} \le T_{\tau,\sigma}(k) \le A_\sigma \Big(\frac{\tau}{2^{\sigma-1}}\Big)^{-\frac{1}{\sigma-1}} \varphi_\sigma(\ln_+ k) + \widetilde{A}_{\tau,\sigma},\ k > 0, \quad (30)$$

for suitable $A_\sigma, B_\sigma > 0$ and $\widetilde{A}_{\tau,\sigma}, \widetilde{B}_{\tau,\sigma} \in \mathbf{R}$. This completes the proof. $\square$

Following [5,6], we introduce the Banach space $\mathcal{B}_{H,\sigma}(K)$, $K \subset\subset U$, with the norm

$$\|\phi\|_{\mathcal{B}_{H,\sigma}(K)} = \sup_{\alpha\in\mathbf{N}^d} \sup_{x\in K} |\partial^\alpha \phi(x)| e^{-\frac{1}{H}\varphi^*_\sigma(H|\alpha|)}, \quad H > 0,$$

where φ^*_σ is Young's conjugate of the function φ_σ introduced in Proposition 2.

We introduce the corresponding Roumieu and Beurling classes as

$$\mathcal{B}_{\{\sigma\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{H\to\infty} \mathcal{B}_{H,\sigma}(U) \quad \text{and} \quad \mathcal{B}_{(\sigma)}(U) = \varprojlim_{K\subset\subset U} \varprojlim_{H\to 0} \mathcal{B}_{H,\sigma}(U),$$

respectively.

Now, we can formulate the main result of the paper.

Theorem 1. *Fix $\sigma > 1$ and let $\varphi_\sigma(t)$ be as in Proposition 2. Moreover, let $T_{\tau,\sigma,h}$ be given by (15). Then, the following is true.*

(i) The function $\omega(k) = \varphi_\sigma(\ln_+ |k|)$ is equivalent to a weight function. Moreover, for every $h > 0$ and $\tau > 0$, the function $\omega_{\tau,h}(k) = T_{\tau,\sigma,h}(|k|)$ is equivalent to a weight function.

(ii) The weight matrices $\mathcal{M}_\sigma = \{p^{\tau p^\sigma}\}_{\tau>0}$ and $\mathcal{N}_\sigma = \{e^{\frac{1}{H}\varphi^*_\sigma(Hp)}\}_{H>0}$ are equivalent. In particular,

$$\mathcal{B}_{(\sigma)}(U) = \mathcal{E}^B_{(\sigma)}(U), \quad \mathcal{B}_{\{\sigma\}}(U) = \mathcal{E}^R_{\{\sigma\}}(U) \quad (31)$$

as locally covex vector spaces. Here, $\mathcal{E}^B_{(\sigma)}(U)$ and $\mathcal{E}^R_{\{\sigma\}}(U)$ are given in (21).

Proof. *(i)* By Proposition 2, it follows that $T_{\tau,\sigma}(|k|) \asymp \varphi_\sigma(\ln_+ |k|)$. Thus, it is sufficient to show that $T_{\tau,\sigma}(|k|)$ is a weight function (see Remark 2).

Since $T_{\tau,\sigma}$ is the function associated with $M^{\tau,\sigma}_p$ in the sense of [1], by [4] (Lemma 12) (see also Remark 7), it is sufficient to show that $m^{\tau,\sigma}_p$ given by (11) satisfies (1), i.e., that there exists $Q \in \mathbf{N}$ such that

$$\liminf_{p\to\infty} \frac{m^{\tau,\sigma}_{Qp}}{m^{\tau,\sigma}_p} > 1. \quad (32)$$

Note that inequalities in (12) imply

$$\frac{m^{\tau,\sigma}_{3p}}{m^{\tau,\sigma}_p} \ge \frac{\Big(\frac{e}{2^\sigma}\Big)^{\frac{\tau(3p)^{\sigma-1}}{2^{\sigma-1}}} (3p)^{\frac{\tau\sigma(3p)^{\sigma-1}}{2^{\sigma-1}}}}{e^{\tau p^{\sigma-1}} p^{\tau\sigma p^{\sigma-1}}} = \Big(\frac{3}{2}\Big)^{\tau\sigma(\frac{3p}{2})^{\sigma-1}} \exp\Big\{\tau\Big((3/2)^{\sigma-1} - 1\Big)p^{\sigma-1}\Big\} p^{\tau\sigma\Big((3/2)^{\sigma-1}-1\Big)p^{\sigma-1}} \to \infty,\ p \to \infty,$$

and (32) follows when $Q = 3$. In addition, Lemma 2 together with Proposition 2 implies that $T_{\tau,\sigma,h}(|k|) \asymp \varphi_\sigma(\ln_+ |k|)$ for all $\tau, h > 0$.

(ii) Note that [1] (Proposition 3.2.) and (23) imply

$$C_2 \exp\{A_{\tau,\sigma}\varphi^*_\sigma(\frac{1}{A_{\tau,\sigma}} p)\} \le p^{\tau p^\sigma} \le C_1 \exp\{B_{\tau,\sigma}\varphi^*_\sigma(\frac{1}{B_{\tau,\sigma}} p)\}, \quad p \in \mathbf{N},$$

for suitable $C_1, C_2 > 0$ (depending on τ and σ). More precisely, if we set

$$H_1 = B^{-1}\tau^{\frac{1}{\sigma-1}}, \quad H_2 = A^{-1}(\tau/2^{\sigma-1})^{\frac{1}{\sigma-1}}, \quad A, B > 0,$$

then (30) implies that

$$C_2' \exp\{\frac{1}{H_2}\varphi^*_\sigma(H_2\, p)\} \le p^{\tau p^\sigma} \le C_1' \exp\{\frac{1}{H_1}\varphi^*_\sigma(H_1\, p)\}, \quad p \in \mathbf{N}. \tag{33}$$

for suitable constants $C_1', C_2' > 0$. Therefore, the matrices $\mathcal{M}_\sigma$ and $\mathcal{N}_\sigma$ are equivalent.

It remains to prove (31). We give the proof for the Roumieu case $\mathcal{B}_{\{\sigma\}}(U) = \mathcal{E}^R_{\{\sigma\}}(U)$, and omit the proof for the Beurling case, since it uses similar arguments.

Let $\phi \in \mathcal{B}_{\{\sigma\}}(U)$. Then, for arbitrary $K \subset\subset U$, there exists $H > 0$ such that $\|\phi\|_{\mathcal{B}_{H,\sigma}(K)} < \infty$. Putting $\tau = (2AH)^{\sigma-1}$, (33) implies

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma,1}(K)} \le C'\|\phi\|_{\mathcal{B}_{H,\sigma}(K)},$$

for some $C' > 0$.

Conversely, if $\phi \in \mathcal{E}^R_{\{\sigma\}}(U)$, then, for arbitrary $K \subset\subset U$, there exists $\tau > 0$ such that $\|\phi\|_{\mathcal{E}_{\tau,\sigma,1}(K)} < \infty$. Choosing $H = B^{-1}\tau^{\frac{1}{\sigma-1}}$, again, by (33), we have

$$\|\phi\|_{\mathcal{B}_{H,\sigma}(K)} \le C''\|\phi\|_{\mathcal{E}_{\tau,\sigma,1}(K)},$$

for suitable $C'' > 0$. This completes the proof. □

Remark 7. *Note that $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ is not a weight sequence in the sense of [4], since it does not satisfy $(M.2)'$. Instead, we used in Proposition 2 estimate (14), which implies $\widetilde{(M.2)}'$ by Remark 3.*

Moreover, in the proof of Theorem 1, we use the part of [4] (Lemma 12) for which it is sufficient to assume $(M.1)$,

$$(M.0) \qquad (\exists C > 0) \quad M_p^{\tau,\sigma} \ge Cp^p, \quad p \in \mathbf{N},$$

(which obviously holds by $\widetilde{(M.4)}'$) and $m_{\tau,\sigma}(\lambda) \to \infty$, $\lambda \to \infty$, which is true by (24) and (26).

We conclude the paper with the following Corollary, which is an immediate consequence of Theorem 1.

Corollary 1. *For each $s > 1$, the function $\omega(t) = \dfrac{\ln_+^s |t|}{\ln^{s-1}(\ln(e + |t|))}$, $t \neq 0$, $\omega(0) = 0$ is equivalent to a weight function.*

4. Discussion

The equivalence of theories of ultradifferentiable functions given by Komatsu's or the Braun–Meise–Taylor approach are well established in most classical situations. A recent approach based on weighted matrices seems to offer a very general construction, see [5,6]. In parallel, it is demonstrated in [7,10,11] that the two-parameter sequences of the form $M_p = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$, provide a useful extension of the Gevrey type spaces.

In this paper, we show that the projective limits of extended Gevrey classes can be viewed as a part of the construction based on the weight matrices. The same conclusion holds when the inductive limits of extended Gevrey classes are replaced by certain slightly larger spaces. At the same time, extended Gevrey classes $\mathcal{E}_{\tau,\sigma}(U)$ for fixed $\tau > 0$ and $\sigma > 1$, can not be characterized by weight matrices used in [5,6] due to the particular role played by the parameter σ.

While finishing the paper, the authors learned about the work in progress "A comparison of two ways to generalize ultradifferentiable classes defined by weight sequences" by J. Jiménez-Garrido, D. N. Nenning, and G. Schindl, which is devoted to a similar topic considered from a different point of view. We thank the authors for their fruitful comments on the first version of this paper.

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