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Modelling Coronavirus and Larvae Pyrausta Data: A Discrete Binomial Exponential II Distribution with Properties, Classical and Bayesian Estimation

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Abstract: In this article, we propose the discrete version of the binomial exponential II distribution for modelling count data. Some of its statistical properties including hazard rate function, mode, moments, skewness, kurtosis, and index of dispersion are derived. The shape of the failure rate function is increasing. Moreover, the proposed model is appropriate for modelling equi-, over- and under-dispersed data. The parameter estimation through the classical point of view has been done using the method of maximum likelihood, whereas, in the Bayesian framework, assuming independent beta priors of model parameters, the Metropolis–Hastings algorithm within Gibbs sampler is used to obtain sample-based Bayes estimates of the unknown parameters of the proposed model. A detailed simulation study is carried out to examine the outcomes of maximum likelihood and Bayesian estimators. Finally, two distinctive real data sets are analyzed using the proposed model. These applications showed the flexibility of the new distribution.

Keywords: probability mass function; binomial exponential II; dispersion index; Bayesian technique; simulation

MSC: 60E05; 62E10; 62F10; 62N05

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1. Introduction

Fitting a probability distribution to real data and synthesizing information from it is a challenging task for statisticians/researchers. Data generated from day-to-day work environments are more complex in nature nowadays, and consequently several lifetime models have been proposed and studied in the literature to analyze these data. The well-known exponential distribution is one of the basic continuous models used to examine continuous data. However, Bakouch et al. [1] developed the binomial exponential II (BiExII) distribution, an extended variant of the ordinary exponential distribution, to provide additional flexibility. The BiExII model is constructed as a distribution of a random sum of independent exponential (Ex) random variables when the sample size has a zero-truncated binomial (Bi) distribution. The cumulative distribution function (CDF) of the BiExII model can be written as

$$G(y;\lambda,\theta) = 1 - \left(1 + \frac{\lambda\theta y}{2-\theta}\right)e^{-\lambda y};\ y > 0, \tag{1}$$

where $0 \leq \theta \leq 1$ is the shape parameter and $\lambda > 0$ is the scale parameter. The probability density function (PDF) corresponding to Equation (1) can be expressed as

$$g(y; \lambda, \theta) = \lambda \left(1 + \frac{(\lambda y - 1)\theta}{2 - \theta}\right) e^{-\lambda y};\ y > 0. \quad (2)$$

As we observe from Equation (1), the Ex distribution is a particular case for $\theta = 0$, whereas for $\theta = 1$, the gamma model with shape parameter 2 and scale parameter λ is a special case. Thus, Equation (2) can be written as

$$g(y; \lambda, \theta) = w\lambda e^{-\lambda y} + (1 - w)\lambda^2 y e^{-\lambda y};\ y > 0, \quad (3)$$

where $w = \frac{2(1-\theta)}{2-\theta}$. Habibi and Asgharzadeh [2] presented a power binomial exponential distribution by applying the power transformation on BiExII random variable. The hazard rate function of the proposed distribution portrays the decreasing, increasing, decreasing-increasing-decreasing and unimodal shapes. Al-babtain et al. [3] developed a new extension of the BiExII model using the Marshall–Olkin (MO-G) family of distributions. They have also discussed a simple type Copula-based construction to derive the bivariate- and multivariate-type distributions. Recently, Zhang et al. [4] first reviewed the two-parameter Poisson binomial-exponential 2 (PBE2) distribution, then they proposed a new integer-valued auto-regressive (INAR) model with PBE2 innovations.

Sometimes reliability/survival experiments yield data which are discrete in nature either due to limitations of measuring instruments or its inherent characteristic. For example, in reliability engineering, the number of successful cycle prior to the failure when device work in cycle, the number of times a device is switched on/off; in survival analysis, the survival times for those suffering from the diseases such as lung cancer or period from remission to relapse may be recorded as number of days/weeks, number of deaths/daily cases due to COVID-19 pandemic observed over a specified duration, etc. Moreover, in many practical problems, the count phenomenon occurs as, for example, the number of occurrences of earthquakes in a calendar year, the number of absences, the number of accidents, the number of kinds of species in ecology, the number of insurance claims, and so on. Therefore, it is reasonable to model such situations by a suitable discrete distribution.

Discretization of continuous models can be done by utilizing various techniques. The most widely used approach is the survival discretization method. One of the important virtues of this methodology is that the developed discrete distribution retains the same functional form of the survival function as that of its continuous counterpart. Due to this feature many reliability characteristics of the distribution remain unchanged. According to this method, for a given continuous random variable (RV) Y with survival function (SF) $S_Y(y) = P(Y \geq y)$, the RV $X = [Y]$ (largest integer less than or equal to Y) will have the probability mass function

$$P(X = x) = S_Y(x) - S_Y(x + 1);\ \ x = 0, 1, 2, 3.... \quad (4)$$

Many authors have used Equation (4) for generating the discrete analogue of the continuous distributions, for instance, discrete Rayleigh distribution (Roy [5]), discrete Burr and Pareto distributions (Krishna and Pundir [6]), discrete gamma distribution (Chakraborty and Chakravarty [7]), discrete modified Weibull distribution (Almalki and Nadarajah [8]), discrete generalized exponential and exponentiated discrete Weibull distributions (Nekoukhou and Bidram [9,10]), discrete extended Weibull distribution (Jia et al. [11]), geometric-zero truncated Poisson distribution (Akdogan et al. [12]), Poisson quasi-Lindley regression model and Poisson–Bilal distribution (Altun [13,14]), discrete Burr–Hatke distribution (El-Morshedy et al. [15]), discrete inverted Nadarajah–Haghighi distributions (Singh et al. [16]), discrete Teissier distribution (Singh et al. [17]), and related references cited therein.

In view of the existing literature, we found that several discrete distributions have been introduced over the past few decades. Yet there is much scope left to introduce new plausible discrete distributions that can adequately capture the diversity of real data. This phenomenon motivates us to provide a flexible discrete model for fitting a wide spectrum of discrete real-world data sets. Therefore, in this paper, we have proposed the discrete analogue of the BiExII model, in the so-called discrete BiExII (DBiExII) distribution using survival discretization method. An important motivation of the proposed study is that the BiExII distribution has manageable and closed-form expressions for various important distributional properties, including probability mass function, cumulative distribution function, moments, etc. Furthermore, discrete data generated from many practical studies, such as mortality experiments, industrial experiments, etc., show constant or increasing failure rates, so the proposed distribution is useful for modelling monotonically increasing failure rate data. Other motivations for developing the BiExII distribution include its ability to analyze not only equi-, over-, and under-dispersed real data, but also a positively skewed, or leptokurtic data set. A final motivation for the new distribution is that the proposed distribution is capable of modelling count data as we will see later, and by this, it provides a well alternative to several discrete distributions for modelling discrete data in applications.

The rest of the article is organized as follows. In Section 2, we have introduced the DBiExII model. Different distributional characteristics are discussed in Section 3. In Section 4, the model parameters are estimated by using maximum likelihood and Bayesian methods. Simulation study is presented in Section 5. The two real data sets (COVID-19 and larvae Pyrausta) are analyzed to show the flexibility of the DBiExII distribution in Section 6. Finally, Section 7 provides some conclusions.

2. The DBiExII Distribution

Using the Equation (4), the probability mass function (PMF) of the DBiExII distribution with positive parameters $0 < p < 1$ and $0 \leq \theta \leq 1$, can be derived as

$$f(x;p,\theta) = \left(1 - p + \frac{\theta(px + p - x)\ln p}{2-\theta}\right)p^x;\ x \in \mathbb{N}_0, \tag{5}$$

where $p = e^{-\lambda}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. The cumulative distribution function (CDF) corresponding to Equation (5) can be expressed as

$$F(x;p,\theta) = f(X \leq x;p,\theta) = \sum_{i=0}^{x} f(i;p,\theta) = 1 - \left(1 - \frac{\theta(x+1)\ln p}{2-\theta}\right)p^{x+1};\ x \in \mathbb{N}_0. \tag{6}$$

The behavior of the CDF of the DBiExII distribution can be described as

$$F(x;p,\theta) = \begin{cases} \frac{-p\theta(\ln p + 1) + 2p + \theta - 2}{\theta - 2}; & x \longrightarrow 0, \\ 1 - p^{x+1}; & \theta \longrightarrow 0, \\ p^{x+1}(x\ln p + \ln p - 1) + 1; & \theta \longrightarrow 1. \end{cases} \tag{7}$$

The behavior of the PMF is given by

$$f(x;p,\theta) = \begin{cases} \frac{-p\theta(\ln p + 1) + 2p + \theta - 2}{\theta - 2}; & x \longrightarrow 0, \\ p^x(1-p); & \theta \longrightarrow 0, \\ p^x[(px + p - x)\ln p - p + 1]; & \theta \longrightarrow 1. \end{cases} \tag{8}$$

The PMF in Equation (5) is log-concave, where $\frac{f(x+1;p,\theta)}{f(x;p,\theta)}$ is a decreasing function in x for all values of the model parameters, and consequently the PMF is unimodal and right-skewed. Figure 1 shows the PMF plots for different values of the parameters.

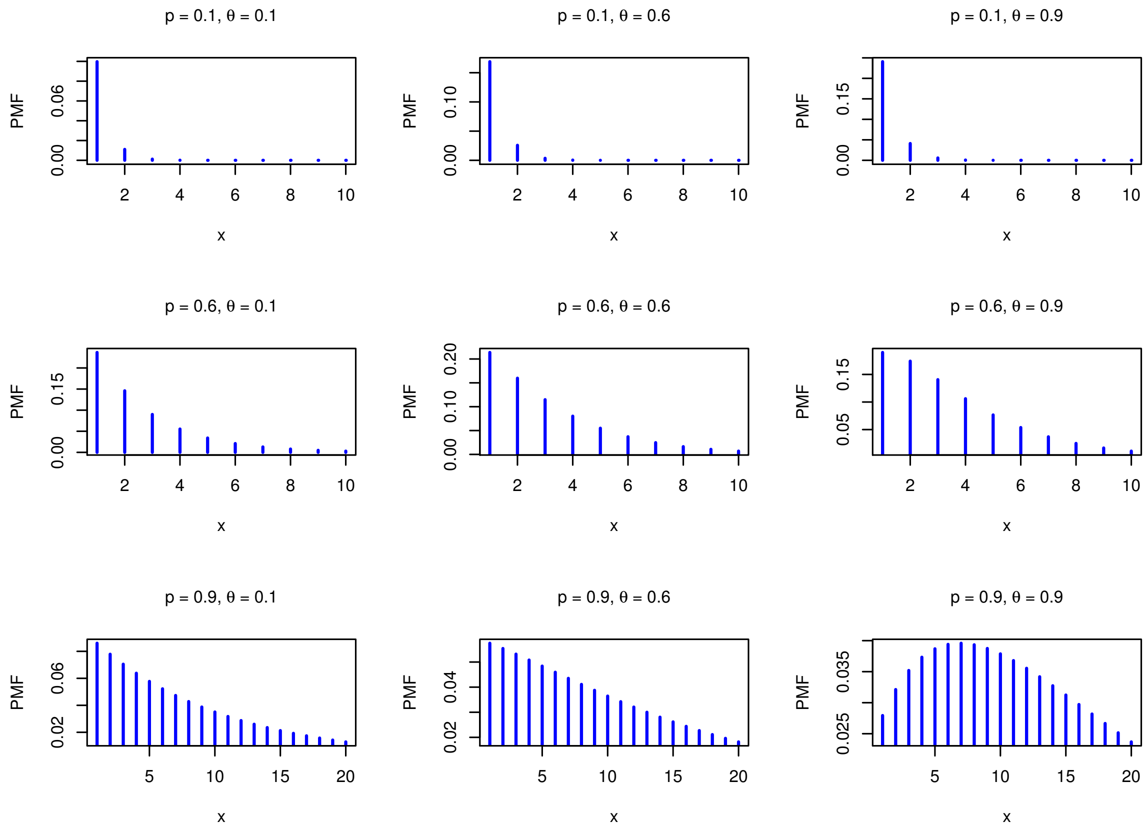

Figure 1. The PMF for the DBiExII model.

The PMF can take unimodal or decreasing-shaped. Assume X has a DBiExII distribution with parameters p and θ. Then, the PMFs of $Z = X^2$ and $U = X + a$ can be formulated, respectively, as

$$f_z(z; p, \theta) = \frac{1}{2\sqrt{z}}\left(1 - p + \frac{\theta(p\sqrt{z} + p - \sqrt{z}) \ln p}{2 - \theta}\right) p^{\sqrt{z}};\ z \in \{0, 1, 4, 9, 16, \ldots\} \quad (9)$$

and

$$f_u(u; p, \theta) = \left(1 - p + \frac{\theta(p(u - a) + p - u + a) \ln p}{2 - \theta}\right) p^{u-a};\ u \in \{a, a+1, a+2, \ldots\}, \quad (10)$$

where a is a positive integer number. The hazard rate function (HRF) of the DBiExII model can be expressed as

$$h(x; p, \theta) = \frac{(2 - \theta)(1 - p) + \theta(px + p - x) \ln p}{2 - \theta(1 + x \ln p)};\ \ x \in \mathbb{N}_0, \quad (11)$$

where $h(x; p, \theta) = \frac{f(x;p,\theta)}{1 - F(x-1;p,\theta)}$. The behavior of the HRF is given by

$$h(x; p, \theta) = \begin{cases} \frac{-p\theta(1 + \ln p) + 2p + \theta - 2}{\theta - 2}; & x \longrightarrow 0, \\ 1 - p; & x \longrightarrow \infty, \\ 1 - p; & \theta \longrightarrow 0, \\ \frac{-(px + p - x) \ln p + p - 1}{x \ln p - 1}; & \theta \longrightarrow 1. \end{cases} \quad (12)$$

Based on the log-concavity properties, the DBiExII distribution has increasing failure rate. For more details around the log-concave function (Gupta and Balakrishnan [18]). Figure 2 shows the HRF plots for different values of the parameters.

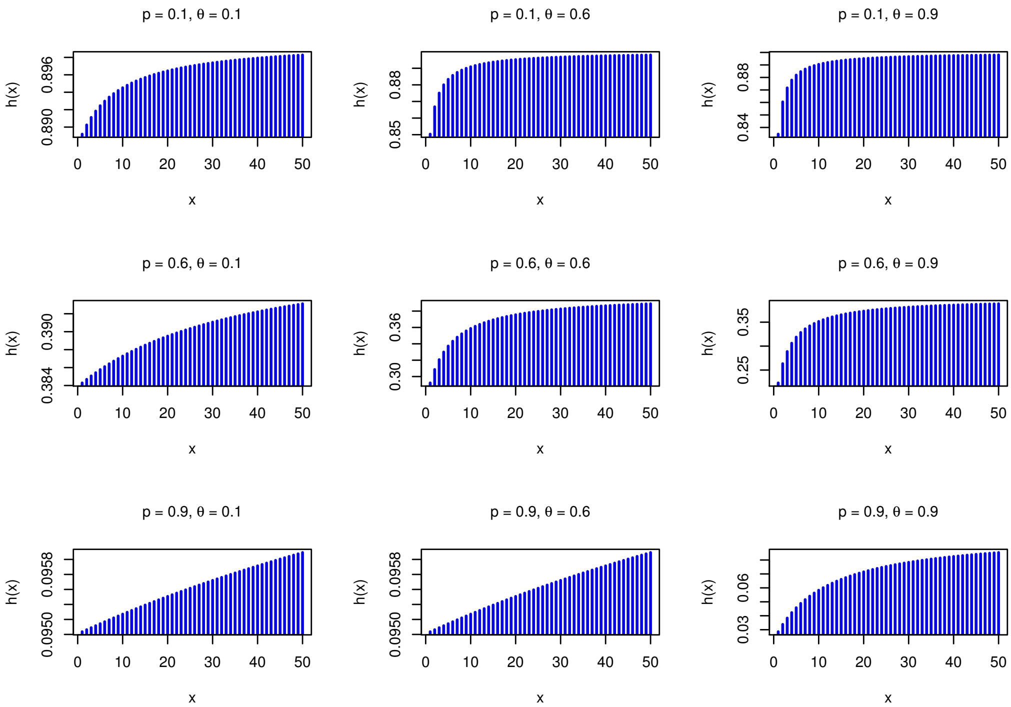

Figure 2. The HRF for the DBiExII model.

It is observed that the HRF takes increasing shape. The second rate of failure (SRF) is given by

$$h^*(x;p,\theta) = \log\left(\frac{\theta x \ln p + \theta - 2}{p[\theta x \ln p + \theta \ln p + \theta - 2]}\right), \tag{13}$$

where $h^*(x;p,\theta) = \log\left(\frac{1 - F(x-1;p,\theta)}{1 - F(x;p,\theta)}\right)$. The behavior of the SRF is given by

$$h^*(x;p,\theta) = \begin{cases} \log\left(\frac{\theta - 2}{p[\theta \ln p + \theta - 2]}\right); & x \longrightarrow 0, \\ \ln \frac{1}{p}; & x \longrightarrow \infty, \\ \ln \frac{1}{p}; & \theta \longrightarrow 0, \\ \log\left(\frac{x \ln p - 1}{p[x \ln p + \ln p - 1]}\right); & \theta \longrightarrow 1. \end{cases} \tag{14}$$

For more details about the difference between the HRF and SRF, we can refer to (Xie et al. [19]).

3. Distributional Statistics

3.1. Mode

If X has a DBiExII model, then the mode can be obtained by solving the following non-linear equation

$$[(1-p)(2-\theta) + \theta(px + p - x)\ln p]p^x \ln p + \theta(p-1)p^x \ln p = 0. \tag{15}$$

Then, the mode of the DBiExII model is given by

$$\mathbf{M}(X) = -\frac{\theta p(\ln p + 2) - 2(p + \theta - 2)}{\theta(p-1)\ln p}. \tag{16}$$

3.2. Moments, Skewness, Kurtosis and Index of Dispersion

Suppose the RV X follows DBiExII distribution. Then, the probability generating function (PGF) can be formulated in an explicit form as

$$\Pi_X(t;p,\theta) = \frac{p\theta(t-1)\ln p + (p-1)(pt-1)(\theta-2)}{(pt-1)^2(\theta-2)}, \tag{17}$$

where $\Pi_X(t;p,\theta) = \sum_{x=0}^{\infty} t^x f(x;p,\theta)$. On Replacing t by e^t in Equation (17), we get the moment generating function (MGF). Thus, the MGF can be proposed as

$$\Pi_X^*(t;p,\theta) = \frac{p\theta(e^t-1)\ln p + (p-1)(pe^t-1)(\theta-2)}{(pe^t-1)^2(\theta-2)}. \tag{18}$$

The first four derivatives of the MGF, with respect to t at $t=0$, yield the first four moments about the origin. So, the first four moments of the DBiExII model are

$$\mathbf{E}(X) = -p\frac{-\theta\ln p + (p-1)(\theta-2)}{(p-1)^2(\theta-2)}, \tag{19}$$

$$\mathbf{E}(X^2) = p\frac{(-3p-1)\theta\ln p + p^2\theta - 2p^2 - \theta + 2}{(p-1)^3(\theta-2)}, \tag{20}$$

$$\mathbf{E}(X^3) = -p\frac{(-7p^2\theta - 10p\theta - \theta)\ln p + (\theta-2)(p-1)(p^2+4p+1)}{(p-1)^4(\theta-2)} \tag{21}$$

and

$$\mathbf{E}(X^4) = p\frac{-150(p^3 + \frac{11}{3}p^2 + \frac{5}{3}p + \frac{1}{15})\ln p + (p^2+10p+1)(p^2-1)(\theta-2)}{(p-1)^5(\theta-2)}. \tag{22}$$

The variance of the DBiExII model can be calculated by using $\mathrm{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2$. Using well-known relations, it is also simple to compute the skewness and kurtosis measures of the DBiExII model. The shape characteristics for both mean and variance of the DBiExII distribution can be formulated, respectively, as

$$\mathbf{E}(X) = \begin{cases} \frac{p}{1-p}; & \theta \longrightarrow 0, \\ \frac{p(-\ln p - p + 1)}{p^2 - 2p + 1}; & \theta \longrightarrow 1 \end{cases} \tag{23}$$

and

$$\mathrm{Variance}(X) = \begin{cases} \frac{p}{p^2 - 2p + 1}; & \theta \longrightarrow 0, \\ \frac{(p^3 - p^2\ln p - p)\ln p + p^3 - 2p^2 + p}{p^4 - 4p^3 + 6p^2 - 4p + 1}; & \theta \longrightarrow 1, \end{cases} \tag{24}$$

The index of dispersion (IxD, for short) is defined as variance to mean ratio, it indicates whether a certain model is suitable for under or over-dispersed data sets. If IxD $< (> 1)$, the model is under- (over-) dispersed. If X has a DBiExII model, then the behavior of the IxD is given by

$$\mathrm{IxD} = \begin{cases} \frac{1}{1-p}; & \theta \longrightarrow 0, \\ \frac{(p\ln p - p^2 + 1)\ln p - p^2 + 2p - 1}{(p^2 - 2p + 1)\ln p + p^3 - 3p^2 + 3p - 1}; & \theta \longrightarrow 1, \end{cases} \tag{25}$$

In order to provide a numerical illustration of the pattern of the moments and associated features, we have calculated these properties for a variety of values of the parameters, and the results can be found in Table 1.

Table 1. Some descriptive statistics for the DBiExII distribution.

Parameter		Measure				
p	θ	Mean	Variance	Skewness	Kurtosis	IxD
0.01	0.01	0.010337	0.010443	9.987279	104.743575	1.010326
	0.1	0.012573	0.012719	9.072254	87.147341	1.011601
	0.3	0.018392	0.018593	7.493631	60.338393	1.010916
	0.5	0.025763	0.025936	6.293304	43.140116	1.006720
	0.7	0.035401	0.035374	5.310417	31.192865	0.999238
	0.9	0.048544	0.047945	4.457520	22.420924	0.987655
	0.99	0.056157	0.055068	4.100374	19.193921	0.980612
0.1	0.01	0.112539	0.125201	3.462044	16.984848	1.112503
	0.1	0.126072	0.141519	3.308182	15.870692	1.122521
	0.3	0.161276	0.182253	2.946320	13.237950	1.130068
	0.5	0.205867	0.230291	2.575399	10.759760	1.118638
	0.7	0.264179	0.287110	2.201012	8.584148	1.086800
	0.9	0.343695	0.353631	1.819239	6.743941	1.028908
	0.99	0.3897517	0.386376	0.386376	6.031577	0.991341
0.3	0.01	0.432275	0.619110	2.369474	10.613745	1.432212
	0.1	0.467367	0.682789	2.322009	10.336483	1.460926
	0.3	0.558652	0.836902	2.161759	9.330162	1.498074
	0.5	0.674280	1.008188	1.954587	8.141058	1.495206
	0.7	0.825485	1.191830	1.721058	7.001209	1.443793
	0.9	1.031674	1.368560	1.478955	6.065819	1.326542
	0.99	1.151101	1.432036	1.376903	5.765703	1.244058
0.5	0.01	1.006966	2.020850	2.119947	9.493563	2.006869
	0.1	1.072962	2.213565	2.095548	9.340058	2.063039
	0.3	1.244640	2.674071	1.982932	8.611448	2.148469
	0.5	1.462098	3.172759	1.819516	7.675631	2.170004
	0.7	1.746466	3.682186	1.630544	6.767094	2.108364
	0.9	2.134240	4.116220	1.447103	6.071141	1.928658
	0.99	2.358842	4.230074	1.386767	5.901186	1.793283
0.9	0.01	9.047650	90.90308	2.002601	9.009693	10.04714
	0.1	9.499076	99.23336	1.988878	8.906269	10.44663
	0.3	10.67337	118.9939	1.899065	8.295979	11.14866
	0.5	12.16081	140.0647	1.757718	7.474355	11.51770
	0.7	14.10593	160.9421	1.592996	6.674835	11.40953
	0.9	16.75836	177.2167	1.445560	6.099409	10.57482
	0.99	18.29467	180.2078	1.413765	5.998139	9.850289
0.99	0.01	99.49999	9999.247	2.000707	9.470644	100.49496
	0.1	104.23675	10914.68	1.985322	8.289761	104.71048
	0.3	116.55852	13085.84	1.896061	8.306480	112.26846
	0.5	132.16610	15400.08	15400.08	6.596502	116.52070
	0.7	152.57602	17691.24	1.593731	9.967532	115.95033
	0.9	180.40772	19472.97	1.444991	17.831995	107.93868
	0.99	196.52806	19796.40	1.409224	−9.710393	100.73070

From Table 1, it is clear that the DBiExII distribution is appropriate for modelling under-, equi-, and over-dispersed data sets. Moreover, the proposed model can be used for modelling positively skewed and leptokurtic data sets.

4. Parameter Estimation

Here, we discuss the point and interval estimation of the unknown parameters of DBiExII distribution using classical and Bayesian estimation.

4.1. Point Estimation through Maximum Likelihood Approach

In this section, we determine the maximum likelihood estimates (MLEs) of the model parameters based on complete sample. Let $X_1, X_2, \cdots X_n$ be a random sample of size n from the DBiExII model. Then, the likelihood function (L) is given by

$$L(\underline{x}; p, \theta) = p^{\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \left(1 - p + \frac{\theta(px_i + p - x_i)\ln p}{2-\theta}\right), \tag{26}$$

and the respective log-likelihood function (l) is

$$l(\underline{x}; p, \theta) = \ln(p) \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\left[1 - p + \frac{\theta[p(x_i+1) - x_i]\ln(p)}{2-\theta}\right]. \tag{27}$$

By differentiating Equation (27) with respect to the parameters p and θ, respectively, we get the non-linear likelihood equations as follows

$$\frac{1}{\widehat{p}} \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{\widehat{\theta} - 2 + \widehat{\theta}\left[(x_i+1)\ln(\widehat{p}) + (x_i+1) - \frac{x_i}{\widehat{p}}\right]}{(1-\widehat{p})(2-\widehat{\theta}) + \widehat{\theta}[\widehat{p}(x_i+1) - x_i]\ln(\widehat{p})} = 0 \tag{28}$$

and

$$\sum_{i=1}^{n} \frac{2[\widehat{p}(x_i+1) - x_i]\ln(\widehat{p})}{(2-\widehat{\theta})\left[(1-\widehat{p})(2-\widehat{\theta}) + \widehat{\theta}[\widehat{p}(x_i+1) - x_i]\ln(\widehat{p})\right]} = 0. \tag{29}$$

Equations (28) and (29) cannot be solved analytically; therefore, the suggested system needs an iterative procedure such as Newton-Raphson to solve these two equations numerically.

4.2. Asymptotic Confidence Interval

Unfortunately, the MLE of the unknown parameter vector $\boldsymbol{\Lambda} = (p, \theta)^T$ has no closed-form expression, so it is not possible to develop the exact confidence interval (CIs) for the parameter vector $\boldsymbol{\Lambda}$. Hence, we derived the asymptotic confidence interval (ACI) for $\boldsymbol{\Lambda}$ by utilizing its asymptotic distribution. For this purpose, let $\hat{\boldsymbol{\Lambda}} = (\hat{p}, \hat{\theta})^T$ be the estimate of $\boldsymbol{\Lambda} = (p, \theta)^T$ and the parameter vector $\boldsymbol{\Lambda}$ has the following Fisher's information matrix,

$$I(\boldsymbol{\Lambda}) = -\mathbf{E}\begin{bmatrix} \frac{\partial^2 l}{\partial p^2} & \frac{\partial^2 l}{\partial p \partial \theta} \\ \frac{\partial^2 l}{\partial \theta \partial p} & \frac{\partial^2 l}{\partial \theta^2} \end{bmatrix}.$$

Thus, under some regularity conditions with large n, $\sqrt{n}(\hat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})$ follows bivariate normal distribution with zero mean vector and variance-covariance matrix as $I^{-1}(\boldsymbol{\Lambda})$. Here, $I^{-1}(\boldsymbol{\Lambda})$ is the inverse of the expected Fisher's information matrix $I(\boldsymbol{\Lambda})$. Due to the complexity of the PMF of the random variable X, the expected values in $I(\boldsymbol{\Lambda})$ is not easily obtainable, therefore we utilized the estimated observed Fisher's information matrix that can be symbolized as

$$I(\hat{\boldsymbol{\Lambda}}) \approx \begin{bmatrix} -\frac{\partial^2 l}{\partial p^2}\Big|_{(\hat{p},\hat{\theta})} & -\frac{\partial^2 l}{\partial p \partial \theta}\Big|_{(\hat{p},\hat{\theta})} \\ -\frac{\partial^2 l}{\partial \theta \partial p}\Big|_{(\hat{p},\hat{\theta})} & -\frac{\partial^2 l}{\partial \theta^2}\Big|_{(\hat{p},\hat{\theta})} \end{bmatrix},$$

where, $\hat{p}$ and $\hat{\theta}$ are the MLEs of p and θ, respectively. The expressions for second order partial derivatives used in above observed Fisher's information matrix are as follows

$$\frac{\partial^2 l}{\partial p^2} = -\frac{1}{p^2}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left[\frac{\theta[(1-p)(2-\theta) + \theta[p(x_i+1) - x_i]\ln(p)]\left(\frac{x_i+1}{p} + \frac{x_i}{p}\right) - \{(2-\theta) - \theta[(x_i+1)(1+\ln(p)) - (x_i/p)]\}^2}{[(1-p)(2-\theta) + \theta[p(x_i+1) - x_i]\ln(p)]^2}\right],$$

$$\frac{\partial^2 l}{\partial \theta^2} = 4\sum_{i=1}^{n}\left[\frac{[p(x_i+1)-x_i]\ln(p)\{(1-p)(2-\theta)-(1-\theta)[p(x_i+1)-x_i]\ln(p)\}}{\{(2-\theta)[(1-p)(2-\theta)+\theta[p(x_i+1)-x_i]\ln(p)]\}^2}\right],$$

and

$$\frac{\partial^2 l}{\partial p \partial \theta} = 2\sum_{i=1}^{n}\left[\frac{(1-p)[p(x_i+1)-x_i]/p+\ln(p)}{[(1-p)(2-\theta)+\theta[p(x_i+1)-x_i]\ln(p)]^2}\right].$$

Hence, $(1-\gamma)\times 100\%$ ACI for parameters $\Lambda_i; i=1,2$ is given by

$$\left(\hat{\Lambda}_i - Z_{1-\gamma/2}\sqrt{Var(\hat{\Lambda}_i)},\, \hat{\Lambda}_i - Z_{1-\gamma/2}\sqrt{Var(\hat{\Lambda}_i)}\right);\quad i=1,2,$$

where $Var(\hat{\Lambda}_i)$ is the $(i,i)^{th}$ diagonal element of $I^{-1}(\mathbf{\Lambda})$ and $Z_{1-\gamma/2}$ is the $(1-\gamma/2)^{th}$ quantile of the standard normal distribution.

4.3. Bayesian Estimation

Over the past few decades, the importance of Bayesian statistics has increased immensely, not only because Bayesian estimators have become much easier to calculate, but also because it is one of the most satisfactory ways of calculating estimates for complex models. In this context, the present section is devoted to the Bayesian estimation of the unknown parameters of DBiExII distribution. Let the prior beliefs regarding the unknown parameters, p and θ are represented through the independent priors as beta distribution of first kind with respective densities as

$$\pi_1(p) \propto p^{a_1-1}\,(1-p)^{b_1-1},\ 0<p<1, \tag{30}$$

$$\pi_2(\theta) \propto \theta^{a_2-1}\,(1-\theta)^{b_2-1},\ 0<\theta<1, \tag{31}$$

where the shape parameters a_1, b_1, a_2 and b_2 are known and non-negative. These parameters are also named as hyper parameters which can be adjusted to indicate the prior information about the unknown parameters of the model. The joint prior distribution of p and θ is

$$\pi_{12}(p,\theta) \propto p^{a_1-1}\,(1-p)^{b_1-1}\theta^{a_2-1}\,(1-\theta)^{b_2-1},\, 0<p<1, 0<\theta<1. \tag{32}$$

Combining the likelihood function in Equation (26) and the joint prior distribution in Equation (32) using Bayes' theorem, the joint posterior distribution of p and θ given data is derived as

$$\pi_0(p,\theta|\underline{x}) \propto p^{\sum_{i=1}^{n} x_i + a_1 - 1}\,(1-p)^{b_1-1}\theta^{a_2-1}\,(1-\theta)^{b_2-1}\prod_{i=1}^{n}\left(1-p+\frac{\theta(px+p-x)\ln(p)}{2-\theta}\right). \tag{33}$$

In theBayesian scenario, a correct statistical decision depends not only on the choice of prior distribution, but it also relies on the selection of appropriate loss function. Therefore, to draw Bayesian inferences on unknown parameters p and θ, we should take care of the question of which type of loss function will be used. Here, we use the most common symmetric loss function, which is as popular as the squared error loss function (SELF). It is symmetric in the sense that it equally penalizes overestimation and underestimation. If $\hat{\kappa}$ is an estimator of κ, then we can define the SELF as

$$Loss(\hat{\kappa},\kappa) = (\hat{\kappa}-\kappa)^2. \tag{34}$$

Thus, in our case, the Bayes estimator (BE) of any function of parameters p and θ , say $\phi(p,\theta)$ under SELF with informative priors (IPs) is obtained as

$$\hat{\phi}(p,\theta|\underline{x}) = E_{p,\theta|\underline{x}}(\phi(p,\theta)) = \int_0^1\int_0^1 \phi(p,\theta)\ \pi_0(p,\theta|\underline{x})dpd\theta. \tag{35}$$

Here, it can be easily observed that the integral in Equation (35) cannot solved analytically, this is because of the complex form of joint posterior distribution in Equation (33).

Therefore, we use popular Markov Chain Monte Carlo (MCMC) technique known as Gibbs sampler (Geman and Geman [20]). The biggest virtue of this algorithm is that it allows us to generate posterior samples for all the parameters using their full conditional posterior distribution. In this context, the full conditional posterior distributions of the parameters p and θ can be written as

$$W(p|\theta, \underline{x}) \propto p^{\sum_{i=1}^{n} x_i + a_1 - 1} (1-p)^{b_1 - 1} \prod_{i=1}^{n} \left(1 - p + \frac{\theta(px + p - x)\ln(p)}{2-\theta}\right) \tag{36}$$

and

$$V(\theta|p, \underline{x}) \propto \theta^{a_2 - 1} (1-\theta)^{b_2 - 1} \prod_{i=1}^{n} \left(1 - p + \frac{\theta(px + p - x)\ln(p)}{2-\theta}\right). \tag{37}$$

Here, it is notable that the generation of parameters draws on p and θ from their respective posterior densities (36) and (37) are not possible through conventional methods of generating samples, therefore we used the Metropolis–Hastings (MH) algorithm advocated by (Devroye [21]). We utilized the following steps of Gibbs sampling algorithm:

1. Plug in with the initial values of p and θ, as $(p^{(0)}, \theta^{(0)})$.
2. Start with $j = 1$.
3. Generate $p^{(j)}$ from the conditional posterior distribution in Equation (36), through MH algorithm with normal proposal distribution.
4. Generate $\theta^{(j)}$ from the conditional posterior density in Equation (37) using MH algorithm with normal proposal distribution.
5. Set $j = j + 1$.
6. Repeat the steps 3–5, a large number of times, say N times, and obtain $p^{(j)}$ and $\theta^{(j)}$, $j = 1, 2, 3, \cdots, N$.

To ensure convergence and avoid the effect of selecting initial values, the first M draws are eliminated. Then, the remaining values $p^{(j)}$ and $\theta^{(j)}$, $j = M+1, M+2, \cdots, N$, represent the required posterior samples, which can be utilized to draw the Bayesian conclusions about the unknown population constraints. Hence, the Bayes estimates of p and θ under SELF, respectively, obtained as

$$\hat{p}_{BE} = \frac{1}{N-M} \sum_{j=M+1}^{N} p^{(j)} \quad \text{and} \quad \hat{\theta}_{BE} = \frac{1}{N-M} \sum_{j=M+1}^{N} \theta^{(j)}. \tag{38}$$

If a prior distribution provides limited or no information about the parameter, it is called a non-informative prior (NIP). The reasoning for utilizing NIPs is often said to be 'to let the data speak for themselves,' so that posterior inferences are ineffective by information external to the sample data. However, finding a suitable prior distribution for a parameter is a daunting task. Although there is a vast literature on how to select an appropriate prior for a parameter of interest (Berger [22]; Bernado and Smith [23]), the choice of priors remains a challenging issue. Various NIPs are available in the existing literature, but uniform (or flat) prior are widely applied in practice. As suggested by an honourable reviewer, here, we use non-informative uniform priors for p and θ and they can be obtained by putting $a_1 = b_1 = a_2 = b_2 = 1$ in Equations (30) and (31). Hence, proceeding similarly as we do in the case of IPs, we can obtain the Bayes estimators with SELF under NIPs.

4.4. Highest Posterior Density (HPD) Credible Interval

A Bayesian counterpart of the classical confidence interval is called a credible or probability interval and it can be explained in a probabilistic manner. This differs from a traditional confidence interval, which can only be expressed in terms of coverage probability.

We can easily obtain an equal-tail credible interval (α_L, α_U) for a parameter α by simplifying the following equations:

$$\int_0^{\alpha_L} \pi(\alpha|\underline{x})d\alpha = \frac{\gamma}{2} \text{ and } \int_0^{\alpha_U} \pi(\alpha|\underline{x})d\alpha = 1 - \frac{\gamma}{2},$$

where $\pi(\alpha|\underline{x})$ denotes the posterior distribution of the unknown parameter α and γ represents the level of significance. We can construct several such credible intervals, but to derive the best among them, we select the interval with the smallest width. The shortest interval, let us say (α_L^h, α_U^h), is the credible interval that fulfills the following two conditions:

$$(i). \int_{\alpha_L^h}^{\alpha_U^h} \pi(\alpha|\underline{x})d\alpha = 1 - \gamma. \quad (ii). \text{ For any } \alpha_1 \in (\alpha_L^h, \alpha_U^h) \text{ and } \alpha_2 \notin (\alpha_L^h, \alpha_U^h) \text{ we have } \pi(\alpha_1|\underline{x}) > \pi(\alpha_2|\underline{x}).$$

This type of credible interval is called the highest posterior density (HPD) credible interval. Since, the mathematical derivation of the HPD credible intervals for the unknown parameters of DBiExII distribution is difficult to obtain due to the non-closure form of the posterior distributions, therefore, we use an algorithm suggested by Chen and Shao [24]. For this purpose, consider the MCMC samples, $p^{(j)}$ and $\theta^{(j)}$, $j = M+1, M+2, \cdots, N$, generated in the previous section. Now, ordered these generated values as $p_{(M+1)} < p_{(M+2)} < \ldots < p_{(N)}$ and $\theta_{(M+1)} < \theta_{(M+2)} < \ldots < \theta_{(N)}$. Thus, the $(1-\gamma) \times 100\%$ HPD credible intervals for p and θ are respectively given as

$$\Big(p_{(M+j^*)}, p_{(M+j^*+[(1-\gamma)(N-M)])}\Big) \text{and} \Big(\theta_{(M+j^{**})}, \theta_{(M+j^{**}+[(1-\gamma)(N-M)])}\Big),$$

where j^* and j^{**} are respectively chosen so that

$$p_{(M+j^*+[(1-\gamma)(N-M)])} - p_{(M+j^*)} = \min_{M \le j \le (N-M)-[(1-\gamma)(N-M)]} \Big(p_{(M+j+[(1-\gamma)(N-M)])} - p_{(M+j)}\Big)$$

and

$$\theta_{(M+j^{**}+[(1-\gamma)(N-M)])} - \theta_{(M+j^{**})} = \min_{M \le j \le (N-M)-[(1-\gamma)(N-M)]} \Big(\theta_{(M+j+[(1-\gamma)(N-M)])} - \theta_{(M+j)}\Big).$$

5. Numerical Illustration through Simulated Data

In this section, we conduct a Monte Carlo simulation analysis to evaluate the behavior of classical and Bayesian procedures for estimating the unknown parameters of the DBiExII distribution. This study consists of the following steps:

Step 1. Generate 10,000 samples of size $n \in$\{20, 40, 60, 100\} from DBiExII distribution using Equation (6) with the arbitrary sets of parameters $(p, \theta) \in$\{(0.3, 0.3), (0.3, 0.8), (0.8, 0.3), (0.5, 0.5), (0.8, 0.8)\}.

Step 2. Compute the MLEs, Bayes estimates (with NIPs and IPs), 95% asymptotic and HPD confidence intervals for each of the 10,000 samples.
In the case of Bayesian estimation, it is important to note that we have calculated the Bayes estimates with IPs and NIPs under SELF. In IPs, the prior densities for the parameters p and θ are taken to be $Beta_1(a_1, b_1)$ and $Beta_1(a_2, b_2)$ distributions, respectively. The hyper-parameters in these prior densities have been selected in such a manner that the mean of a parameter's prior density is almost equal to the corresponding assumed value of that parameter, whereas, all hyper-parameters are set to 1 for Bayes estimation under NIPs. Using the algorithm described in Section 4.3, we produced 21,000 realizations of the Markov chain of p and θ from their full conditional posterior distributions in order to calculate the required Bayesian quantities. To counteract the impact of the parameters beginning values, the first 1000 burn-in values for each chain have been eliminated. Additionally, we have stored every tenth observation to reduce the autocorrelation between draws. By plotting MCMC runs, posterior densities, and the autocorrelation function for

each pair of true parameters, the convergence of the produced chains is investigated. For the sake of simplicity, we have only included these graphs for $(p, \theta) = (0.3, 0.3)$ in Figure 3. After the convergence testing, we have utilized simulated posterior samples to compute Bayes estimates and HPD intervals.

Step 3. Compute the average estimate (AE), root mean squared error (RMSE), and average absolute bias (AB) for MLEs and Bayes estimates (with SELF under IPs and NIPs), whereas for 95% asymptotic and HPD confidence intervals, we calculate average lower confidence limit (ALCL), average upper confidence limit (AUCL), average width (AW), and coverage probability (CP).

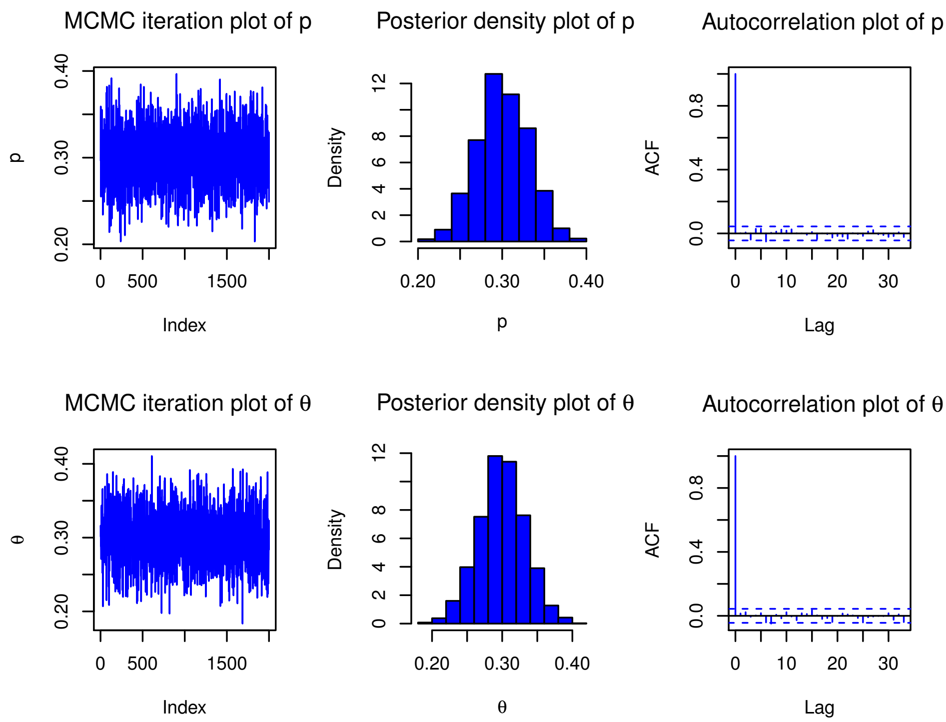

Figure 3. MCMC diagnostics plots for DBiExII(0.3,0.3) distribution.

All the calculations have been done with R software. The outcomes of the simulation study are given in Tables 2–7. From these tables, the following important conclusions can be drawn:

i. The RMSE of both estimators decreases with increasing sample size. This validates the consistency property of the estimators. Moreover, as the value of n increases, the absolute bias decreases toward zero.

ii. Bayes estimators obtained under IPs show smaller RMSE as compared to the MLEs and Bayes estimators with NIPs.

iii. Under NIPs, the Bayesian method becomes the first choice of estimation in the absence of prior information. This is due to the fact that the Bayes estimates under NIPs have the smallest estimation errors when compared to the MLEs.

iv. The average width of classical and Bayesian intervals becomes smaller as the sample size increases. Moreover, the HPD interval with IPs outperforms asymptotic and HPD intervals under NIPs in terms of the width of the intervals. The CPs are close to the corresponding nominal levels in both classical and Bayesian intervals.

v. In both estimation processes, the estimation of θ is more sensitive than the estimation of p since it results in more estimation error relative to the other parameter.

Table 2. Simulation results based on MLEs.

(p,θ)	n	AE(p)	RMSE(p)	AB(p)	AE(θ)	RMSE(θ)	AB(θ)
(0.3, 0.3)	25	0.2504	0.1169	0.0978	0.4972	0.4472	0.3982
	50	0.2673	0.0960	0.0791	0.4282	0.3878	0.3420
	100	0.2812	0.0774	0.0643	0.3693	0.3278	0.2900
	200	0.2916	0.0620	0.0523	0.3271	0.2755	0.2440
(0.3, 0.8)	25	0.3072	0.0924	0.0734	0.7318	0.3242	0.2404
	50	0.3065	0.0741	0.0578	0.7522	0.2702	0.1951
	100	0.3033	0.0557	0.0432	0.7772	0.1987	0.1435
	200	0.3014	0.0393	0.0308	0.7899	0.1342	0.1003
(0.8, 0.3)	25	0.7696	0.0682	0.0521	0.4364	0.3751	0.3348
	50	0.7806	0.0511	0.0400	0.3852	0.3287	0.2949
	100	0.7900	0.0380	0.0308	0.3361	0.2816	0.2536
	200	0.7958	0.0297	0.0247	0.3001	0.2447	0.2183
(0.5, 0.5)	25	0.4734	0.1072	0.0874	0.5325	0.3600	0.3194
	50	0.4895	0.0844	0.0699	0.4624	0.3235	0.2780
	100	0.4985	0.0663	0.0546	0.4691	0.2802	0.2297
	200	0.5031	0.0533	0.0431	0.4907	0.2389	0.1857
(0.8, 0.8)	25	0.7991	0.0419	0.0330	0.7361	0.2762	0.1899
	50	0.7991	0.0315	0.0248	0.7667	0.2127	0.1435
	100	0.7990	0.0223	0.0174	0.7863	0.1376	0.0958
	200	0.8001	0.0153	0.0121	0.7906	0.0894	0.0663

Table 3. Simulation results based on asymptotic confidence intervals.

(p,θ)	n	ALCL(p)	AUCL(p)	AW(p)	CP(p)	ALCL(θ)	AUCL(θ)	AW(θ)	CP(θ)
(0.3, 0.3)	25	0.0182	0.5754	0.5572	0.9014	0.0463	0.9963	0.9501	0.9107
	50	0.0422	0.5385	0.4963	0.9493	0.0775	0.9892	0.9117	0.9431
	100	0.0785	0.5095	0.4310	0.9577	0.0808	0.9779	0.8971	0.9555
	200	0.1175	0.4815	0.3640	0.9584	0.0746	0.9644	0.8898	0.9632
(0.3, 0.8)	25	0.0739	0.5565	0.4827	0.9237	0.2095	0.9997	0.7902	0.9495
	50	0.1343	0.4814	0.3471	0.9386	0.3320	0.9993	0.6673	0.9586
	100	0.1856	0.4215	0.2358	0.9323	0.4398	0.9997	0.5598	0.9585
	200	0.2225	0.3804	0.1579	0.9561	0.5386	0.9950	0.4564	0.9462
(0.8, 0.3)	25	0.5989	0.9095	0.3106	0.9033	0.0966	0.9739	0.8773	0.9487
	50	0.6464	0.8962	0.2498	0.9305	0.1191	0.9883	0.8691	0.9383
	100	0.6818	0.8863	0.2045	0.9446	0.1378	0.9954	0.8577	0.9682
	200	0.7093	0.8757	0.1664	0.9645	0.0836	0.9180	0.8344	0.9783

Table 3. *Cont.*

(p,θ)	n	ALCL(p)	AUCL(p)	AW(p)	CP(p)	ALCL(θ)	AUCL(θ)	AW(θ)	CP(θ)
(0.5, 0.5)	25	0.2060	0.7288	0.5228	0.9302	0.1591	0.9978	0.8387	0.9542
	50	0.2715	0.6981	0.4266	0.9231	0.1609	0.9947	0.8337	0.9681
	100	0.3275	0.6642	0.3367	0.9425	0.1619	0.9905	0.8286	0.9592
	200	0.3735	0.6298	0.2563	0.9592	0.1796	0.9499	0.7703	0.9536
(0.8, 0.8)	25	0.6981	0.8908	0.1927	0.9655	0.3656	0.9997	0.6341	0.9359
	50	0.7332	0.8624	0.1292	0.9571	0.4697	0.9996	0.5298	0.9448
	100	0.7552	0.8423	0.0871	0.9564	0.5557	0.9972	0.4415	0.9569
	200	0.7701	0.8301	0.0600	0.9546	0.6251	0.9559	0.3309	0.9607

Table 4. Simulation results based on Bayes estimates with non-informative priors.

(p,θ)	n	AE(p)	RMSE(p)	AB(p)	AE(θ)	RMSE(θ)	AB(θ)
(0.3, 0.3)	25	0.2998	0.0118	0.0095	0.2998	0.0089	0.0072
	50	0.2999	0.0110	0.0090	0.2998	0.0078	0.0063
	100	0.3002	0.0093	0.0074	0.3001	0.0065	0.0052
	200	0.3006	0.0076	0.0061	0.3005	0.0051	0.0041
(0.3, 0.8)	25	0.2981	0.0117	0.0094	0.8064	0.0161	0.0126
	50	0.2987	0.0114	0.0091	0.8057	0.0141	0.0111
	100	0.2988	0.0106	0.0085	0.8062	0.0124	0.0099
	200	0.2994	0.0086	0.0069	0.8055	0.0107	0.0086
(0.8, 0.3)	25	0.7986	0.0122	0.0097	0.2962	0.0064	0.0051
	50	0.7989	0.0116	0.0094	0.2963	0.0061	0.0049
	100	0.7992	0.0116	0.0093	0.2998	0.0057	0.0046
	200	0.7993	0.0098	0.0078	0.3012	0.0056	0.0046
(0.5, 0.5)	25	0.4995	0.0115	0.0092	0.4999	0.0115	0.0090
	50	0.4997	0.0115	0.0091	0.4999	0.0099	0.0079
	100	0.5000	0.0098	0.0079	0.5002	0.0084	0.0067
	200	0.5005	0.0081	0.0065	0.5004	0.0065	0.0053
(0.8, 0.8)	25	0.7988	0.0123	0.0099	0.8054	0.0190	0.0151
	50	0.7990	0.0123	0.0099	0.8051	0.0154	0.0123
	100	0.7992	0.0107	0.0086	0.8035	0.0124	0.0106
	200	0.7992	0.0090	0.0073	0.8033	0.0097	0.0086

Table 5. Simulation results based on HPD intervals with non-informative priors.

(p,θ)	n	ALCL(p)	AUCL(p)	AW(p)	CP(p)	ALCL(θ)	AUCL(θ)	AW(θ)	CP(θ)
(0.3, 0.3)	25	0.2543	0.3462	0.0919	0.9231	0.2132	0.3868	0.1736	0.9061
	50	0.2561	0.3438	0.0877	0.9389	0.2133	0.3859	0.1725	0.9394
	100	0.2601	0.3411	0.0809	0.9469	0.2151	0.3860	0.1709	0.9482
	200	0.2640	0.3358	0.0718	0.9604	0.2156	0.3843	0.1687	0.9597
(0.3, 0.8)	25	0.2552	0.3451	0.0899	0.8974	0.7150	0.8856	0.1706	0.8968
	50	0.2574	0.3420	0.0846	0.9151	0.7162	0.8833	0.1671	0.9289
	100	0.2615	0.3383	0.0768	0.9394	0.7179	0.8803	0.1624	0.949
	200	0.2662	0.3336	0.0673	0.9451	0.7216	0.8772	0.1556	0.9679
(0.8, 0.3)	25	0.7596	0.8386	0.0789	0.9341	0.2133	0.3868	0.1736	0.9281
	50	0.7647	0.8333	0.0686	0.9409	0.2132	0.3863	0.1731	0.9326
	100	0.7703	0.8271	0.0569	0.9498	0.2136	0.3863	0.1727	0.9649
	200	0.7758	0.8212	0.0454	0.9562	0.2142	0.3856	0.1714	0.9585
(0.5, 0.5)	25	0.4545	0.5454	0.0909	0.9457	0.4139	0.5865	0.1726	0.9149
	50	0.4568	0.5427	0.0859	0.9462	0.4145	0.5854	0.1709	0.9446
	100	0.4614	0.5396	0.0783	0.9574	0.4162	0.5845	0.1683	0.9527
	200	0.4652	0.5338	0.0687	0.9656	0.4173	0.5820	0.1648	0.9533
(0.8, 0.8)	25	0.7620	0.8361	0.0742	0.9346	0.7153	0.8848	0.1694	0.9353
	50	0.7671	0.8303	0.0632	0.9451	0.7171	0.8825	0.1654	0.9492
	100	0.7730	0.8247	0.0516	0.9563	0.7200	0.8788	0.1587	0.9560
	200	0.7787	0.8196	0.0409	0.9512	0.7259	0.8745	0.1486	0.9512

Table 6. Simulation results based on Bayes estimates with informative priors.

(p,θ)	n	AE(p)	RMSE(p)	AB(p)	AE(θ)	RMSE(θ)	AB(θ)
(0.3, 0.3)	25	0.2982	0.0110	0.0093	0.2963	0.0088	0.0071
	50	0.2989	0.0107	0.0086	0.2966	0.0076	0.0063
	100	0.2991	0.0091	0.0073	0.2967	0.0060	0.0047
	200	0.2992	0.0073	0.0059	0.2968	0.0049	0.0038
(0.3, 0.8)	25	0.2997	0.0111	0.0089	0.7993	0.0153	0.0122
	50	0.2997	0.0111	0.0089	0.7996	0.0136	0.0109
	100	0.2999	0.0104	0.0084	0.7998	0.0116	0.0093
	200	0.3001	0.0084	0.0068	0.8003	0.0089	0.0072
(0.8, 0.3)	25	0.7996	0.0115	0.0091	0.2999	0.0058	0.0046
	50	0.7998	0.0112	0.0089	0.3001	0.0050	0.0040
	100	0.8005	0.0111	0.0089	0.3005	0.0049	0.0039
	200	0.8007	0.0096	0.0077	0.3002	0.0044	0.0035

Table 6. *Cont.*

(p,θ)	n	AE(p)	RMSE(p)	AB(p)	AE(θ)	RMSE(θ)	AB(θ)
(0.5, 0.5)	25	0.4998	0.0110	0.0084	0.4997	0.0110	0.0088
	50	0.5120	0.0107	0.0075	0.4991	0.0092	0.0073
	100	0.4995	0.0091	0.0069	0.5001	0.0080	0.0065
	200	0.5022	0.0074	0.0060	0.5002	0.0065	0.0051
(0.8, 0.8)	25	0.7991	0.0120	0.0096	0.7998	0.0184	0.0146
	50	0.7992	0.0120	0.0096	0.7999	0.0151	0.0120
	100	0.8056	0.0106	0.0084	0.8002	0.0112	0.0098
	200	0.8009	0.0088	0.0070	0.8008	0.0091	0.0077

Table 7. Simulation results based on HPD intervals with informative priors.

(p,θ)	n	ALCL(p)	AUCL(p)	AW(p)	CP(p)	ALCL(θ)	AUCL(θ)	AW(θ)	CP(θ)
(0.3, 0.3)	25	0.2534	0.3445	0.0911	0.9172	0.2129	0.3807	0.1678	0.9179
	50	0.2548	0.3418	0.0870	0.9320	0.2129	0.3796	0.1666	0.9460
	100	0.2591	0.3394	0.0803	0.9501	0.2143	0.3794	0.1651	0.9548
	200	0.2636	0.3349	0.0713	0.9486	0.2152	0.3785	0.1632	0.9665
(0.3, 0.8)	25	0.2543	0.3435	0.0892	0.9061	0.7245	0.8883	0.1639	0.9082
	50	0.2563	0.3402	0.0838	0.9282	0.7250	0.8859	0.1610	0.9222
	100	0.2607	0.3368	0.0761	0.9584	0.7277	0.8842	0.1565	0.9444
	200	0.2662	0.3328	0.0666	0.9696	0.7300	0.8800	0.1501	0.9589
(0.8, 0.3)	25	0.7613	0.8396	0.0783	0.9321	0.2125	0.3802	0.1678	0.9292
	50	0.7666	0.8346	0.0680	0.9401	0.2127	0.3799	0.1672	0.9431
	100	0.7713	0.8277	0.0564	0.9562	0.2131	0.3796	0.1665	0.9448
	200	0.7772	0.8222	0.0450	0.9574	0.2136	0.3792	0.1657	0.9620
(0.5, 0.5)	25	0.4546	0.5448	0.0902	0.9437	0.4163	0.5833	0.1670	0.9264
	50	0.4563	0.5416	0.0853	0.9500	0.4162	0.5819	0.1657	0.9364
	100	0.4608	0.5385	0.0778	0.9699	0.4184	0.5817	0.1634	0.9549
	200	0.4657	0.5338	0.0682	0.9662	0.4200	0.5802	0.1601	0.9686
(0.8, 0.8)	25	0.7626	0.8362	0.0737	0.9112	0.7238	0.8864	0.1626	0.9394
	50	0.7678	0.8305	0.0627	0.9422	0.7255	0.8842	0.1587	0.9459
	100	0.7734	0.8247	0.0513	0.9570	0.7267	0.8793	0.1526	0.9417
	200	0.7790	0.8196	0.0406	0.9760	0.7311	0.8745	0.1434	0.9639

6. Data Analysis

In this section, we illustrate the flexibility of the DBiExII distribution for modelling various types of data sets generated from different fields. We have compared the fits of the DBiExII model with some well-known existing models having one or two parameters and these are reported in Table 8. The fitted models are compared using famous criteria, namely, $-l$ (Negative log-likelihood), Akaike information criterion (AIC) "$AIC = 2c - 2L$", with its corrected value (CAIC) "$CAIC = [2nc/(n - c - 1)] - 2L$", and Chi-square ($\chi^2$) statistic with associated P-value, where L is the maximized likelihood function evaluated at MLEs, n is the sample size, and c is the number of parameters in the model.

Table 8. The competitive models.

Distribution	Abbreviation	Author(s)
Discrete Exponential (Geometric)	DEx	-
Generalized Discrete Exponential	GDEx	Gómez-Déniz [25]
Discrete generalized exponential type II	DGExII	Nekoukhou et al. [26]
Discrete Rayleigh	DR	Roy [5]
Discrete inverse Rayleigh	DIR	Hussain and Ahmad [27]
Discrete Bilal	DBe	Eliwa et al. [28]
Discrete Burr–Hatke	DBH	El-Morshedy et al. [15]
Discrete Pareto	DPa	Krishna and Pundir [9]
Discrete inverse Weibull	DIW	Jazi et al. [29]
Discrete Burr type II	DBX-II	Para and Jan [30]
Discrete log-logistic	DLogL	Para and Jan [31]

6.1. Data Set I: COVID-19

The data are reported in (https://www.worldometers.info/coronavirus/country/south-korea/, 5 February to 13 June 2022) and represents the daily new deaths in South Korea country. The MLEs with their corresponding standard errors (Std-er), confidence interval (C. I) for the parameter(s) and goodness of fit statistic for data set I are listed in Tables 9–11.

Table 9. The MLEs with their corresponding Std-ers and C. I for data set I.

Parameter $\rightarrow$	p			θ		
Model $\downarrow$	**MLE**	**Std-er**	**C. I**	**MLE**	**Std-er**	**C. I**
DBiExII	0.5966	0.0429	[0.5123, 0.6808]	0.6436	0.1674	[0.3155, 0.9718]
DEx	0.7039	0.0225	[0.6598, 0.7479]	–	–	–
GDEx	0.6449	0.0439	[0.5588, 0.7309]	1.6124	0.5023	[0.6274, 2.5969]
DGExII	0.6739	0.0336	[0.6079, 0.7398]	1.2149	0.1947	[0.8333, 1.5966]
DR	0.9306	0.0061	[0.9186, 0.9426]	–	–	–
DIR	0.1768	0.0329	[0.1122, 0.2414]	–	–	–
DBe	0.7487	0.0143	[0.7206, 0.7767]	–	–	–
DBH	0.9315	0.0269	[0.8789, 0.9842]	–	–	–
DPa	0.4152	0.0332	[0.3500, 0.4803]	–	–	–
DIW	0.2338	0.0381	[0.1591, 0.3086]	1.2658	0.1134	[1.0436, 1.4879]
DB-XII	0.6225	0.0487	[0.5271, 0.7179]	2.3359	0.3772	[1.5967, 3.0751]
DLogL	2.0210	0.1890	[1.6505, 2.3915]	1.7457	0.1523	[1.4472, 2.0443]

From Tables 10 and 11, the DBiExII model is the best distribution among all tested models. Figure 4 shows the profile of the log-likelihood function for data set I and this figure announces that the parameters are unimodal functions.

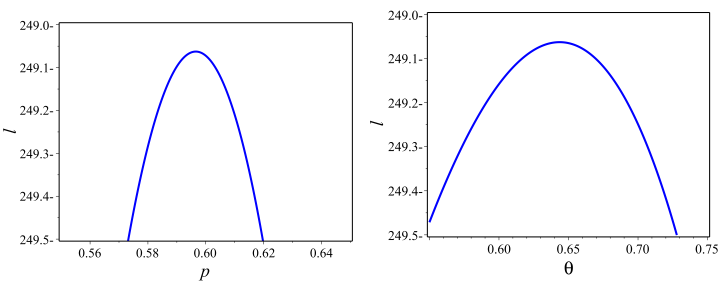

Figure 4. The profile of l function for the model parameters based on data set I.

Table 10. The goodness of fit statistic for data set I.

No.	**Observed**	**Expected Frequency**						
X	**Frequency**	**DBiExII**	**DEx**	**DR**	**DIR**	**DBe**	**DBH**	**DPa**
0	32	31.3801	36.1263	8.4645	21.5662	19.2478	65.1759	55.6679
1	27	25.9175	25.4287	22.0300	57.5405	30.7382	21.5347	19.8889
2	17	19.7549	17.8988	27.6341	21.5260	25.5994	10.6342	10.3779
3	14	14.3464	12.5987	25.2606	8.8444	17.8606	6.2813	6.4241
4	8	10.0864	8.8679	18.3967	4.3528	11.4844	4.1105	4.3897
5	7	6.9287	6.2420	11.0489	2.4366	7.0553	2.8746	3.2002
6	6	4.6771	4.3937	5.5663	1.4943	4.2138	2.1058	2.4424
7	5	3.1145	3.0926	2.3750	0.9801	2.4707	1.5963	1.9288
8	5	2.0515	2.1768	0.8635	0.6767	1.4305	1.2423	1.5640
9	1	3.7429	5.1745	0.3604	2.5824	1.8993	6.4444	16.1161
Total	**122**	**122**	**122**	**122**	**122**	**122**	**122**	**122**
$-l$		**249.0626**	250.3056	279.9239	278.0657	255.5355	277.0495	279.8059
AIC		**502.1251**	502.6112	561.8477	558.1313	513.0710	556.0990	561.6119
CAIC		**502.2260**	502.6445	561.8811	558.1647	513.1043	556.1323	561.6452
χ^2		**2.2108**	8.4698	89.7303	57.2829	18.5571	43.5311	64.2454
***p*-value**		**0.8193**	0.2057	<0.0001	<0.0001	0.0023	<0.0001	<0.0001

Table 11. The goodness of fit statistic for data set I "Contin".

No.	**Observed**	**Expected Frequency**					
X	**Frequency**	**DBiExII**	**GDEx**	**DGExII**	**DIW**	**DB-XII**	**DLogL**
0	32	31.3801	31.0604	31.2734	28.5245	34.1655	27.6305
1	27	25.9175	25.7405	27.2030	38.1383	35.8539	32.81328
2	17	19.7549	19.8882	19.7995	18.3045	17.0802	20.7934
3	14	14.3464	14.5504	13.8789	9.9187	9.0893	12.3449
4	8	10.0864	10.2257	9.5696	6.0537	5.5023	7.6045
5	7	6.9287	6.9849	6.5406	4.0199	3.6500	4.9354
6	6	4.6771	4.6791	4.4475	2.8368	2.5837	3.3619
7	5	3.1145	3.0936	3.0147	2.0949	1.9183	2.3864
8	5	2.0515	2.0277	2.0393	1.6026	1.4770	1.7534
9	1	3.7429	3.7495	4.2335	10.5061	10.6798	8.3763
Total	**122**	**122**	**122**	**122**	**122**	**122**	**122**
$-l$		**249.0626**	249.2000	249.5807	262.3222	263.5383	256.7394
AIC		**502.1251**	502.3999	503.1614	528.6444	531.0766	517.4788
CAIC		**502.2260**	502.5008	503.2622	528.7453	531.1774	517.5796
χ^2		**2.2108**	2.3634	2.3941	12.3012	14.1272	5.5044
***p*-value**		**0.8193**	0.7969	0.7924	0.0152	0.0069	0.2393

Figure 5 shows the estimated PMFs for data set I.

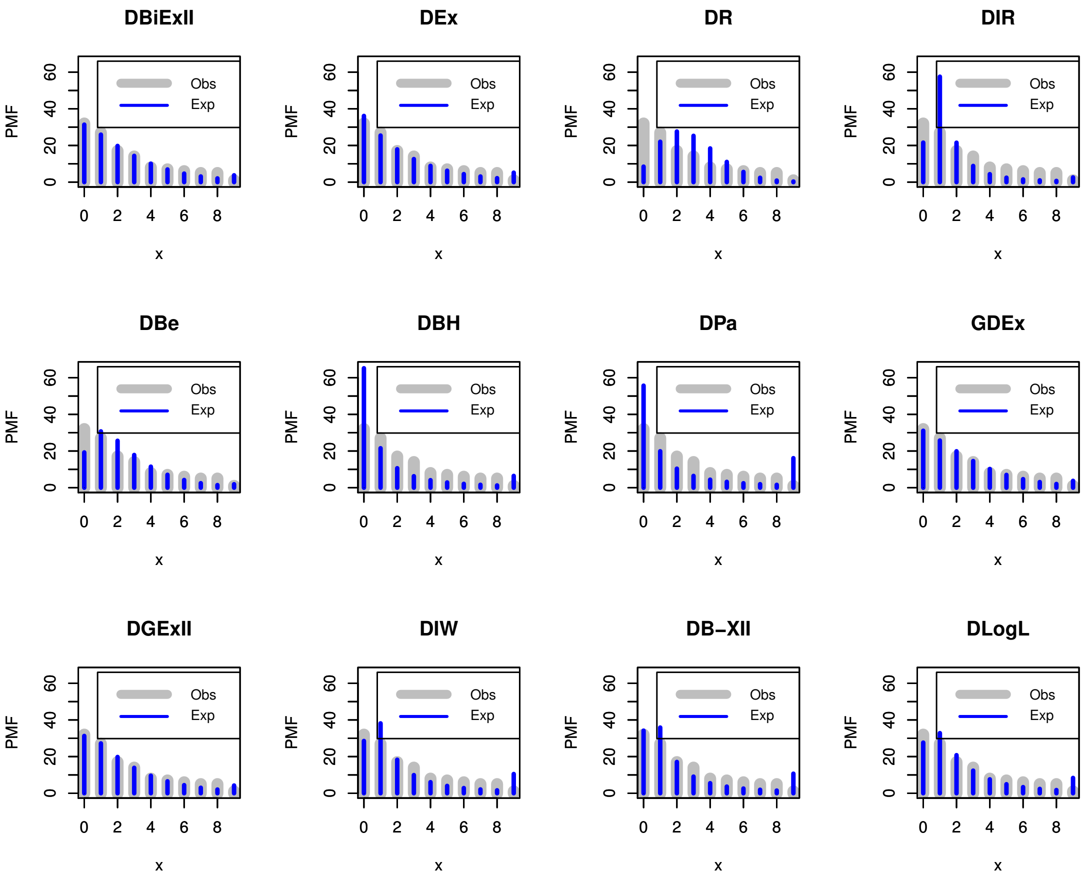

Figure 5. The estimated PMFs for data set I.

Table 12 lists some information about data set I based on the DBiExII model.

Table 12. Some descriptive statistics for data set I.

Type ↓ Measures →	Mean	Variance	IxD	Skewness	Kurtosis
Theoretical	2.37742	6.41498	2.69829	1.66410	6.95252
Empirical	2.37705	5.757485	2.42211	0.98615	3.07782

From Table 12, the empirical mean, variance, IxD are close to theoretical ones. Data set I suffers from overdispersion phenomena, and most of the distribution is at the left with leptokurtic character.

6.2. Data Set II: Larvae Pyrausta

This data set is the biological experiment data which represents the number of European corn borer larvae Pyrausta in field (Holt et al. [32]). The MLEs with their corresponding Std-ers, confidence intervals (C. Is) for the parameter(s) and goodness of fit statistic for data set II are listed in Tables 13–15.

Table 13. The MLEs with their corresponding Std-ers and C. Is for data set II.

Parameter →	p			θ		
Model ↓	**MLE**	**Std-er**	**C. I**	**MLE**	**Std-er**	**C. I**
DBiExII	0.2576	0.0377	[0.1837, 0.3315]	0.6439	0.1472	[0.3556, 0.9324]
DEx	0.3933	0.0211	[0.3518, 0.4347]	–	–	–
GDEx	0.3179	0.0407	[0.2375, 0.3971]	1.5917	0.3804	[0.8462, 2.3373]
DGExII	0.3379	0.0366	[0.2661, 0.4096]	1.3317	0.2231	[0.8943, 1.7690]
DR	0.6216	0.0175	[0.5873, 0.6559]	–	–	–
DIR	0.5747	0.0274	[0.5209, 0.6285]	–	–	–
DBe	0.4809	0.0149	[0.4516, 0.5102]	–	–	–
DBH	0.6555	0.0356	[0.5877, 0.7232]	–	–	–
DPa	0.1913	0.0182	[0.1556, 0.2271]	–	–	–
DIW	0.5744	0.0276	[0.5202, 0.6285]	2.0178	0.1672	[1.6902, 2.3455]
DB-XII	0.2858	0.0267	[0.2335, 0.3381]	2.0128	0.1892	[1.6419, 2.3836]
DLogL	0.8846	0.0466	[0.7933, 0.9759]	2.2880	0.1786	[1.9379, 2.6381]

Table 14. The goodness of fit statistic for data set II.

No.	**Observed**	**Expected Frequency**						
X	**Frequency**	**DBiExII**	**DEx**	**DR**	**DIR**	**DBe**	**DBH**	**DPa**
0	188	186.7622	196.5842	122.5868	186.2094	171.2839	217.8157	221.0309
1	83	88.0289	77.3084	153.0275	95.8946	108.7492	59.7847	50.3079
2	36	32.9622	30.4022	43.8937	22.5578	32.8374	23.5898	19.9369
3	14	11.1415	11.9559	4.3308	8.3142	8.4488	10.8491	10.0986
4	2	3.5529	4.7018	0.1589	3.9247	2.0489	5.4276	5.8896
5	1	1.5523	3.0475	0.0023	7.0993	0.6318	6.5331	16.7361
Total	**324**	**324**	**324**	**324**	**324**	**324**	**324**	**324**
$-l$		**355.9001**	357.8779	404.4854	366.2275	360.5431	369.7014	387.8939
AIC		**715.8001**	717.7558	810.9708	734.4551	723.0862	741.4027	777.7877
CAIC		**715.8375**	717.7682	810.9832	734.4675	723.0986	741.4152	777.8001
$Ø^2$		**2.1769**	5.0847	67.3905	16.1573	11.1293	27.3908	57.9946
p value		**0.3367**	0.1657	<0.0001	0.0011	0.0038	<0.0001	<0.0001

From Tables 14 and 15, the DBiExII distribution is the best model among all tested models. Figure 6 shows the profile of the log-likelihood function for data set II, and from this figure, it is clear that the parameters are unimodal functions.

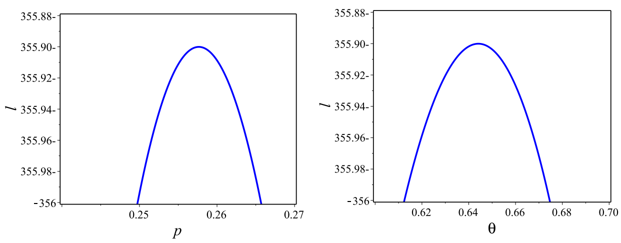

Figure 6. The profile of l function for the model parameters based on data set II.

Table 15. The goodness of fit statistic for data set II "Contin".

No.	Observed	Expected Frequency					
X	Frequency	DBiExII	GDEx	DGExII	DIW	DsB-XII	DLogL
0	188	186.7622	186.2301	187.1074	186.0956	188.0009	184.5744
1	83	88.0289	88.7678	88.5925	96.4464	93.2129	96.0252
2	36	32.9622	32.8333	31.7641	22.4638	24.9509	24.7251
3	14	11.1415	10.9727	10.9246	8.2235	8.7052	8.7293
4	2	3.5529	3.5407	3.7126	3.8629	3.7882	3.9026
5	1	1.5523	1.6554	1.8988	6.9078	5.3419	6.0434
Total	**324**	**324**	**324**	**324**	**324**	**324**	**324**
$-l$		**355.9001**	356.0700	356.4021	366.2218	362.6942	363.8550
AIC		**715.8001**	716.1400	716.8042	736.4436	729.3883	731.7100
CAIC		**715.8375**	716.1774	716.8416	736.4810	729.4257	731.7474
Ø^2		**2.1769**	2.4604	3.0032	19.7147	10.5252	12.0788
p **value**		**0.3367**	0.2922	0.2228	<0.0001	0.0052	0.0024

Figure 7 shows the estimated PMFs for data set II.

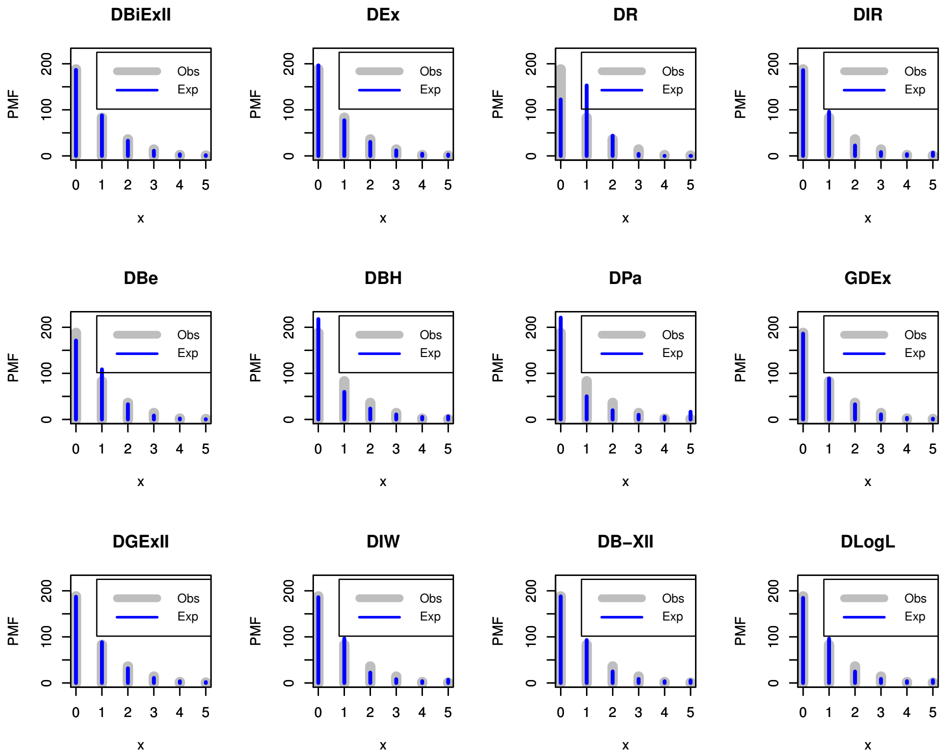

Figure 7. The estimated PMFs for data set II.

Table 16 lists some information around data set II under the DBiExII model.

Table 16. Some descriptive statistics for data set II.

Type ↓ Measures →	Mean	Variance	IxD	Skewness	Kurtosis
Theoretical	0.64798	0.88666	1.36834	1.83365	7.44446
Empirical	0.64814	0.84795	1.30828	1.52091	5.23804

From Table 16, the theoretical mean, variance, IxD and skewness are close to empirical ones. Data set II suffers from overdispersion phenomena, and most of the distribution is at the left with leptokurtic character.

6.3. Bayesian Estimation for Data Sets I and II

In this sub-section, we calculate Bayes estimates and HPD credible intervals for the unknown parameters of DBiExII distribution under the real data set I and II. Due to the lack of prior information on these parameters for the given data sets, we assume the uniform NIPs for the unknown parameters of the model. Doing the same as we did in the simulation study in Section 5, we have computed the Bayes estimates with their associated posterior standard errors (PStd-ers) and HPD credible intervals for the unknown parameters p and θ under the data sets I and II. The resulted values are tabulated in Table 17.

Table 17. Bayes estimates for data sets I and II.

Data Set	p		θ	
	Estimate (PStd-er)	HPD (Width)	Estimate (PStd-er)	HPD (Width)
Data set I	0.5970 (0.0179)	[0.5626, 0.6326] (0.0700)	0.6423 (0.0425)	[0.5639, 0.7278] (0.1638)
Data set II	0.2578 (0.0159)	[0.2251, 0.2876] (0.0624)	0.6443 (0.0412)	[0.5662, 0.7278] (0.1616)

For both data sets, when we have compared the Bayes estimates with the results of MLEs, we found that both methods work well, but Bayesian estimation with NIPs is superior than method of maximum likelihood in terms of estimation errors and length of the confidence intervals.

7. Conclusions

The present article introduced a new two-parameter discrete model, called discrete binomial exponential II distribution. We have discussed several important properties of the proposed model. One of the key advantages of this newly developed model is that it can model a variety of data (over-, equi-, and under-dispersed, positively skewed, leptokurtic, and increasing failure time data). Two well-known estimation techniques, the method of maximum likelihood and Bayesian estimation, have been used to derive the point and interval estimators of the unknown parameters of the DBiExII distribution.

A detailed Monte Carlo simulation study has been performed to test the behaviour of different point and interval estimators with respect to sample size and parametric values. The results of this numerical study show that both estimation methods work satisfactorily, but Bayesian estimation under beta priors dominates the method of maximum likelihood in terms of estimation errors. In the end, the usefulness of the new distribution is illustrated by means of two real data sets to prove its versatility in practical applications. We, therefore, believe that the DBiExII distribution may be a better alternative to some popular existing discrete models and may be widely applicable for modelling real-life data sets in various fields. With regard to future work, the researchers may use the new model to propose a bi-variate distribution based on the shock model approach for modelling bi-variate data. In addition, a regression model and a first-order integer-valued auto-regressive process can be studied in detail.

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Data Availability Statement: The data sets are available in the paper.

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