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Categorically Closed Unipotent Semigroups

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Abstract: Let $\mathcal{C}$ be a class of T_1 topological semigroups, containing all Hausdorff zero-dimensional topological semigroups. A semigroup X is $\mathcal{C}$-*closed* if X is closed in any topological semigroup $Y \in \mathcal{C}$ that contains X as a discrete subsemigroup; X is *injectively* $\mathcal{C}$-*closed* if for any injective homomorphism $h : X \to Y$ to a topological semigroup $Y \in \mathcal{C}$ the image $h[X]$ is closed in Y. A semigroup X is *unipotent* if it contains a unique idempotent. It is proven that a unipotent commutative semigroup X is (injectively) $\mathcal{C}$-closed if and only if X is bounded and nonsingular (and group-finite). This characterization implies that for every injectively $\mathcal{C}$-closed unipotent semigroup X, the center $Z(X)$ is injectively $\mathcal{C}$-closed.

Keywords: $\mathcal{C}$-closed semigroup; unipotent semigroup

1. Introduction and Main Results

In many cases, the completeness properties of various objects of general topology or topological algebra can be characterized externally as closedness in ambient objects. For example, a metric space X is complete if and only if X is closed in any metric space containing X as a subspace. A uniform space X is complete if and only if X is closed in any uniform space containing X as a uniform subspace. A topological group G is Raĭkov complete if and only if it is closed in any topological group containing G as a subgroup.

On the other hand, for topological semigroups there are no reasonable notions of (inner) completeness. Nonetheless, one can define many completeness properties of semigroups via their closedness in ambient topological semigroups.

A *topological semigroup* is a topological space X endowed with a continuous associative binary operation $X \times X \to X$, $(x, y) \mapsto xy$.

Definition 1. *Let $\mathcal{C}$ be a class of topological semigroups. A topological semigroup X is called*

- $\mathcal{C}$-*closed if for any isomorphic topological embedding $h : X \to Y$ to a topological semigroup $Y \in \mathcal{C}$, the image $h[X]$ is closed in Y;*
- *injectively $\mathcal{C}$-closed if for any injective continuous homomorphism $h : X \to Y$ to a topological semigroup $Y \in \mathcal{C}$, the image $h[X]$ is closed in Y;*
- *absolutely $\mathcal{C}$-closed if for any continuous homomorphism $h : X \to Y$ to a topological semigroup $Y \in \mathcal{C}$, the image $h[X]$ is closed in Y.*

For any topological semigroup we have the implications:

$$\text{absolutely } \mathcal{C}\text{-closed} \Rightarrow \text{injectively } \mathcal{C}\text{-closed} \Rightarrow \mathcal{C}\text{-closed}.$$

Definition 2. *A semigroup X is defined to be (injectively, absolutely) $\mathcal{C}$-closed if it is X endowed with the discrete topology.*

In this paper, we are interested in the (absolute, injective) $\mathcal{C}$-closedness for the classes:

- $\mathsf{T_1S}$ of topological semigroups satisfying the separation axiom T_1;
- $\mathsf{T_2S}$ of Hausdorff topological semigroups;
- $\mathsf{T_zS}$ of Hausdorff zero-dimensional topological semigroups.

A topological space satisfies the separation axiom T_1 if all its finite subsets are closed. A topological space is *zero-dimensional* if it has a base of the topology consisting of *clopen* (= closed-and-open) sets.

Since $\mathsf{T_zS} \subseteq \mathsf{T_2S} \subseteq \mathsf{T_1S}$, for every semigroup the following implications hold:

$$
\begin{array}{ccccc}
\text{absolutely } \mathsf{T_1S}\text{-closed} & \Longrightarrow & \text{absolutely } \mathsf{T_2S}\text{-closed} & \Longrightarrow & \text{absolutely } \mathsf{T_zS}\text{-closed} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{injectively } \mathsf{T_1S}\text{-closed} & \Longrightarrow & \text{injectively } \mathsf{T_2S}\text{-closed} & \Longrightarrow & \text{injectively } \mathsf{T_zS}\text{-closed} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\mathsf{T_1S}\text{-closed} & \Longrightarrow & \mathsf{T_2S}\text{-closed} & \Longrightarrow & \mathsf{T_zS}\text{-closed.}
\end{array}
$$
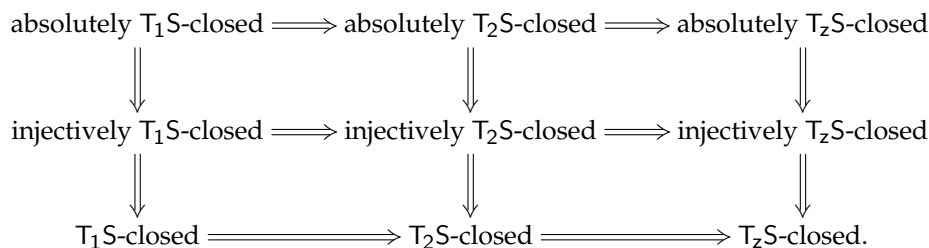
From now on, we assume that $\mathcal{C}$ is a class of topological semigroups such that

$$\mathsf{T_zS} \subseteq \mathcal{C} \subseteq \mathsf{T_1S}.$$

Semigroups having one of the above closedness properties are called categorically closed. Categorically closed topological groups and semilattices were investigated in [1–11] and [12–15], respectively. This paper is a continuation of the papers [3,15,16], which contain inner characterizations of semigroups possessing various categorically closed properties.

In this paper we shall characterize (absolutely and injectively) $\mathcal{C}$-closed *unipotent* semigroups.

A semigroup X is called

- *unipotent* if X contains a unique idempotent;
- *chain-finite* if any infinite set $I \subseteq X$ contains elements $x, y \in I$ such that $xy \notin \{x, y\}$;
- *singular* if there exists an infinite set $A \subseteq X$ such that AA is a singleton;
- *periodic* if for every $x \in X$ there exists $n \in \mathbb{N}$ such that x^n is an idempotent;
- *bounded* if there exists $n \in \mathbb{N}$ such that for every $x \in X$ the n-th power x^n is an idempotent;
- *group-finite* if every subgroup of X is finite;
- *group-bounded* if every subgroup of X is bounded.

The following characterization of $\mathcal{C}$-closed commutative semigroups was proved in the paper [16].

Theorem 1. *A commutative semigroup is $\mathcal{C}$-closed if and only if it is chain-finite, periodic, nonsingular and group-bounded.*

For unipotent semigroups, this characterization can be simplified as follows:

Theorem 2. *A unipotent semigroup X is $\mathcal{C}$-closed if and only if X is bounded and nonsingular.*

Another principal result of this paper is the following characterization of injectively $\mathcal{C}$-closed unipotent semigroups.

Theorem 3. *A unipotent commutative semigroup X is injectively $\mathcal{C}$-closed if and only if X is bounded, nonsingular and group-finite.*

Example 1. *For an infinite cardinal κ, the* Taimanov semigroup T_κ *is the set κ endowed with the semigrop operation*

$$xy = \begin{cases} 1, & \text{if } x \neq y \text{ and } x, y \in \kappa \setminus \{0,1\}; \\ 0, & \text{otherwise.} \end{cases}$$

The semigroup T_κ was introduced by Taimanov in [17]. Its algebraic and topological properties were investigated by Gutik [18] who proved that the semigroup T_κ is injectively $\mathsf{T_1S}$-closed. The same also follows for Theorem 3 because the semigroup T_κ is unipotent, bounded, nonsingular and group-finite. The Taimanov semigroups witness that there exist injectively $\mathsf{T_1S}$-closed unipotent semigroups of arbitrarily high cardinality.

For a semigorup X, let

$$Z(X) \overset{\text{def}}{=} \{z \in X : \forall x \in X \ (xz = zx)\}$$

be the *center* of X. The center of an (injectively) $\mathcal{C}$-closed semigroup has the following properties, proven in Lemmas 5.1, 5.3, 5.4 of [16] (and Theorem 1.7 of [19]).

Theorem 4. *The center $Z(X)$ of any (injectively) $\mathcal{C}$-closed semigroup is chain-finite, periodic, nonsingular (and group-finite).*

Corollary 1. *The center $Z(X)$ of an injectively $\mathcal{C}$-closed unipotent semigroup X is injectively $\mathcal{C}$-closed.*

Proof. By Theorem 4, the semigroup $Z(X)$ is chain-finite, periodic, nonsingular, and group-finite. By Theorem 1, the semigroup $Z(X)$ is $\mathcal{C}$-closed. By Theorem 2, $Z(X)$ is bouned. If $Z(X)$ is empty, then $Z(X)$ is injectively $\mathcal{C}$-closed. So, we assume that $Z(X) \neq \varnothing$. Being bounded, the semigroup $Z(X)$ contains an idempotent. Being a subsemigroup of the unipotent semigroup X, the semigroup $Z(X)$ is unipotent. By Theorem 3, the unipotent bounded nonsingular group-finite semigroup $Z(X)$ is injectively $\mathcal{C}$-closed. $\square$

Another corollary of Theorem 3 describes the center of an absolutely $\mathcal{C}$-closed unipotent semigroup.

Corollary 2. *The center $Z(X)$ of an absolutely $\mathcal{C}$-closed unipotent semigroup X is finite and hence absolutely $\mathcal{C}$-closed.*

Proof. By Theorem 4, the semigroup $Z(X)$ is chain-finite, periodic, nonsingular, and group-finite. If $Z(X)$ is empty, then $Z(X)$ is finite and hence absolutely $\mathcal{C}$-closed. So, we assume that $Z(X)$ is not empty. Being periodic, the semigroup $Z(X)$ contains an idempotent e. Since X is unipotent, e is a unique idempotent of the semigroups X and $Z(X)$. Let H_e be the maximal subgroup of the semigroup $Z(X)$. The group H_e is finite because $Z(X)$ is group-finite. By Theorem 1.7 of [19], the complement $Z(X) \setminus H_e$ is finite and hence the set $Z(X)$ is finite, too. $\square$

Corollaries 1 and 2 suggest the following open problems.

Problem 1.

1. *Is the center of a $\mathcal{C}$-closed semigroup $\mathcal{C}$-closed?*
2. *Is the center of an injectively $\mathcal{C}$-closed semigroup injectively $\mathcal{C}$-closed?*
3. *Is the center of an absolutely $\mathcal{C}$-closed semigroup absolutely $\mathcal{C}$-closed?*

2. Preliminaries

We denote by ω the set of finite ordinals and by $\mathbb{N} \overset{\text{def}}{=} \omega \setminus \{0\}$ the set of positive integer numbers.

For an element a of a semigroup X the set

$$H_a \stackrel{\text{def}}{=} \{x \in X : (xX^1 = aX^1) \wedge (X^1 x = X^1 a)\}$$

is called the *$\mathcal{H}$-class* of a. Here $X^1 \stackrel{\text{def}}{=} X \cup \{1\}$ where 1 is an element such that $1x = x = x1$ for all $x \in X^1$.

By Corollary 2.2.6 [20], for every idempotent e of a semigroup X its $\mathcal{H}$-class H_e coincides with the maximal subgroup of X, containing the idempotent e.

For a subset A of a semigroup X and a positive integer number n, let

$$\sqrt[n]{A} = \{x \in X : x^n \in A\} \quad \text{and} \quad \sqrt[\mathbb{N}]{A} = \bigcup_{n \in \mathbb{N}} \sqrt[n]{A} = \{x \in X : A \cap x^{\mathbb{N}} \neq \varnothing\},$$

where

$$x^{\mathbb{N}} = \{x^n : n \in \mathbb{N}\}$$

is the *monogenic semigroup* generated by x.

The following lemma is proven in [16], 3.1.

Lemma 1. *For any idempotent e of a semigroup, $(\sqrt[\mathbb{N}]{H_e} \cdot H_e) \cup (H_e \cdot \sqrt[\mathbb{N}]{H_e}) \subseteq H_e$.*

3. Proof of Theorem 2

Theorem 2 will be derived from the following lemmas.

Lemma 2. *Let X be a periodic commutative semigroup with a unique idempotent e and trivial maximal subgroup H_e. If X is not bounded, then there exists an infinite subset $A \subseteq X$ such that $AA = \{e\}$.*

Proof. To derive a contradiction, assume that X is not bounded but for every infinite set $A \subseteq X$ we have $AA \neq \{e\}$. Taking into account that X is periodic and unipotent, we conclude that $X = \sqrt[\mathbb{N}]{H_e}$. By Lemma 1, the maximal subgroup $H_e = \{e\}$ is an ideal in X.

Inductively we shall construct a sequence of points $(x_k)_{k \in \omega}$ and a sequence of positive integer numbers $(n_k)_{k \in \omega}$ such that for every $k \in \omega$ the following conditions are satisfied:

(i) $x_k^{n_k} \notin \{e\} \cup \{x_i^{n_i} : i < k\}$;

(ii) $x_k^{2n_k} = e$;

(iii) $\max_{i<k} |x_i^{n_i} X| < n_k$.

To start the inductive construction, take any $x_0 \in X \setminus \{e\}$ and let n_0 be the smallest number such that $x_0^{n_0+1} = e$. Such number n_0 exists as X is periodic. Since $\{e\} = H_e$ is an ideal in X, it follows from $x_0^{n_0+1} = e$ and $2n_0 \geq n_0 + 1$ that $x_0^{2n_0} = e$. Assume that for some $k \in \mathbb{N}$, we have chosen sequences $(x_i)_{i<k}$ and $(n_i)_{i<k}$. For every $i < k$, consider the set $x_i^{n_i} X$ and observe that for every $a, b \in X$ the inductive condition (ii) implies

$$(x_i^{n_i} a)(x_i^{n_i} b) = x_i^{2n_i} ab = eab = e.$$

This means that $(x_i^{n_i} X)^2 = \{e\}$ and by our assumption, the set $x_i^{n_i} X$ is finite. Since X is unbounded, there exists an element $x_k \in X$ and a number $m_k > k + \max_{i<k} |x_i^{n_i} X|$ such that $x_k^{k+m_k} \neq e$ but $x_k^{1+k+m_k} = e$. Since the set $\{x_k^j : m_k \leq j \leq m_k + k\}$ consists of $k+1$ points, there exist a number $n_k \in \{m_k, \dots, m_k + k\}$ such that $x_k^{n_k} \notin \{x_i^{n_i} : i < k\}$. It follows from $x_k^{k+m_k} \neq e = x_k^{1+k+m_k}$ and $2n_k \geq m_k + m_k \geq 1 + k + m_k > n_k$ that $x_k^{n_k} \neq e = x_k^{2n_k}$. This completes the inductive step.

After completing the inductive construction, consider the infinite set $A = \{x_k^{n_k} : k \in \omega\}$. We claim that $x_i^{n_i} x_k^{n_k} = e$ for any $i \leq k$. For $i = k$ this follows from the inductive condition (ii). So, assume that $i < k$. By the induction condition (iii) and the Pigeonhole

Principle, there exist two positive numbers $j < j' \leq |x_i^{n_i} X| + 1 \leq n_k$ such that $x_i^{n_i} x_k^j = x_i^{n_i} x_k^{j'}$. Let $d = j' - j \leq n_k$ and observe that $x_i^{n_i} x_k^j = x_i^{n_i} x_k^{j+d}$. Then

$$x_i^{n_i} x_k^{j+2d} = x_i^{n_i} x_k^{j+d} x_k^d = x_i^{n_i} x_k^j x_k^d = x_i^{n_i} x_k^{j+d} = x_i^{n_i} x_k^j.$$

Proceeding by induction, we can prove that $x_i^{n_i} x_k^j = x_i^{n_i} x_k^{j+pd}$ for every $p \in \omega$. Since X is periodic and $\{e\} = H_e$ is an ideal in X, there exists $p \in \mathbb{N}$ such that $x_k^{j+pd} = e$ and hence $x_i^{n_i} x_k^j = x_i^{n_i} x_k^{j+pd} = e$. Then, $x_i^{n_i} x_k^{n_k} = x_i^{n_i} x_k^j x_k^{n_k-j} = e x_k^{n_k-j} = e$ and hence $AA = \{e\}$, which contradicts our assumption. $\square$

Lemma 3. *Let X be a periodic commutative semigroup with a unique idempotent e and bounded maximal subgroup H_e. If X is not bounded, then there exists an infinite subset $A \subseteq X$ such that $AA = \{e\}$.*

Proof. Since H_e is bounded, there exists a number $p \in \mathbb{N}$ such that $x^p = e$ for all $x \in H_e$. Assuming that X is not bounded, we conclude that the subsemigroup $P = \{x^p : x \in X\}$ of X is not bounded. We claim that $P \cap H_e = \{e\}$. Indeed, for every $x \in X$ with $x^p \in H_e$, we have $xe \in H_e$ by Lemma 1 and hence $x^p = x^p e = (xe)^p = e$. Since the maximal subgroup of P is trivial, one can apply Lemma 2 and find an infinite set $A \subseteq P \subseteq X$ such that $AA = \{e\}$. $\square$

Our final lemma implies Theorem 2.

Lemma 4. *For a unipotent commutative semigroup X, the following conditions are equivalent:*

1. *X is $\mathcal{C}$-closed;*
2. *X is periodic, nonsingular and group-bounded;*
3. *X is bounded and not singular.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 1, and (3) $\Rightarrow$ (2) is trivial. The implication (2) $\Rightarrow$ (3) follows from Lemma 3. $\square$

4. Proof of Theorem 3

In this section we prove Lemmas 5 and 6 implying the "only if" and "if" parts of the characterization Theorem 3, respectively.

Lemma 5. *If a unipotent semigroup X is injectively $\mathcal{C}$-closed, then its center $Z(X)$ is bounded, nonsingular, and group-finite.*

Proof. By Theorem 4, the semigroup $Z(X)$ is periodic, nonsingular and group-finite. If $Z(X)$ is empty, then $Z(X)$ is bounded. If $Z(X)$ is not empty, then by the periodicity, $Z(X)$ contains an idempotent and hence is unipotent, being a subsemigroup of the unipotent semigroup X. By Lemma 4, $Z(X)$ is bounded. $\square$

Lemma 6. *Every bounded nonsingular group-finite unipotent commutative subsemigroup X of a T_1 topological semigroup Y is closed and discrete in Y.*

Proof. Replacing Y by the closure of X, we can assume that X is dense in Y.

Claim 1. *For every $x \in X$ and $y \in Y$ there exists a neighborhood $U \subseteq Y$ of y such that the set $x(U \cap X)$ is finite.*

Proof. To derive a contradiction, assume that there exists $a \in X$ and $y \in Y$ such that for every neighborhood $U \subseteq Y$ of y the set $a(U \cap X)$ is infinite. The periodicity of X ensures

that $X = \bigcup_{\ell \in \mathbb{N}} \sqrt[\ell]{H_e}$ where $\sqrt[\ell]{H_e} = \{x \in X : x^\ell \in H_e\} \subseteq \sqrt[\ell+1]{H_e}$, see Lemma 1. This lemma also implies that the set

$$\sqrt[1]{H_e} \cdot Y = H_e Y = H_e \overline{X} \subseteq \overline{H_e X} = \overline{H_e} = H_e$$

is finite.

Let k be the largest number such that for every $x \in \sqrt[k]{H_e}$ there exists a neighborhood $U \subseteq Y$ of y such that the set $x(U \cap X)$ is finite. Since $1 \le k < \min\{\ell : a \in \sqrt[\ell]{H_e}\}$, the number k is well-defined.

Subclaim 1. *For every $x \in \sqrt[k]{H_e}$ there exists a neighborhood $V \subseteq Y$ of y such that the set xV is a singleton in X.*

Proof. By the choice of k, for every $x \in \sqrt[k]{H_e}$ there exists a neighborhood $U \subseteq Y$ of y such that the set $x(U \cap X)$ is finite and hence closed in the T_1-space Y. Then

$$xy \in xU \subseteq x\overline{(U \cap X)} \subseteq \overline{x(U \cap X)} = x(U \cap X) \subseteq X.$$

Since the space Y is T_1 there exists an open neighborhood W of xy such that $W \cap x(U \cap X) = \{xy\}$. By the continuity of the semigroup operation, the point y has an open neighborhood $V \subseteq U$ such that $xV \subseteq W$. Then,

$$xV \subseteq x\overline{V \cap X} \subseteq \overline{x(V \cap X)} \subseteq \overline{xV \cap x(U \cap X)} = \overline{\{xy\}} = \{xy\} \subseteq X.$$

$\square$

By the maximality of k, there exists $b \in \sqrt[k+1]{H_e}$ such that for every neighborhood $V \subseteq Y$ of y the set $b(V \cap X)$ is infinite. It follows from $b \in \sqrt[k+1]{H_e}$ that $b^{k+1} \in H_e$ and hence $(b^2)^k = b^{2k} = b^{k+1}b^{k-1} \in H_e X^1 \subseteq H_e$ and hence $b^2 \in \sqrt[k]{H_e}$. By Subclaim 1, there exists a neighborhood $U \subseteq Y$ of y such that the set b^2U is a singleton in X. Choose any $u \in U \cap X$. By Lemma 1, $(b^2u)^k = b^{2k}u^k \in H_e X \subseteq H_e$ and $b^2y = b^2u \in \sqrt[k]{H_e}$. By Subclaim 1, there exists a neighborhood $V \subseteq U$ such that $(b^2y)V$ is a singleton in X. Then, the set $A = b(V \cap X)$ is infinite but

$$AA \subseteq b^2VV \subseteq b^2UV = (b^2y)V$$

is a singleton. However, this contradicts the nonsingularity of X. $\square$

Claim 2. *For every $x \in X$ and $y \in Y$ there exists a neighborhood $V \subseteq Y$ of y such that xV is a singleton in X.*

Proof. By Claim 1, there exists a neighborhood $U \subseteq Y$ of y such that the set $x(U \cap X)$ is finite and hence closed in the T_1-space Y. Then,

$$xy \in xU \subseteq x\overline{(U \cap X)} \subseteq \overline{x(U \cap X)} = x(U \cap X) \subseteq X.$$

Since the space Y is T_1, there exists an open neighborhood W of xy such that $W \cap x(U \cap X) = \{xy\}$. By the continuity of the semigroup operation, the point y has an open neighborhood $V \subseteq U$ such that $xV \subseteq W$. Then,

$$xV \subseteq x\overline{V \cap X} \subseteq \overline{x(V \cap X)} \subseteq \overline{xV \cap x(U \cap X)} = \overline{\{xy\}} = \{xy\} \subseteq X.$$

$\square$

Claim 3. *For every $n \in \mathbb{N}$, $x \in X$ and $y \in Y$, there exists a neighborhood $V \subseteq Y$ of y such that xV^n is a singleton in X.*

Proof. For $n = 1$ the statement follows from Claim 2. Assume that for some $n \in \mathbb{N}$ we know that for every $x \in X$ and $y \in Y$ there exists a neighborhood $U \subseteq Y$ of y such that xU^n is a singleton $\{a\}$ in X. By Claim 2, there exists a neighborhood $V \subseteq U$ of y such that aV is a singleton in X. Then $xV^{n+1} \subseteq xV^nV = aV$ is a singleton in X. $\square$

Claim 4. *For every $k \in \mathbb{N}$ the subspace $\sqrt[k]{H_e} = \{x \in X : x^k \in H_e\}$ of Y is discrete.*

Proof. To derive a contradiction, assume that for some $k \in \mathbb{N}$ the subspace $\sqrt[k]{H_e}$ is not discrete and let k be the smallest number with this property. Since $\sqrt[1]{H_e} = H_e$ is finite, $k > 1$. Let y be a non-isolated point of $\sqrt[k]{H_e}$. It follows that $y^k \in H_e$ and $(y^2)^{k-1} = y^k y^{k-2} \in H_e X^1 \subseteq H_e$ and hence $y^2 \in \sqrt[k-1]{H_e}$. By the minimality of k, the space $\sqrt[k-1]{H_e}$ is discrete. By the continuity of the semigroup operation, there exists a neighborhood $V_0 \subseteq Y$ of y such that $V_0 V_0 \cap \sqrt[k-1]{H_e} = \{y^2\}$. By Claim 2, we can additionally assume that $V_0 y = \{y^2\}$.

By induction we shall construct a sequence of points $(x_n)_{n \in \omega}$ in $\sqrt[k]{H_e}$ and a decreasing sequence $(V_n)_{n \in \omega}$ of open sets in Y such that for every $n \in \omega$ the following conditions are satisfied:

(i) $x_n \in V_n \cap \sqrt[k]{H_e} \setminus \{x_i\}_{i<n}$;
(ii) $y \in V_{n+1} \subseteq V_n$ and $x_n V_{n+1} = \{y^2\}$.

Assume that for some $n \in \omega$ we have chosen a neigborhood V_n of y and a sequence of points $\{x_i\}_{i<n}$. Since y is a non-isolated point of $\sqrt[k]{H_e}$, there exists a point x_n satisfying the inductive condition (i). Observe that $x_n y \in V_0 y = \{y^2\}$. By Claim 2, there exists a neighborhood $V_{n+1} \subseteq V_n$ of y such that $x_n V_{n+1} = \{x_n y\} = \{y^2\}$. This completes the inductive step.

After completing the inductive construction, we obtain the infinite set $A \stackrel{\text{def}}{=} \{x_n\}_{n \in \omega} \subseteq X_k$ such that $AA = \{y^2\}$. However, this contradicts the nonsingularity of X. $\square$

Claim 5. *For every $k \in \mathbb{N}$ the set $\sqrt[k]{H_e}$ is closed in Y.*

Proof. To derive a contradiction, assume that for some k the set $\sqrt[k]{H_e}$ is not closed in Y. We can assume that k is the smallest number with this property. Since $\sqrt[1]{H_e} = H_e$ is finite, $k > 1$ and hence $\{x^2 : x \in \sqrt[k]{H_e}\} \subseteq \sqrt[k-1]{H_e}$. Fix any point $y \in \overline{\sqrt[k]{H_e}} \setminus \sqrt[k]{H_e}$ and observe that

$$y^2 \in \overline{\{x^2 : x \in \sqrt[k]{H_e}\}} \subseteq \overline{\sqrt[k-1]{H_e}} = \sqrt[k-1]{H_e} \subseteq \sqrt[k]{H_e},$$

see Lemma 1. By Claim 4, the space $\sqrt[k]{H_e}$ is discrete. Consequently, there exists a neighborhood $U \subseteq Y$ of y such that $UU \cap \sqrt[k]{H_e} = \{y^2\}$. Since $y \in \overline{\sqrt[k]{H_e}} \setminus \sqrt[k]{H_e}$, the set $A = U \cap \sqrt[k]{H_e}$ is infinite and $AA \subseteq UU \cap \sqrt[k]{H_e} = \{y^2\}$, which contradicts the nonsingularity of X. $\square$

The boundedness of X implies that $X = \sqrt[k]{H_e}$ for some $k \in \mathbb{N}$. By Claims 4 and 5, the set $\sqrt[k]{H_e} = X$ is closed and discrete in Y. $\square$

5. Conclusions

This paper studies the categorical closedness properties of unipotent commutative semigroups. The main results of our work are Theorem 2, characterizing $\mathcal{C}$-closed unipotent commutative semigroups, and Theorem 3, characterizing injectively $\mathcal{C}$-closed unipotent semigroups. Theorem 3 implies Corollary 1 (and Corollary 2), in that the center of an injective (and absolutely) $\mathcal{C}$-closed unipotent semigroup remains injectively (and absolutely) $\mathcal{C}$-closed. Since each periodic commutative semigroup decomposes into a disjoint union of unipotent subsemigroups, the information on the categorical closedness properties of unipotent semigroups can shed some light on the categorical closedness properties of commutative semigroups and will be applied in our subsequent papers devoted to studying categorically closed semigroups.

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