axioms

MDPI

Article

A Unit Half-Logistic Geometric Distribution and Its Application in Insurance

Ahmed T. Ramadan [1,†], Ahlam H. Tolba [2,*,†] and Beih S. El-Desouky [2,†]

1 Department of Basic Sciences, High Raya Institute, Damietta 34511, Egypt
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 33516, Egypt
* Correspondence: a.hamdy6@yahoo.com
† These authors contributed equally to this work.

Abstract: A new one parameter distribution recently was proposed for modelling lifetime data called half logistic-geometric (HLG) distribution. In this paper, appropriate transformation is considered for HLG distribution and a new distribution is derived called unit half logistic-geometric (UHLG) distribution for modelling bounded data in the interval (0, 1). Some important statistical properties are investigated with a closed form quantile function. Some methods of parameter estimation are introduced to evaluate the distribution parameter and a simulation study is introduced to compare these different methods. A real data application in the insurance field is introduced to show the flexibility of the new distribution modelling such data comparing with other distributions.

Keywords: half logistic-geometric distribution; unit distributions; parameter estimation methods; regression models; risk survey data

Citation: Ramadan, A.T.; Tolba, A.H.; El-Desouky, B.S. A Unit Half-Logistic Geometric Distribution and Its Application in Insurance. *Axioms* **2022**, *11*, 676. https://doi.org/10.3390/axioms11120676

Academic Editor: Delfim F. M. Torres

Received: 8 August 2022
Accepted: 9 September 2022
Published: 28 November 2022

1. Introduction

Modelling data sets bounded in the interval (0, 1) has become very important in recent times and is used in many fields to deal with survival and failure rates of products, see [1,2]. Therefore, many unit distributions bounded in the interval (0, 1) arise because of its flexibility dealing with such probabilistic models. In addition, many fields such as medical, actuarial and finance sciences are in desperate need of these kinds of distributions. As a result, many researchers have proposed unit distributions. For instance, Abd El-Monsef et al. [3] proposed a new two-parameter unit-omega distribution with flexible probability density function (pdf) and hazard function. Moreover, Altun et al. [4] studied a distribution called unit-improved second-degree Lindley distribution modelling data in the interval (0, 1). Moreover, Altun et al. [5] proposed a more flexible model, log-Bilal distribution, as an alternative to beta and unit-Lindley regression models. Additionally, Bayes et al. [6] proposed a new regression model for the relationship between one or more covariates and a response beta variable conditional mean. Cordeiro et al. [7] recently offered some statistical methods of lifetime and survival models.

Both beta [8] and Kumaraswamy [9] are common distributions for modelling such data in the unit interval (0, 1) and as a result, beta and Kumaraswamy [10] regression models have been extended to study the behaviour of variables in the presence of covariance. As an alternative to the beta regression model, Gómez-Déniz et al. [11] proposed a new Log–Lindley distribution model with useful applications in econometric analysis and actuarial settings. In addition, Korkmaz et al. [12] modified the Burr-XII distribution and obtained a new two-parameter distribution on the unit interval called the unit Burr-XII distribution and showed that it had better modelling capabilities than other competing models. Moreover, Mazucheli et al. [13] not only proposed a unit-Weibull two-parameter distribution, modelling data on the unit interval (0, 1), and proposed some useful statistics for this distribution, but also they recently (in 2020) considered the unit-Weibull distribution [14] as an alternative to the Kumaraswamy distribution for the modelling of quantiles and demonstrated the suitability for modelling quantiles in accounting, health and other social sciences. Moreover, [15] discussed modelling the COVID-19 mortality rate with a new versatile modification of the log-logistic distribution and [16] introduced an extended Cosine generalized family of distributions for reliability modelling: characteristics and applications with simulation study. Moreover, Mazucheli et al. [17] introduced the unit-Lindley distribution and investigated some of important statistical properties. In addition, Mitnik and Baek [10] presented two median-dispersion re-parameterizations of the Kumaraswamy distribution to facilitate its use in regression models. Furthermore, Mousa et al. [18] presented a new regression model named as the unit gamma distribution as an alternative to the beta regression model; [19,20] presented the competing risks models and regression competing risks models with Weibull lifetime distributions. Tadikamalla [21] proposed more flexible unit-gamma distribution aimed at modelling data in the unit interval (0, 1) too.

Recently (in 2020), Liu and Balakrishnan [22] proposed a new simple one-parameter half logistic-geometric (HLG) distribution useful for analyzing lifetime data with some interesting properties. The probability density function (pdf) and cumulative distribution function (cdf) of HLG distribution are defined, respectively, as

$$g(y) = \frac{\theta}{\theta + (2-\theta)e^{-y}}, \quad y > 0, 0 < \theta < 1, \tag{1}$$

$$G(y) = \frac{\theta(1-e^{-y})}{\theta + (2-\theta)e^{-y}}, \quad y > 0, 0 < \theta < 1. \tag{2}$$

The aim of this paper is to derive a new flexible distribution modelling data in the unit interval (0, 1). The negative exponential function transformation was used to derive the new distribution named unit half logistic-geometric (UHLG) distribution with some attractive properties: *(i)* statistical functions of the UHLG distribution have closed form expressions; *(ii)* statistical properties of the UHLG distribution were derived in simple expressions; *(iii)* the UHLG distribution presents more flexibility, dealing with bounded unit interval data more than other distributions, as shown later in Section 6; *(iv)* because of its flexibility, a new regression model was introduced considering parameterizing the UHLG distribution in terms of its quantile function in a closed form expression.

The remainder of this paper is summarized as follows: In Section 2, the UHLG distribution is introduced with some important statistical properties. Some methods of parameter estimations are introduced in Section 3 to evaluate the unknown parameter and a simulation study is introduced to compare these different methods in Section 4. In addition, a regression model is introduced as an alternative to some other regression models in Section 5. Finally, an application with real data about risk survey is given in Section 6.

2. Unit Half Logistic-Geometry Distribution

In this section, the UHLG distribution is derived and some important functions are illustrated.

2.1. Cumulative and Density Functions of UHLG Distribution

In Equation (2), replacing e^{-y} with x, we obtain a new random variable X following the UHLG distribution with cdf as follows

$$F(x) = 1 - \frac{\theta(1-x)}{\theta + (2-\theta)x}, \quad 0 < x < 1, \theta > 0. \tag{3}$$

As a result, the quantile function $Q(u|\theta)$ of UHLG distribution can be given as

$$Q(u|\theta) = \frac{u\theta}{2 - 2u + u\theta}, \quad 0 < u < 1, \theta > 0. \tag{4}$$

Taking the first derivative of the cdf given in Equation (3) with respect to x, the pdf is obtained as follows

$$f(x) = \frac{2\theta}{(\theta + (2-\theta)x)^2}, \quad 0 < x < 1, \theta > 0. \tag{5}$$

Theorem 1. *The pdf of the UHLG distribution is*

(i) *decreasing function if* $0 < \theta < 2$,
(ii) *increasing function if* $\theta > 2$,
(iii) *constant when* $\theta = 2$.

Proof. The first derivative of the pdf with respect to x is given by

$$\frac{df}{dx} = -\frac{4\theta(\theta-2)}{(x(\theta-2)-\theta)^3},$$

and it is clear that

(i) When $0 < \theta < 2$, the first derivative is negative, which implies that the pdf of the UHLG distribution is decreasing;
(ii) When $\theta > 2$, the first derivative is positive, which implies that the pdf of the UHLG distribution is increasing;
(iii) Lastly, when $\theta = 2$, the pdf of the UHLG distribution is constant and equal to 1.

□

Figures 1 and 2 show the cdf and pdf functions, respectively, of the UHLG distribution at different values of θ.

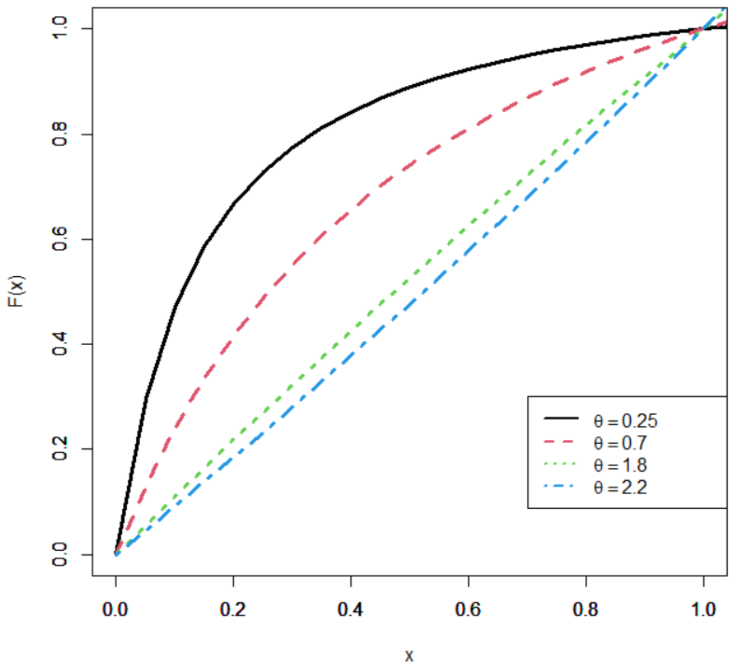

Figure 1. cdf of UHLG distribution at different values of θ.

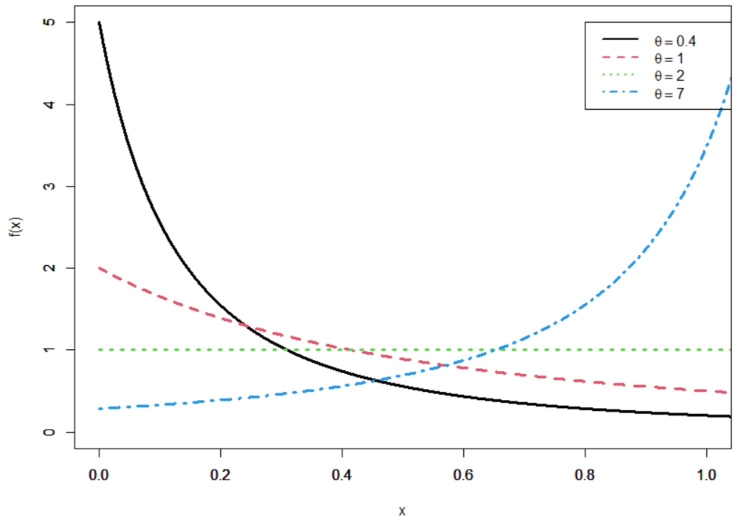

Figure 2. pdf of UHLG distribution at different values of θ.

2.2. Survival and Hazard Functions of the UHLG Distribution

The survival and the hazard rate functions of the UHLG distribution can take the next formulas, respectively,

$$S(x) = 1 - F(x) = \frac{\theta(1-x)}{\theta + (2-\theta)x}, \quad 0 < x < 1, \theta > 0, \tag{6}$$

$$H(x) = \frac{f(x)}{S(x)} = \frac{2}{(x-1)((\theta-2)x-\theta)}, \quad 0 < x < 1, \theta > 0. \tag{7}$$

The next theorem shows the different shapes of the UHLG hazard function with respect to θ.

Theorem 2. *The hazard rate function of the UHLG distribution is*

(i) *bathtub (U-shaped) function if* $0 < \theta < 1$,
(ii) *increasing function if* $\theta \geq 1$.

Proof. The first derivative of the hazard function with respect to x is given by

$$\frac{dH}{dx} = \frac{4\theta(1-x) - 4(1-2x)}{(x-1)^2((2-\theta)x+\theta)^2}.$$

Clearly, $\frac{dH}{dx}$ and $\psi(\theta) = 4\theta(1-x) - 4(1-2x)$ have the same signs.

The function $\psi(\theta)$ has a root equal to $\frac{1-\theta}{2-\theta}$. Then:

(i) When $0 < \theta < 1$, the sign of $\psi(\theta)$ changes from negative to positive, which implies that the function $H(x)$ is decreasing first and increasing second (bathtub shape) with minimum value equal to $\frac{1-\theta}{2-\theta}$;
(ii) When $\theta \geq 1$, the sign of $\psi(\theta)$ is always positive, which implies that the function $H(x)$ is increasing.

This completes the proof. □

Figures 3 and 4 show the cdf and pdf functions respectively of the UHLG distribution at different values of θ.

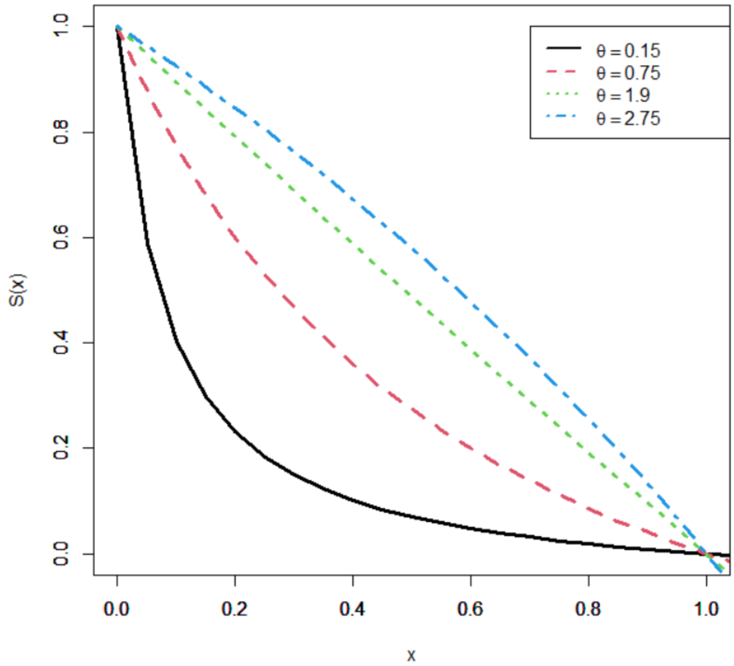

Figure 3. Survival function of UHLG distribution at different values of θ.

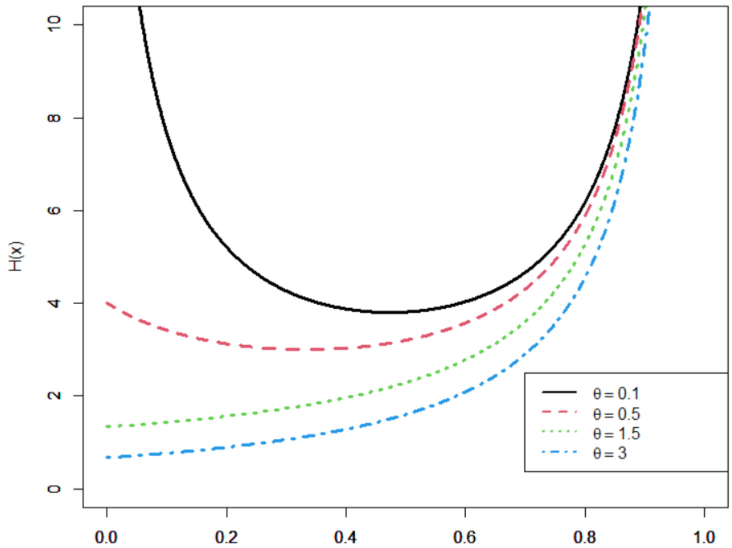

Figure 4. Hazard rate function of UHLG distribution at different values of θ.

2.3. Moments

Consider X as a random variable following the UHLG distribution with pdf given in Equation (5). Then, the rth moment about zero, μ_r', can be given as

$$\begin{aligned}\mu_r' &= \int_0^1 x^r f(x)dx = \int_0^1 x^r \frac{2\theta}{(\theta + (2-\theta)x)^2}dx \\ &= \frac{2}{\theta}\int_0^1 \frac{x^r}{(1+\left(\frac{2-\theta}{\theta}\right)x)^2}dx.\end{aligned}$$

Using Equation (3.194.1) in [23], where $u = 1, \mu = r+1, \beta = \frac{2-\theta}{\theta}$, and $\nu = 2$, the final form of μ_r' is given by

$$\mu_r' = \frac{2}{\theta(r+1)} \quad {}_2F_1\left(2, r+1; r+2; \frac{\theta-2}{\theta}\right), \tag{8}$$

where ${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{s=0}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)s!} z^s$ is the hypergeometric function.

The first four moments about zero can be given by putting $r = 1, 2, 3$, and 4 in Equation (8).

Coefficients of skewness and kurtosis can be derived from moments about zero as follows

$$skewness = \frac{\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}{(\sqrt{\mu_2' - (\mu_1')^2})^3},$$

$$kurtosis = \frac{\mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4}{(\mu_2' - (\mu_1')^2)^2}.$$

Figure 5 shows the mean and the variance plots and it is clear that the mean is always increasing but the variance is increasing when $0 \leq \theta < 2$ and decreasing otherwise.

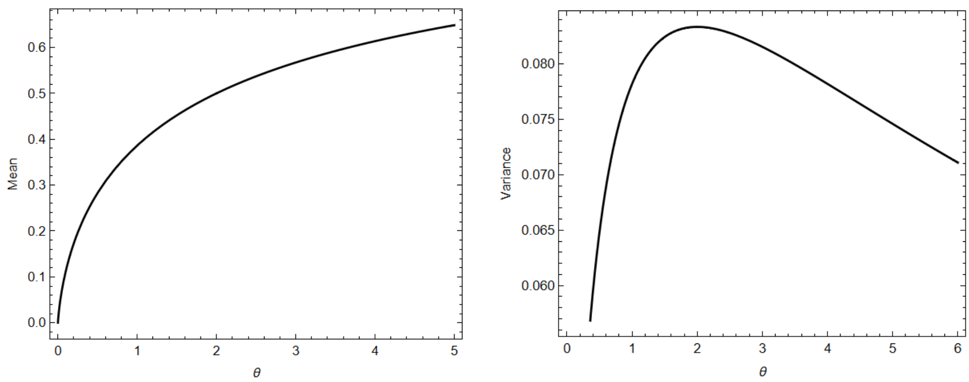

Figure 5. The mean and the variance plots for UHLG distribution.

Moreover, Figure 6 shows the skewness and the kurtosis plots and it is clear that the skewness is positive when $0 \leq \theta < 2$ and negative otherwise and the kurtosis is positive when $0 \leq \theta < 0.1$ and negative otherwise.

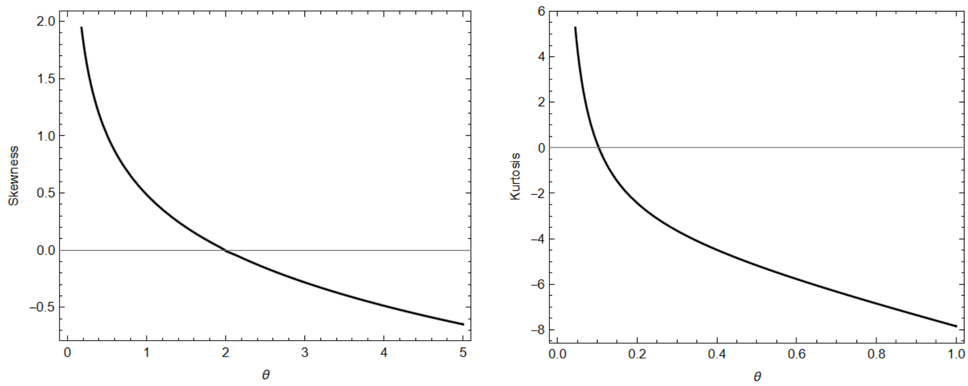

Figure 6. The skewness and the kurtosis plots for UHLG distribution.

2.4. Incomplete Moments and Related Measures

In this subsection, the rth incomplete moment, $m_r(y)$, is introduced with some related measures, for instance, mean deviation about mean and median and Bonferroni and Lorenz curves.

The rth incomplete moment of the UHLG distribution is given by

$$m_r(y) = \int_0^y x^r f(x)dx = \int_0^y x^r \frac{2\theta}{(\theta + (2-\theta)x)^2}dx$$
$$= \frac{2}{\theta}\int_0^y \frac{x^r}{(1+\left(\frac{2-\theta}{\theta}\right)x)^2}dx.$$

Using Equation (3.194.1) in [23], where $u = 1, \mu = r+1, \beta = \frac{2-\theta}{\theta}$, and $\nu = 2$, the final form of $m_r(y)$ is given by

$$m_r(y) = \frac{2y^{r+1}}{\theta(r+1)} \quad {}_2F_1\left(2, r+1; r+2; \frac{(\theta-2)y}{\theta}\right), \tag{9}$$

where $0 < y < 1$.

Some important statistical measures are defined based on the moments and the incomplete moments, such as the mean deviation about the mean $D(\mu_1')$ and about the median D(M). These measures can be expressed as

$$D(\mu_1') = 2\mu_1' F(\mu_1') - 2m_1(\mu_1'),$$

and

$$D(M) = \mu_1' - 2m_1(M),$$

where $M = Q(0.5)$.

Another related measure is the mean residual life (MRL), which is defined as the expected value of the remaining lifetimes after a fixed time point t. It can be defined in terms of the moments and incomplete moments as

$$mrl(t) = \frac{\mu_1' - m_1(t)}{\bar{F}(t)} - t.$$

Moreover, the mean inactivity time which represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$ is given by

$$mit(t) = t - \frac{m_1(t)}{F(t)}.$$

Other important applications of the moments and incomplete moments are related to Bonferroni and Lorenz curves of X, which can be defined by

$$B(\pi) = \frac{m_1(q)}{\pi\mu_1'},$$

and

$$L(\pi) = \frac{m_1(q)}{\mu_1'},$$

respectively, where $q = Q(\pi)$ follows from Equation (4) for a given probability π. The importance of Bonferroni and Lorenz curves is due to the wide variety of the potential applications of these curves. These curves can be applied in financial studies, medicine and insurance.

Other measure can be defined based on the moments and incomplete moments. For a complete list of these measures see [7].

2.5. Stress Strength Parameter

According to [24,25], the reliability, R, of a component arises when its strength is greater than its stress. Let $X \sim UHLG(\theta_1)$ represent the strength of the component and

$Y \sim UHLG(\theta_2)$ represent its stress. It is said that the component is functioning when the condition $Y < X$ is held and then its reliability is given by

$$\begin{aligned} R = P(Y < X) &= \int_0^1 f_X(x|\theta_1)F_Y(x|\theta_2)dx \\ &= \int_0^1 \frac{2}{\theta_1}\frac{1}{\left(1+\frac{2-\theta_1}{\theta_1}x\right)^2}\left(1-\frac{1-x}{1+\frac{2-\theta_2}{\theta_2}x}\right)dx \\ &= 1-\frac{2}{\theta_1}\int_0^1 (1-x)\left(1-\frac{\theta_1-2}{\theta_1}x\right)^{-2}\left(1-\frac{\theta_2-2}{\theta_2}x\right)^{-1}dx. \end{aligned}$$

Using Equation (3.211) in [23], where $\lambda = 1, \mu = 2, u = \frac{\theta_1-2}{\theta_1}, e = 2, \nu = \frac{\theta_2-2}{\theta_2}$ and $\sigma = 1$, the integral is given by

$$R = \frac{\theta_1-1}{\theta_1}F_1(1,2,1,3;\frac{\theta_1-2}{\theta_1},\frac{\theta_2-2}{\theta_2}), \tag{10}$$

where $F_1(a,b,c,d;x,y) = \frac{\Gamma(d)}{\Gamma(a)\Gamma(d-a)}\int_0^1 \frac{u^{a-1}(1-u)^{d-a-1}}{(1-ux)^b(1-uy)^c}du$ is the Appell hypergeometric function (see [26]).

2.6. Stochastic Ordering

The stochastic order arises when we have two independent continuous random variables, X_1, X_2, such that $X_1 < X_2$; we say that X_2 is stochastically smaller than X_1, $X_2 <_{lr} X_1$ if $\frac{f_1(x)}{f_2(x)}$ is a non-decreasing function of x. For more details see ([22,27,28]).

Proposition 1. *Let X_1, X_2 be two independent random variables such that $X_1 \sim UHLG(\theta_1)$, $X_2 \sim UHLG(\theta_2)$. If $\theta_1 \geq \theta_2$, then $X_2 <_{lr} X_1$.*

Proof. The first derivative of $\frac{f_1(x)}{f_2(x)}$ is given by

$$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{4\theta_1(\theta_1-\theta_2)((\theta_2-2)x-\theta_2)}{\theta_2((\theta_1-2)x-\theta_1)^3}.$$

It is obvious that if $\theta_1 \geq \theta_2$, then $\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right)$ is non-negative and as a sequence, $\frac{f_1(x)}{f_2(x)}$ is a non-decreasing function of x and this completes the proof. □

3. Parameter Estimation

In this section, six methods of estimations are used to estimate θ, the parameter of UHLG distribution. These methods are maximum likelihood estimation method (MLE), Bayesian estimation method (BE), Cramer–Von-Mises method (CVME), least squares method (LSE), method of moments (MME), weighted least squares method (WLSE).

3.1. Maximum Likelihood Estimation Method (MLE)

Given a random sample $(x_1, x_2, \ldots, x_n)$ from the UHLG distribution, the likelihood estimation function, L, can be given as follows

$$\begin{aligned} L &= \prod_{i=1}^n f(x_i;\theta) \\ &= \frac{(2\theta)^n}{\prod_{i=1}^n(\theta+(2-\theta)x_i)^2}, \end{aligned} \tag{11}$$

and the logarithmic likelihood function is

$$l = \log[L] = n\log[2\theta] - 2\sum_{i=1}^{n}(\log[\theta + (2-\theta)x_i]), \tag{12}$$

and the first derivatives of l with respect to θ are given by

$$\frac{dl}{d\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\frac{1-x_i}{\theta + (2-\theta)x_i}. \tag{13}$$

3.2. Bayesian Estimation Method

The Bayesian estimation (BE) method is used to fit the probability model to a set of data and summarize the results by the probability distribution of the model parameters. The data come from the prior distribution and the likelihood, L, function and give the posterior distribution.

Suppose that we have a non-informative prior distribution $u(\theta) = \frac{1}{\theta}$.

Therefore, the posterior distribution function, $g(\theta)$, is given by

$$\begin{aligned} g(\theta) &= \frac{u(\theta) \quad L}{\int_0^1 u(\theta) \quad L \quad d\theta} \\ &= \frac{\frac{1}{\theta}\left(\frac{(2\theta)^n}{\prod_{i=1}^{n}(\theta+(2-\theta)x_i)^2}\right)}{\int_0^1 \frac{1}{\theta}\left(\frac{(2\theta)^n}{\prod_{i=1}^{n}(\theta+(2-\theta)x_i)^2}\right)d\theta}. \end{aligned} \tag{14}$$

According to the squared error loss function, Bayes estimate, $\hat{\theta}$, is the posterior mean of θ with pdf given in Equation (14) as follows

$$\begin{aligned} \hat{\theta} &= \int_0^1 \theta \quad g(\theta) \quad d\theta \\ &= \int_0^1 \frac{\frac{(2\theta)^n}{\prod_{i=1}^{n}(\theta+(2-\theta)x_i)^2}}{\int_0^1 \frac{1}{\theta}\left(\frac{(2\theta)^n}{\prod_{i=1}^{n}(\theta+(2-\theta)x_i)^2}\right)d\theta}d\theta. \end{aligned} \tag{15}$$

Since these integrals cannot be obtained analytically, alternative methods are assumed to obtain the estimate. For this purpose, the Markov Chain Monte Carlo (MCMC) method is used. The Metropolis Hastings algorithm [29] is a modification version of the MCMC technique and can be used for this purpose.

The posterior distribution of θ can be written as

$$\pi(\theta) \propto \theta^{n-1}\prod_{i=1}^{n}(\theta + (2-\theta)x_i)^{-2}.$$

The following algorithm uses Metropolis Hastings steps with a normal proposal for updating the parameter θ and then obtains the Bayesian estimate of θ.

- Step 1: Start with an arbitrary initial value $\theta^{(0)}$ where $g(\theta^{(0)}|x) > 0$ and set $k = 1$.
- Step 2: Generate a proposal θ^* from normal distribution, i.e., $q(\theta) = N(\theta^{(k-1)}, var(\theta^{(k-1)}))$.
- Step 3: Calculate the acceptance probability function

$$\rho = Min\left(1, \frac{\pi(\theta^*)q(\theta^{(k-1)})}{\pi(\theta^{(k-1)})q(\theta^*)}\right).$$

- Step 4: Generate $U \sim uniform(0,1)$.
- Step 5: If $U \leq \rho$ put $\theta^{(k)} = \theta^*$; otherwise put $\theta^{(k)} = \theta^{(k-1)}$.

- Step 6: Repeat steps (2) and (5) N times to have $\theta^{(k)}, k = 1, \ldots, N$.

Using the simulated posterior sample, the Bayesian estimate of θ is given as: $\hat{\theta} = \frac{1}{N-N_0}\sum_{k=N_0+1}^{N}\theta^{(k)}$ where N_0 represents the number of burn-in periods of Markov chain discarded to remove the effect of the selected initial value of θ. For more details, see [30].

3.3. Cramer-Von-Mises Method

In the CVME method the distance between the cumulative distribution function and the experimental distribution function is reduced, which can be summarized as follows

$$CMV(\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_{(i)},\theta) - \frac{2i-1}{2n}\right)^2, \tag{16}$$

and the first derivative with respect to θ is given by

$$\begin{aligned}\frac{\partial CMV(\theta)}{\partial\theta} &= 2\sum_{i=1}^{n}\left(F(x_{(i)},\theta) - \frac{2i-1}{2n}\right)F'_{\theta}(x_{(i)},\theta) = 0\\ &= 2\sum_{i=1}^{n}\left(1 - \frac{\theta(1-x_{(i)})}{\theta+(2-\theta)x_{(i)}} - \frac{2i-1}{2n}\right)\\ &\left(\frac{\theta+(2-\theta)x_{(i)}(x_{(i)}-1) - 2\theta x_{(i)}(x_{(i)}-1)(1-\theta)}{(\theta+(2-\theta)x_{(i)})^2}\right) = 0.\end{aligned} \tag{17}$$

$\hat{\theta}$ is the value of θ that minimizes Equation (17).

3.4. Least Squares Method

In this method, the sum of the offsets or residuals of points from the plotted curve is minimized which can be summarized as follows

$$LS(\theta) = \sum_{i=1}^{n}\left(F(x_{(i)},\theta) - \frac{i}{n+1}\right)^2, \tag{18}$$

and the first derivative with respect to θ is given by

$$\begin{aligned}\frac{\partial LS(\theta)}{\partial\theta} &= 2\sum_{i=1}^{n}\left(F(x_{(i)},\theta) - \frac{i}{n+1}\right)F'_{\theta}(x_{(i)},\theta) = 0\\ &= 2\sum_{i=1}^{n}\left(1 - \frac{\theta(1-x_{(i)})}{\theta+(2-\theta)x_{(i)}} - \frac{i}{n+1}\right)\\ &\left(\frac{\theta+(2-\theta)x_{(i)}(x_{(i)}-1) - 2\theta x_{(i)}(x_{(i)}-1)(1-\theta)}{(\theta+(2-\theta)x_{(i)})^2}\right) = 0.\end{aligned} \tag{19}$$

$\hat{\theta}$ is the value of θ that minimizes Equation (19).

3.5. Method of Moments

This method can be obtained by equating the population moments with the sample moments as follows

$$\begin{aligned}\frac{\sum_{i=1}^{n} x_i}{n} &= \mu'_1\\ &= \frac{1}{\theta} \quad {}_2F_1\left(2, 2; 3; \frac{\theta-2}{\theta}\right).\end{aligned} \tag{20}$$

3.6. Weighted Least Squares Method

This method is assumed to be a generalization to the LSE method and is given as follows

$$WLS(\theta) = \sum_{i=1}^{n} \frac{(n+2)(n+1)^2}{i(n-i+1)} \left(F(x_{(i)}, \theta) - \frac{i}{n+1} \right)^2, \tag{21}$$

and the first derivative with respect to θ is given by

$$\begin{aligned} \frac{\partial WLS(\theta)}{\partial \theta} &= 2 \sum_{i=1}^{n} \frac{(n+2)(n+1)^2}{i(n-i+1)} \left(F(x_{(i)}, \theta) - \frac{i}{n+1} \right) F'_{\theta}(x_{(i)}, \theta) = 0 \\ &= 2 \sum_{i=1}^{n} \frac{(n+2)(n+1)^2}{i(n-i+1)} \left(1 - \frac{\theta(1-x_{(i)})}{\theta + (2-\theta)x_{(i)}} - \frac{i}{n+1} \right) \\ & \left(\frac{\theta + (2-\theta)x_{(i)}(x_{(i)}-1) - 2\theta x_{(i)}(x_{(i)}-1)(1-\theta)}{(\theta + (2-\theta)x_{(i)})^2} \right) = 0. \end{aligned} \tag{22}$$

$\hat{\theta}$ is the value of θ that minimizes Equation (22).

Equations (13), (15), (17), (19), (20) and (22) have no analytic closed form when equating by zero, so numerical methods are used to give solutions.

4. Simulation Study

In this section, a simulation study is performed to show the effectiveness of the previous estimation methods of $\hat{\theta}$. All observations follow UHLG distribution. In this study, some statistics of $\hat{\theta}$ including mean estimated (ME), average bias (AB) and mean squared error (MSE) using previous estimation methods are calculated. Different values of θ are assumed here and the study is performed 2000 times at samples sizes 20, 70, 100, 150 and 200.

The statistical measurements ME, AB and MSE can be given, respectively, as follows

$$\text{ME} = \frac{1}{2000} \sum_{i=1}^{2000} \hat{\theta}_i, \qquad \text{AB} = \frac{1}{2000} \sum_{i=1}^{2000} (\theta - \hat{\theta}_i), \qquad \text{and} \qquad \text{MSE} = \frac{1}{2000} \sum_{i=1}^{2000} (\theta - \hat{\theta}_i)^2.$$

Tables 1 and 2 show some statistics of $\hat{\theta}$ including ME, AB and MSE using the previous estimation methods.

Tables 1 and 2 show the following notes in general:

- ME converges to θ when the sample size, n, increases;
- AB tends to zero when the sample size, n, increases;
- MSE decreases when the sample size, n, increases;
- In general, MLE and BE methods are the best estimation methods compared with the previous methods.

Table 1. Some statistics of $\hat{\theta}$ including ME, AB and MSE using MLE, BE and CVME methods.

θ	n	$MLE_{\hat{\theta}}$			$BE_{\hat{\theta}}$			CVMEs		
		ME	AB	MSE	ME	AB	MSE	ME	AB	MSE
0.1	20	0.115	0.015	0.002	0.109	0.009	0.004	0.148	0.049	0.058
	70	0.103	0.003	0.001	0.106	0.006	0.001	0.150	0.050	0.048
	100	0.102	0.002	0.001	0.101	0.001	0.000	0.147	0.047	0.046
	150	0.101	0.001	0.001	0.102	0.002	0.000	0.162	0.062	0.042
	200	0.101	0.001	0.001	0.103	0.003	0.000	0.150	0.050	0.036
0.5	20	0.573	0.073	0.052	0.539	0.039	0.065	0.535	0.035	0.075
	70	0.516	0.016	0.011	0.532	0.032	0.020	0.549	0.049	0.065
	100	0.509	0.009	0.008	0.507	0.007	0.009	0.525	0.025	0.051
	150	0.503	0.003	0.005	0.509	0.009	0.007	0.531	0.031	0.044
	200	0.504	0.004	0.004	0.515	0.015	0.006	0.543	0.043	0.034
0.9	20	1.031	0.131	0.168	0.967	0.067	0.215	0.911	0.011	0.096
	70	0.929	0.029	0.037	0.951	0.051	0.067	0.932	0.032	0.087
	100	0.916	0.016	0.026	0.911	0.011	0.035	0.927	0.027	0.074
	150	0.905	0.005	0.016	0.914	0.014	0.027	0.942	0.042	0.062
	200	0.907	0.007	0.013	0.924	0.024	0.019	0.922	0.022	0.055

Table 2. Some statistics of $\hat{\theta}$ including ME, AB and MSE using LSE, MME and WLSE methods.

θ	n	LSEs			MMEs			WLSEs		
		ME	AB	MSE	ME	AB	MSE	ME	AB	MSE
0.1	20	0.149	0.049	0.107	0.294	0.194	0.352	0.144	0.044	0.116
	70	0.155	0.055	0.088	0.295	0.195	0.327	0.146	0.046	0.089
	100	0.163	0.063	0.076	0.256	0.156	0.297	0.161	0.061	0.089
	150	0.151	0.051	0.069	0.245	0.145	0.237	0.161	0.061	0.066
	200	0.150	0.050	0.053	0.271	0.171	0.211	0.161	0.061	0.042
0.5	20	0.524	0.024	0.074	0.979	0.479	1.724	0.625	0.125	0.312
	70	0.538	0.038	0.067	0.890	0.390	1.702	0.616	0.116	0.302
	100	0.537	0.037	0.061	0.938	0.438	1.664	0.625	0.125	0.291
	150	0.540	0.040	0.059	0.859	0.359	1.568	0.621	0.121	0.285
	200	0.542	0.042	0.056	0.915	0.415	1.497	0.614	0.114	0.264
0.9	20	0.909	0.009	0.083	1.225	0.325	2.049	1.065	0.165	0.608
	70	0.930	0.030	0.082	1.131	0.231	2.049	1.084	0.184	0.582
	100	0.920	0.020	0.079	1.164	0.264	1.962	1.064	0.164	0.571
	150	0.935	0.035	0.074	1.098	0.198	1.924	1.080	0.180	0.554
	200	0.900	0.000	0.063	1.085	0.185	1.852	1.084	0.184	0.521

5. Unit Half Logistic-Geometry Quantile Regression Model

In this section, a new regression model for bounded unit intervals is introduced as an alternative to some other regression models such as log Bilal, beta and Kumarswammy regression models.

Consider the re-parameterization,

$$\theta = \frac{2\mu(\tau - 1)}{\tau(\mu - 1)}, \tag{23}$$

where $\mu = Q(\tau|\theta)$.

Substituting from Equation (23) into Equations (3) and (5), we obtain

$$F(x) = 1 - \frac{\frac{2\mu(\tau-1)}{\tau(\mu-1)}(1-x)}{\frac{2\mu(\tau-1)}{\tau(\mu-1)} + \left(2 - \frac{2\mu(\tau-1)}{\tau(\mu-1)}\right)x}, \tag{24}$$

and

$$f(x) = \frac{\frac{4\mu(\tau-1)}{\tau(\mu-1)}}{\left(\frac{2\mu(\tau-1)}{\tau(\mu-1)} + \left(2 - \frac{2\mu(\tau-1)}{\tau(\mu-1)}\right)x\right)^2}, \tag{25}$$

where $0 < x < 1$ and $0 < \mu < 1$.

Let $X_i, i = 1, 2, \ldots, n$ be n independent random variables such that $X_i \sim UHLG(\mu_i; \tau)$. The UHLG quantile regression is given as

$$g(\mu_i) = \delta^T t_i, \tag{26}$$

where $t_i = (1, t_{1i}, t_{2i}, \ldots, t_{pi})$ is the vector of covariates and $\delta = (\delta_0, \delta_1, \ldots, \delta_p)^T$ is the regression coefficients vector.

The logit link function used to link the covariates to the mean of response variable can be given as follows

$$g(\mu_i) = \log\left[\frac{\mu_i}{1-\mu_i}\right]. \tag{27}$$

From Equations (26) and (27), we have

$$\mu_i = \frac{e^{\delta^T t_i}}{1 + e^{\delta^T t_i}}, \quad i = 1, 2, \ldots, n. \tag{28}$$

5.1. Maximum Likelihood Estimates Method

The unknown parameter $\delta = (\delta_0, \delta_1, \ldots, \delta_p)^T$ is estimated under the classical approach MLE method, expressed as

$$\begin{aligned} l(\delta) = {} & n\log[4\mu] + n\log[\tau-1] - n\log[\tau] - n\log[\mu-1] - \\ & 2\sum_{i=1}^{n}\left(\log\left[\frac{2\mu(\tau-1)}{\tau(\mu-1)} + \left(2 - \frac{2\mu(\tau-1)}{\tau(\mu-1)}\right)x_i\right]\right), \end{aligned} \tag{29}$$

where δ is the vector of unknown parameters. By maximization of l given in Equation (29), we obtain $\hat{\delta}$, the MLE_s of δ. Maximization can be obtained with the R program using the functions (optim and Maxlik), see [31,32].

5.2. Residual Analysis

To check the suitability of the regression model, a residual analysis is needed. To do that, Cox–Snell, $\hat{e}_i$ [33] and the randomized quantile residuals, $\hat{r}_i$ [34], are given, respectively, as follows

$$\hat{e}_i = -\log[\bar{G}(x_i, \mu_i, \theta)],$$
$$\hat{r}_i = \Phi^{-1}[G(x_i, \mu_i, \theta)],$$

where $\bar{G}(x_i, \mu_i, \theta) = 1 - G(x_i, \mu_i, \theta)$ is the survival function of the UHLG regression model, and $\Phi^{-1}[.]$ is the inverse cumulative function of the standard normal distribution.

6. Application

In this section, a real data set is used to show the ability of the UHLG distribution in modelling bounded data sets. It is compared with other unit distributions. These distributions are log Bilal regression model (LB), beta regression model (B) and Kumaraswamy regression model (K). The pdfs of these models are given, respectively, as follows

$$f_{LB}(x) = \frac{3\tau(\mu-1)}{\mu(\tau-1)} x^{\frac{\tau(\mu-1)}{\mu(\tau-1)}-1} \left(1 - x^{\frac{\tau(\mu-1)}{2\mu(\tau-1)}}\right), \quad (30)$$

$$f_B(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\mu)\Gamma(\alpha(1-\mu))} x^{\alpha\mu-1}(1-x)^{(1-\mu)\alpha-1}, \quad (31)$$

and

$$f_K(x) = \frac{\alpha \log[0.5]}{\log[1-\mu^\alpha]} x^{\alpha-1}(1-x^\alpha)^{\frac{\log[0.5]}{\log[1-\mu^\alpha]}-1}. \quad (32)$$

where $\alpha > 0, x \in (0,1)$, and $\mu \in (0,1)$ represent the mean in Equation (31) and the median in Equations (30) and (32).

To show the differences between these models and the UHLG regression model, some statistics such as Akaike information criterion (AIC), corrected Akaike information criterion (AICC), Bayesian information criterion (BIC), Kolmogorov–Smirnov (K-S) and p-value are calculated.

Risk Survey Data

Insurance can be defined as a contract, represented by a policy in which those insured by an insurance company receive protection against potential losses. The company aggregates the risk of the largest number of customers to make payments more at discounted rates for the insured. Insurance policies are used to protect against the risk of financial losses, whether large or small, which may result from damage to the insured or what he owns, or from civil liability for damage to another party. Some of the most prevalent types of insurance are life, death and property insurance.

Risk management is an important and necessary aspect of insurance. Risk surveys are an effective way to identify, quantify and therefore manage risk by collecting information, perceptions and insights from managers across an organization.

The data set represents a questionnaire sent to 374 risk managers in large U.S.-based organizations. Seventy-three of the managers returned the completed survey. The data were used before by Mazucheli et al. [13]. Four important topics were solicited including captive insurance, decision making, organizational data and evaluating and identifying exposures. The data were described as follows:

- Firm cost (y) is the mean variable and represents the cost of the firm's cost management effectiveness;
- Assume (x_1) represents the firm's retention strategy;
- Cap (x_2) represents the indicator with value 1 if the firm uses a captive insurer and the value 0 otherwise;
- Sizelog (x_3) represents the log of firm's size;
- Indcost (x_4) represents the risk in the firm's industry;
- Central (x_5) represents the strategy of the firm's centralization;
- Analy (x_6) represents the degree of importance of using analytical tools.

First, a univariate regression model was used to model the risk survey data to test the goodness of fit of UHLG distribution over some other distributions such as log Bilal, beta and Kumaraswamy distributions with pdfs given, respectively, as

$$f_1(x) = \frac{6}{\theta} x^{\frac{2}{\theta}-1}(1 - x^{\frac{1}{\theta}}), \quad (33)$$

$$f_2(x) = \frac{\Gamma(\theta+\alpha)}{\Gamma(\alpha)\Gamma(\alpha)} x^{\theta-1}(1-x)^{\alpha-1}, \quad (34)$$

and

$$f_3(x) = \theta\alpha x^{\theta-1}(1-x^\theta)^\alpha, \quad (35)$$

where $\theta, \alpha > 0, x \in (0,1)$.

A comparison of ML estimates between some various unit distributions and some statistics for the previous data is given in Table 3 and it is obvious that unit half logistic gives the best fit to data.

Table 3. ML estimates and some statistics for the risk survey data.

Model	$\hat{\theta}$	$\hat{\alpha}$	AIC	AICC	BIC	K-S	*p*-Value
unit half logistic	0.132	-	−177.02	−177.01	−174.78	0.1191	0.2515
log Bilal	3.464	-	−149.388	−149.332	−147.098	0.2241	0.0013
beta	0.613	3.799	−148.24	−148.06	−143.65	0.1805	0.0172
Kumaraswamy	7.350	2.300	−150.01	−149.84	−144.59	0.9586	0.0000

Now, a multivariate regression model is used to show the impact of assume, cap, sizelog, indcost, central and analy components on the firm cost component.

The logit link function for μ_i is assumed for all fitted regression models as it ensures that the estimated mean lies between 0 and 1 as follows

$$logit(\mu_i) = \delta_0 + \delta_1 x_{i1} + \delta_2 x_{i2} + \delta_3 x_{i3} + \delta_4 x_{i4} + \delta_5 x_{i5} + \delta_6 x_{i6}, \quad i = 1, \ldots, 73. \tag{36}$$

Tables 4 and 5 give the results of fitting data. The ML estimates of parameters θ, α and $\delta_i, i = 0, 1, \ldots, 6$ are listed in these tables. In addition, the corresponding standard error (SE) and *p*-value are also given. Moreover, AIC and BIC statistics are given for each regression model.

Table 4. ML estimates for the regression model parameters with some other statistics fitting risk survey data (comparison between UHLG, Beta and Kumaraswamy regression models).

coeffs.	UHLG			Beta			Kumaraswamy		
	Est.	SE	*p*-Value	Est.	SE	*p*-Value	Est.	SE	*p*-Value
δ_0	4.128	1.438	<0.0000	1.888	0.944	<0.0000	−1.866	2.55	<0.0000
δ_1	−0.012	0.149	<0.0000	−0.012	0.120	<0.0000	0.429	0.447	<0.0000
δ_2	0.018	0.635	<0.0000	0.178	0.472	<0.0000	0.026	1.174	<0.0000
δ_3	−0.918	0.456	<0.0000	−0.511	0.334	<0.0000	−0.090	0.788	<0.0000
δ_4	2.145	0.953	<0.0000	1.236	0.513	<0.0000	−1.028	1.711	<0.0000
δ_5	−0.092	0.389	<0.0000	−0.012	0.204	<0.0000	0.088	0.722	<0.0000
δ_6	0.005	0.189	<0.0000	−0.004	0.085	<0.0000	−0.056	0.356	<0.0000
α	-	-	-	6.33	0.436	<0.0000	0.241	0.204	<0.0000
AIC	−192.34			−159.4			−190.1		
BIC	−176.31			−141.1			−171.8		

Table 5. ML estimates for the regression model parameters with some other statistics fitting risk survey data (comparison between UHLG, Unit Weibull and Unit Omega regression models).

coeffs.	UHLG			log Bilal		
	Est.	SE	*p*-Value	Est.	SE	*p*-Value
δ_0	4.128	1.438	<0.0000	−1.704	0.963	<0.0000
δ_1	−0.012	0.149	<0.0000	0.005	0.011	<0.0000
δ_2	0.018	0.635	<0.0000	−0.061	0.189	<0.0000
δ_3	−0.918	0.456	<0.0000	0.298	0.100	<0.0000
δ_4	2.145	0.953	<0.0000	−0.727	0.400	<0.0000
δ_5	−0.092	0.389	<0.0000	0.020	0.070	<0.0000
δ_6	0.005	0.189	<0.0000	−0.001	0.017	<0.0000
AIC	−192.34			−151.46		
BIC	−176.31			−135.42		

Based on Tables 4 and 5, we can notice the following:

- All covariates have an impact on the firm's cost management effectiveness;
- The UHLG regression model explains the greatest difference by using fewer parameters (-AIC = 192.34 and -BIC = 176.31);
- UHLG regression model gives the best fit to the data compared to the other models.

Figure 7 shows the PP plots of the theoretical and empirical probabilities of the Cox–Snell residuals for different regression models fitting the risk survey data.

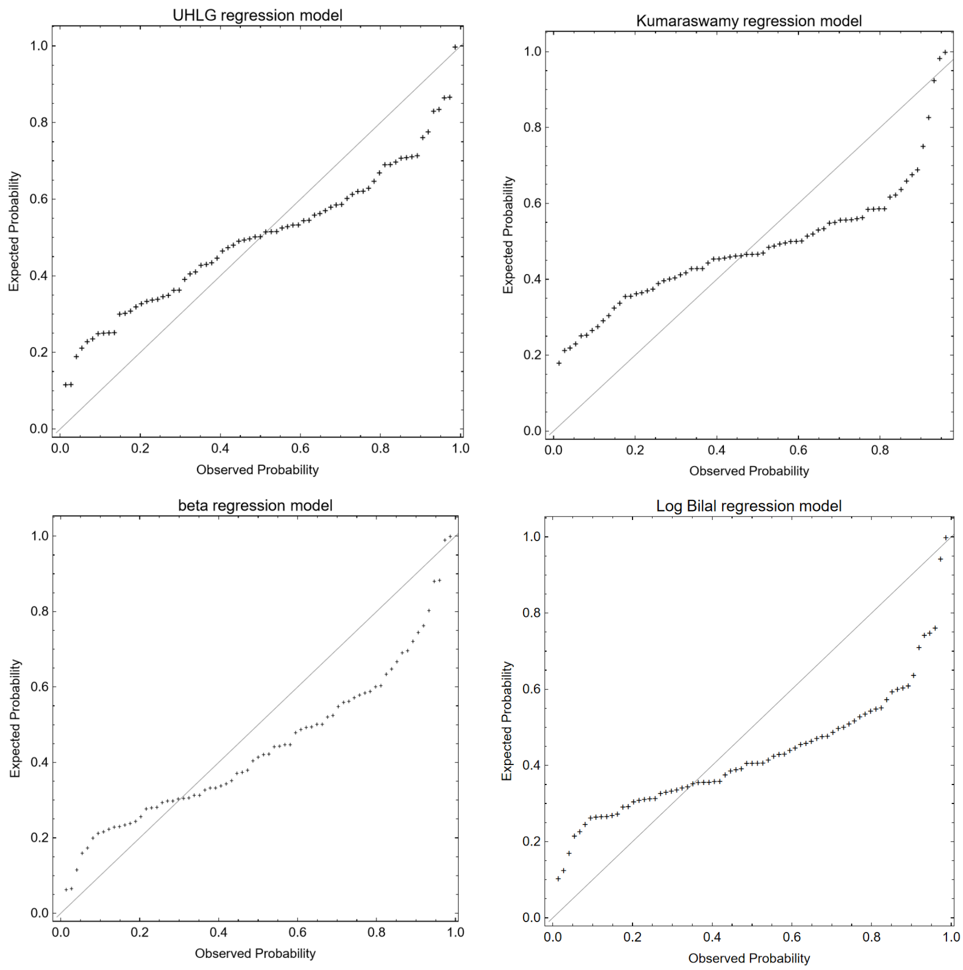

Figure 7. PP plots of the theoretical and empirical probabilities of the Cox–Snell residuals fitting the risk survey data.

7. Conclusions

A new unit distribution is proposed to deal with data lying in the unit interval (0, 1). This distribution is called unit half-logistic geometric distribution with some flexible statistical properties. The new distribution is assumed to be alternative to some other distributions including beta, Kumaraswamy and log Bilal distributions. Some important statistical properties like moments, mean inactivity, mean residual, stress strength, stochastic ordering and other properties are given. In addition, different estimation methods are used estimating the parameter. Moreover, a new quantile regression model is introduced using UHLG distribution. Finally, an application on a real data set is performed to clarify the usefulness of this distribution and its regression model. The data come from a questionnaire sent to some large organizations in the united states. The p-value of the UHLG distribution was the biggest among other distributions. Moreover, the UHLG regression model explained the greatest difference by using fewer parameters.

Modelling bounded data sets lying in the (0, 1) interval became very important recently. Therefore, we are in desperate need of new unit distributions modelling such data. In the future, more unit distributions are needed to give the best fit of data from medical, actuarial and finance science fields.

Author Contributions: Conceptualization, A.T.R. and A.H.T.; methodology, A.T.R., A.H.T. and B.S.E.-D.; software, A.T.R.; validation, A.H.T.; formal analysis, A.T.R., A.H.T.and B.S.E.-D.; investigation, A.T.R. and A.H.T.; resources, A.T.R., A.H.T. and B.S.E.-D.; data curation, A.T.R. and A.H.T.; writing—original draft preparation, A.H.T., B.S.E.-D. and A.T.R.; writing—review and editing, A.H.T., A.T.R. and B.S.E.-D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are available in this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ramadan, A.T.; Tolba, A.H.; El-Desouky, B.S. Generalized power Akshaya distribution and its applications. *Open J. Model. Simul.* **2021**, *4*, 323–338. [CrossRef]
2. El-Sagheer, R.M.; Tolba, A.H.; Jawa, T.M.; Sayed-Ahmed, N. Inferences for Stress-Strength Reliability Model in the Presence of Partially Accelerated Life Test to Its Strength Variable. *Comput. Intell. Neurosci.* **2022**, *5*, 4710536. [CrossRef] [PubMed]
3. Abd, El-Monsef, M.M.E.; El-Awady, M.M.; Seyam, M.M. A new quantile regression model for modelling child mortality. *Int. J. Biomath.* **2022**, *10*, 142–149.
4. Altun, E.; Cordeiro, G.M. The unit-improved second-degree Lindley distribution: Inference and regression modelling. *Comput. Stat.* **2020**, *35*, 259–279. [CrossRef]
5. Altun, E.; El-Morshedy, M.; Eliwa, M.S. A new regression model for bounded response variable: An alternative to the beta and unit-Lindley regression models. *PLoS ONE* **2021**, *16*, e0245627. [CrossRef] [PubMed]
6. Bayes, C.L.; Bazán, J.L.; De Castro, M. A quantile parametric mixed regression model for bounded response variables. *Stat. Its Interface* **2017**, *10*, 483–493. [CrossRef]
7. Cordeiro, G.M.; Silva, R.B.; Nascimento, A.D.C. *Recent Advances in Lifetime and Reliability Models*; Bentham Science Publishers: Oak Park, IL, USA, 2020; Volume 17, pp. 93–112.
8. Ferrari, S.; Cribari-Neto, F. Beta regression for modelling rates and proportions. *J. Appl. Stat.* **2004**, *31*, 799–815. [CrossRef]
9. Kumaraswamy, P. A generalized probability density function for double-bounded random processes. *J. Hydrol.* **1980**, *46*, 79–88. [CrossRef]
10. Mitnik, P.A.; Baek, S. The Kumaraswamy distribution: Median-dispersion re-parameterizations for regression modelling and simulation-based estimation. *Stat. Pap.* **2013**, *54*, 177–192. [CrossRef]
11. Gómez-Déniz, E.; Sordo, M.A.; Calderín-Ojeda, E. The Log-Lindley distribution as an alternative to the beta regression model with applications in insurance. *Insur. Math. Econ.* **2014**, *54*, 49–57. [CrossRef]
12. Korkmaz, M.Ç.; Chesneau, C. On the unit Burr-XII distribution with the quantile regression modelling and applications. *Comput. Appl. Math.* **2021**, *40*, 1–26. [CrossRef]
13. Mazucheli, J.; Menezes, A.F.B.; Ghitany, M.E. The unit-Weibull distribution and associated inference. *J. Appl. Probab. Stat.* **2018**, *13*, 1–22.
14. Mazucheli, J.; Menezes, A.F.B.; Fernandes, L.B.; De Oliveira, R.P.; Ghitany, M.E. The unit-Weibull distribution as an alternative to the Kumaraswamy distribution for the modelling of quantiles conditional on covariates. *J. Appl. Stat.* **2020**, *47*, 954–974. [CrossRef] [PubMed]
15. Muse, A.H.; Tolba, A.H.; Fayad, E.; Ali, O.A.A.; Nagy, M.; Yusuf, M. Modelling the COVID-19 mortality rate with a new versatile modification of the log-logistic distribution. *Comput. Intell. Neurosci.* **2021**, *26*, 203–224. [CrossRef] [PubMed]
16. Mahmood, Z.; Jawa, M.T.; Sayed-Ahmed, N.; Khalil, E.M.; Muse, A.H.; Tolba, A.H. An Extended Cosine Generalized Family of Distributions for Reliability Modelling: Characteristics and Applications with Simulation Study. *Math. Probl. Eng.* **2022**, *3*, 112–128.
17. Mazucheli, J.; Menezes, A.F.B.; Chakraborty, S. On the one parameter unit-Lindley distribution and its associated regression model for proportion data. *J. Appl. Stat.* **2019**, *46*, 700–714. [CrossRef]
18. Mousa, A.M.; El-Sheikh, A.A.; Abdel-Fattah, M.A. A gamma regression for bounded continuous variables. *Adv. Appl. Stat.* **2016**, *49*, 305–321. [CrossRef]
19. Sarhan, A.M.; El-Gohary, A.I.; Mustafa, A.; Tolba, A.H. Statistical analysis of regression competing risks model with covariates using Weibull sub-distributions. *Int. J. Reliab. Appl.* **2019**, *2*, 73–88.
20. Sarhan, A.M.; El-Gohary, A.I.; Tolba, A.H. Statistical Analysis of a Competing Risks Model with Weibull Sub-Distributions. *Appl. Math.* **2017**, *11*, 1671–1690. [CrossRef]

21. Tadikamalla, P.R. On a family of distributions obtained by the transformation of the gamma distribution. *J. Stat. Comput. Simul.* **1981**, *13*, 209–214. [CrossRef]
22. Liu, K.; Balakrishnan, N. Recurrence relations for moments of order statistics from half logistic-geometric distribution and their applications. *Commun. Stat.-Simul. Comput.* **2020**, *17*, 1–19. [CrossRef]
23. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*; Academic Press: Cambridge, MA, USA, 2014; Volume 11.
24. Mohamed, R.A.H.; Tolba, A.H.; Almetwally, E.M.; Ramadan, D.A. Inference of Reliability Analysis for Type II Half Logistic Weibull Distribution with Application of Bladder Cancer. *Axioms* **2022**, *8*, 386. Available online: https://www.mdpi.com/2075-1680/11/8/386 (accessed on 6 August 2022). [CrossRef]
25. Shanmugam, R. *The Stress-Strength Model and Its Generalizations: Theory and Applications*; Taylor & Francis: Boca Raton, FL, USA, 2004; Volume 16, pp. 84–105.
26. Frank, W.J.O. *NIST Handbook of Mathematical Functions*; Cambridge University Press: Cambridge, UK, 2010; Volume 17, pp. 224–251.
27. Balakrishnan, N. Order statistics from the half logistic distribution. *J. Stat. Comput. Simul.* **1985**, *20*, 287–309. [CrossRef]
28. Belzunce, F.; Riquelme, C.M.; Mulero, J. *An Introduction to Stochastic Orders*; Academic Press: Cambridge, MA, USA, 2015; Volume 12, pp. 165–187.
29. Abushal, T.A.; Kumar, J.; Muse, A.H.; Tolba, A.H. Estimation for Akshaya Failure Model with Competing Risks under Progressive Censoring Scheme with Analyzing of Thymic Lymphoma of Mice Application. *Complexity* **2022**, *2022*, 5151274. [CrossRef]
30. McCool, J.I. *Using the Weibull Distribution: Reliability, Modelling and Inference*; John Wiley & Sons: Hoboken, NJ, USA, 2012; Volume 950, pp. 43–66.
31. Zhang, P.; Qiu, Z.; Shi, C. Simplexreg: An R package for regression analysis of proportional data using the simplex distribution. *J. Stat. Softw.* **2016**, *71*, 51–76. [CrossRef]
32. Nurunnabi, A.A.M.; Hadi, A.S.; Imon, A.H.M.R. Procedures for the identification of multiple influential observations in linear regression. *J. Appl. Stat.* **2014**, *41*, 1315–1331. [CrossRef]
33. Cox, D.R.; Snell, E.J. A general definition of residuals. *J. R. Stat. Soc. Ser. B Methodol.* **1968**, *30*, 248–265. [CrossRef]
34. Dunn, P.K.; Smyth, G.K. Randomized quantile residuals. *J. Comput. Graph. Stat.* **1996**, *5*, 121–143.